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A Fredholm Integral Equation via a Fixed Point Result in Controlled Metric Spaces

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Abstract. In this article, we establish various fixed point results in the structure of controlled metric spaces. By the help of that results, we give efficient solution of a Fredholm type integral equation. Moreover, some examples are given to show the conceptual depth within this approach.

1 Introduction

More than a century ago in 1903, Fredholm [6] did wonders in the theory of non linear integral equations by presenting an integration expression with fixed limits:

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$$k(\omega) = \Upsilon(\omega) + \lambda \int_p^q \Phi(\omega, t) k(t) \, \mathrm{d}t. \tag{1}$$

In this expression, p and q are constants and λ is a parameter. Given $\Upsilon(\omega)$ and the kernel function $\Phi(\omega, t)$, the problem is typically to find the function $k(\omega)$. Integral equations of Fredholm have been widely used in different fields of mathematics, e.g, physical mathematics, computational mathematics and approximation theory. Motivated by his great work, many researchers have focused their work on solving the Fredholm integral equation by applying the technique of fixed point theory. Rus in [15] used Krasnoselskii's fixed point theorem on cones in order to obtain positive continuous solutions of the following Fredholm integral equation

$$k(\omega) = \Upsilon(\omega) + \int_0^T \Phi(\omega, t) f(k(t)) \, \mathrm{d}t, \qquad \omega \in [0, T], \qquad T > 0. \tag{2}$$

using limit type conditions for f in $(0, \infty)$. In [4], Berenguer et al. established a method of a numerical approximation of the fixed point of an operator, especially the integral one associated with a non linear Fredholm integral equation, that uses the properties of a classical Schauder basis in the Banach space $\mathbb{C}([a, b] \times [a, b])$. In the same direction, Pathak et al. in [13] proved a fixed point result for a pair of weakly compatible mappings, and by the use of that result they proved the existence of a solution of a system of non linear integral equations. For more related works, you can see [8, 14, 2, 1]. Recently, Karapinar et al. in [9] proved fixed point results over extended b-metric spaces and applied that results to find a solution of the equation (1). The in-depth analysis of a non linear expression has resulted one of the best principle, called the Banach contraction principle (BCP). Researchers have been greatly benefited from generalizations of this principle. BCP has been widely utilized by many researchers in their vital researches. In nut shell, many portion of fixed point theory contains BCP generalization. Below, we recall some basic generalizations of it.

Definition 1.1 ([11])**.** A mapping $T : S_1 \cup S_2 \to S_1 \cup S_2$ is called cyclic, if $T(S_1) \subset S_2$ and $T(S_2) \subset S_1$, where S_1, S_2 are nonempty subsets of a metric space (W, d).

Definition 1.2 ([11])**.** A cyclic map $T : S_1 \cup S_2 \to S_1 \cup S_2$ is called cyclic contraction, if for all $\omega_1 \in S_1$, $\omega_2 \in S_2$, there exists $\kappa \in (0, 1)$ such that

$$d(T\omega_1, T\omega_2) \le \kappa d(\omega_1, \omega_2).$$

Definition 1.3 ([10])**.** A cyclic map $T : S_1 \cup S_2 \to S_1 \cup S_2$ is called a cyclic orbital contraction, if for all $\omega_1 \in S_1$, there exists $\kappa \in (0, 1)$ such that

$$d(T^{2n}\omega_1, T\omega_2) \le \kappa d(T^{2n-1}\omega_1, \omega_2), \tag{3}$$

where $\omega_2 \in S_1$, $n \in \mathbb{N}$ and S_1, S_2 are closed subsets of W.

Definition 1.4 ([16])**.** A mapping $T : W \to W$ on a metric space (W, d) is called F-contraction, if there exists $\Omega > 0$ such that

$$\Omega + F(d(T\omega_1, T\omega_2)) \leq F(d(\omega_1, \omega_2)), \tag{4}$$

for all $\omega_1, \omega_2 \in W$ with $d(\omega_1, \omega_2) > 0$, where $F : [0, \infty) \to \mathbb{R}$ is function satisfying the following axioms:

(I) F is strictly increasing;

(II) for each sequence $\{a_n\} \subset (0, \infty)$ of positive real numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;

(III) for each sequence $\{a_n\} \subset (0, \infty)$ of positive real numbers, $\lim_{n\to\infty} a_n = 0$, there exists $\kappa \in (0, 1)$ such that $\lim_{n\to\infty} (a_n)^\kappa F(a_n) = 0$;

We denote by $\mathcal{F}$ the family of all functions F satisfying $(I) - (III)$.

In 2015, Cosentino et al. [**?**] introduced the following Definition, and proved fixed point results over multi valued F-contraction in b-metric [5, 3] spaces.

Definition 1.5. Let $\kappa \geq 1$ be a real number. Then by $\mathcal{F}_\kappa$ we denote the family of all functions $F : (0, \infty) \to \mathbb{R}$, which satisfies the following axioms:

(I) F is strictly non-decreasing;

(II) for each sequence $\{a_n\} \subset (0, \infty)$ of positive real numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$;

(III) for each sequence $\{a_n\} \subset (0, \infty)$ of positive real numbers, $\lim_{n\to\infty} a_n = 0$, there exists $l \in (0, 1)$ such that $\lim_{n\to\infty} (a_n)^l F(a_n) = 0$;

(IV) For each sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers such that

$$\tau + F(\kappa a_n) \leq F(a_{n-1}), \quad \textit{for all } n \in \mathbb{N} \textit{ and some } \tau > 0,$$

then

$$\tau + F(\kappa^n a_n) \leq F(\kappa^{n-1} a_{n-1}), \quad \textit{for all } n \in \mathbb{N}.$$

Recently in [12], Mlaiki et al. introduced controlled type metric space as a generalization of a b-metric space, which is different from extended b-metric spaces [7].

Definition 1.6 ([12])**.** Let W be a nonempty set and $\alpha : W \times W \to [1, \infty)$. Then a mapping $d_\alpha : W \times W \to [0, \infty)$ is called a controlled metric, if for all $\omega_1, \omega_2, \omega_3 \in W$, it satisfies the following axioms:

(i) $d_\alpha(\omega_1, \omega_2) = 0$ iff $\omega_1 = \omega_2$,

(ii) $d_\alpha(\omega_1, \omega_2) = d_\alpha(\omega_2, \omega_1)$,

(iii) $d_\alpha(\omega_1, \omega_3) \leq \alpha(\omega_1, \omega_2) d_\alpha(\omega_1, \omega_2) + \alpha(\omega_2, \omega_3) d_\alpha(\omega_2, \omega_3)$.

The pair (W, d_α) is called a controlled metric space.

Remark 1.1 Every b-metric space is a controlled metric space, if we take $\alpha(\omega_1, \omega_2) = s \geq 1$ for all $\omega_1, \omega_2 \in W$. Generally, a controlled metric space is not an extended b-metric space, if we take same functions $\alpha = \theta$ as follows:

Example 1.7 ([12]). Let $W = \{1, 2, \ldots\ldots\}$. Define $d_\alpha : W \times W \to [0, \infty)$ as:

$$d_\alpha(\omega_1, \omega_2) = \begin{cases} 0, & \text{if } \omega_1 = \omega_2; \\ \frac{1}{\omega_1}, & \text{if } \omega_1 \text{ is even and } \omega_2 \text{ is odd;} \\ \frac{1}{\omega_2}, & \text{if } \omega_1 \text{ is odd and } \omega_2 \text{ is even;} \\ 1, & \text{otherwise.} \end{cases}$$

Hence (W, d_α) is a controlled metric space, where $\alpha : W \times W \to [1, \infty)$ is defined as:

$$\alpha(\omega_1, \omega_2) = \begin{cases} \omega_1, & \text{if } \omega_1 \text{ is even and } \omega_2 \text{ is odd;} \\ \omega_2, & \text{if } \omega_1 \text{ is odd and } \omega_2 \text{ is even;} \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, d_α is not an extended b-metric for the same function $\alpha = \theta$.

In this paper, we introduce and establish new sort of contractions, called controlled cyclic orbital contractions and controlled cyclic orbital F-contractions in the setting of controlled metric spaces. By the help of that results, we are in the position to consider real-life applications in a very general structure such as simple and efficient solution of a Fredholm integral equation in the setting of a controlled metric space. Moreover, some examples are given to show the conceptual depth within this approach.

2 Main Results

In this section, we introduce the notion of controlled cyclic orbital contractions.

Definition 2.1. Let S_1, S_2 be nonempty subsets of a controlled metric space (W, d_α). Then a cyclic map $T : S_1 \cup S_2 \to S_1 \cup S_2$ is called a controlled cyclic orbital contraction, if for some $\omega_1 \in S_1$, there exists $\kappa \in (0, 1)$ such that

$$d_\alpha(T^{2n}\omega_1, T\omega_2) \leq \kappa d_\alpha(T^{2n-1}\omega_1, \omega_2), \tag{5}$$

where $\omega_2 \in S_1$ and $n \in \mathbb{N}$.

Theorem 2.2. *Let S_1 and S_2 be nonempty closed subsets of a complete controlled metric space (W, d_α) such that d_α is a continuous functional. Let $T : S_1 \cup S_2 \to S_1 \cup S_2$ be a controlled cyclic orbital contraction. Also suppose that for any $\omega_0 \in W$,*

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\alpha(\omega_{i+1}, \omega_{i+2})}{\alpha(\omega_i, \omega_{i+1})} \alpha(\omega_{i+1}, \omega_m) < \frac{1}{k}, \tag{6}$$

where $\omega_n = T^n\omega_0$, $n = 1, 2, \ldots$. Then $S_1 \cap S_2$ is nonempty and T has a unique fixed point.

Proof. Suppose there exists $\omega_0 \in S_1$ satisfying equation (5). Define an iterative sequence $\{\omega_n\}$ starting from ω_0 as follows:

$$\omega_1 = T\omega_0, \ \omega_2 = T\omega_1 = T(T\omega_0) = T^2\omega_0, \ \ldots, \omega_n = T^n\omega_0, \ldots$$

From equation (5), we have

$$d_\alpha(T^2\omega_0, T\omega_0) \leq \kappa d_\alpha(T\omega_0, \omega_0).$$

Recursively, we have

$$\begin{aligned} d_\alpha(T^3\omega_0, T^2\omega_0) &\leq \kappa d_\alpha(T^2\omega_0, T\omega_0) \\ &\leq \kappa^2 d_\alpha(T\omega_0, \omega_0). \end{aligned}$$

As for any $n \in \mathbb{N}$, either n or $n+1$ is even, so we have

$$d_\alpha(T^{n+1}\omega_0, T^n\omega_0) \leq \kappa^n d_\alpha(T\omega_0, \omega_0),$$

that is,

$$d_\alpha(\omega_{n+1}, \omega_n) \leq \kappa^n d_\alpha(\omega_1, \omega_0). \tag{7}$$

From the triangle inequality and equation (7) for $m > n$, we have

$$\begin{aligned} d_\alpha(\omega_n, \omega_m) &\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \alpha(\omega_{n+1}, \omega_m) d_\alpha(\omega_{n+1}, \omega_m) \\ &\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \alpha(\omega_n, \omega_m)\alpha(\omega_{n+1}, \omega_{n+2}) d_\alpha(\omega_{n+1}, \omega_{n+2}) \\ &+ \alpha(\omega_n, \omega_m)\alpha(\omega_{n+2}, \omega_m) d_\alpha(\omega_{n+2}, \omega_m) \\ &\vdots \\ &\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1}) d_\alpha(\omega_i, \omega_{i+1}) \\ &+ \prod_{j=n+1}^{m-1} \alpha(\omega_j, \omega_m)\alpha(\omega_{m-1}, \omega_m) d_\alpha(\omega_{m-1}, \omega_m) \\ &\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \sum_{i=n+1}^{m-1} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1}) d_\alpha(\omega_i, \omega_{i+1}) \\ &\leq [\alpha(\omega_n, \omega_{n+1})\kappa^n + \sum_{i=n+1}^{m-1} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1})\kappa^i] d_\alpha(\omega_1, \omega_0). \end{aligned}$$

Using (6) for any $\omega_0 \in S_1$, we get the series $\sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1})\kappa^i$ converges by the ratio test for each $m \in \mathbb{N}$. Let $S = \sum_{i=1}^{\infty} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1})\kappa^i$ and

$$S_n = \sum_{p=1}^{n} \left(\prod_{q=1}^{p} \alpha(\omega_q, \omega_m) \right) \alpha(\omega_p, \omega_{p+1})\kappa^p.$$

Thus, for $m > n$, we have

$$d_\alpha(\omega_n, \omega_m) \leq [\alpha(\omega_n, \omega_{n+1})\kappa^n + (S_{m-1} - S_n)]d_\alpha(\omega_1, \omega_0). \tag{8}$$

Condition (6) implies that $\lim_{n\to\infty} S_n$ exists and hence the real sequence $\{S_n\}$ is Cauchy. Finally, if we take limit in the inequality (8) as $n, m \to \infty$, we deduce that

$$\lim_{n,m\to\infty} d_\alpha(\omega_n, \omega_m) = 0, \tag{9}$$

that is, $\{\omega_n\}$ is a Cauchy sequence. Hence as a result, there exists $\rho \in S_1 \cup S_2$ such that $\omega_n \to \rho$. Now, note that $\{\omega_{2n}\} = \{T^{2n}\omega_0\}$ is a sequence in S_1 and $\{\omega_{2n-1}\} = \{T^{2n-1}\omega_0\}$ is a sequence in S_2 and both converge to ρ. As the sets S_1 and S_2 are closed in W and $\rho \in S_1 \cap S_2$, hence $S_1 \cap S_2$ is nonempty. Next, we prove that ρ is a fixed point of T. The continuity of d_α yields that

$$\begin{aligned} d_\alpha(\rho, T\rho) &= \lim_{n\to\infty} d_\alpha(T^{2n}\omega, T\rho) \\ &\leq \kappa \lim_{n\to\infty} d_\alpha(T^{2n-1}\omega, \rho) \\ &= 0. \end{aligned}$$

Thus ρ is a fixed point of T. For the uniqueness, assume there exists $\varrho \in S_1 \cap S_2$, $\rho \neq \varrho$ such that $T\varrho = \varrho$. Now,

$$\begin{aligned} d_\alpha(\rho, \varrho) &= d_\alpha(T\rho, T\varrho) \\ &\leq \kappa d_\alpha(\rho, \varrho) \\ &< d_\alpha(\rho, \varrho). \end{aligned}$$

Which is a contradiction. Hence, $\rho = \varrho$ and ρ is the unique fixed point of T. □

Remark 2.3. Since a controlled metric space is not in general an extended b-metric space. Therefore Theorem 2.2 is different from Theorem 1 of Karapinar et al. [9].

Remark 2.4. (1) For $\alpha(\omega_1, \omega_2) = s \geq 1$, the above theorem reduces to the b-metric space.

(2) For $\alpha(\omega_1, \omega_2) = 1$, the above theorem reduces to the main result of Karpagam et al. [10].

Example 2.5. Let $W = \mathbb{R}$. Define $d_\alpha : W \times W \to [0, \infty)$ by $d_\alpha(\omega_1, \omega_2) = (\omega_1 - \omega_2)^2$. Then, clearly (W, d_α) is a complete controlled metric space with $\alpha : W \times W \to [1, \infty)$ defined as $\alpha(\omega_1, \omega_2) = 3\omega_1 + 2\omega_2 + 2$. Let $S_1 = [0, \frac{1}{3}]$ and $S_2 = [\frac{1}{4}, 1]$. Define a mapping $T : S_1 \cup S_2 \to S_1 \cup S_2$ as

$$T\omega_1 = \begin{cases} \frac{1}{3}, & 0 \leq \omega_1 \leq \frac{1}{4}; \\ \frac{1}{2}(1 - \omega_1), & \frac{1}{4} < \omega_1 \leq 1. \end{cases}$$

First, we have to show that T is a cyclic map.

(i) If $\omega_1 = 0 \in S_1$, then $T0 = \frac{1}{3} \in S_2$.
(ii) If $\omega_1 = \frac{1}{3} \in S_1$, then $T\frac{1}{3} = \frac{1}{3} \in S_2$.
(iii) If $\omega_1 = \frac{1}{4} \in S_2$, then $T\frac{1}{4} = \frac{1}{3} \in S_1$.
(iv) If $\omega_1 = 1 \in S_2$, then $T1 = 0 \in S_1$.

Hence $T(S_1) \subseteq S_2$, $T(S_2) \subseteq S_1$ and T is a cyclic map. Next fix any $\omega_1 \in S_1$. Let $\omega_1 = 0$, then we have

$$T\omega_1 = \frac{1}{3}, \ T^2\omega_1 = T(T\omega_1) = \frac{1}{3}, \ \dots.$$

Thus $T^n\omega_1 = \frac{1}{3}$, therefore $T^{2n}\omega_1 = \frac{1}{3}$ and $T^{2n-1}\omega_1 = \frac{1}{3}$. For ω_2, we will take the following cases:

Case 1: If $\omega_2 = 0$, $T0 = \frac{1}{3}$, then $d_\alpha(T^{2n}\omega_1, T\omega_2) = d_\alpha(\frac{1}{3}, \frac{1}{3}) = 0$ and $d_\alpha(T^{2n-1}\omega_1, \omega_2) = d_\alpha(\frac{1}{3}, 0) = \frac{1}{9}$. Hence, for $\kappa = \frac{1}{4} \in (0, 1)$

$$d_\alpha(T^{2n}\omega_1, T\omega_2) \leq \kappa d_\alpha(T^{2n-1}\omega_1, \omega_2).$$

Case 2: If $\omega_2 = \frac{1}{3}$, $T\frac{1}{3} = \frac{1}{3}$, then $d_\alpha(T^{2n}\omega_1, T\omega_2) = d_\alpha(\frac{1}{3}, \frac{1}{3}) = 0$ and $d_\alpha(T^{2n-1}\omega_1, \omega_2) = d_\alpha(\frac{1}{3}, \frac{1}{3}) = 0$. Hence, for $\kappa = \frac{1}{4} \in (0, 1)$

$$d_\alpha(T^{2n}\omega_1, T\omega_2) \leq \kappa d_\alpha(T^{2n-1}\omega_1, \omega_2).$$

Case 3: If $0 < \omega_2 < \frac{1}{3}$, we will take subcases:

Subcase A: If $0 < \omega_2 \leq \frac{1}{4}$, $T\omega_2 = \frac{1}{3}$, then $d_\alpha(T^{2n}\omega_1, T\omega_2) = d_\alpha(\frac{1}{3}, \frac{1}{3}) = 0$ and $d_\alpha(T^{2n-1}\omega_1, \omega_2) = d_\alpha(\frac{1}{3}, \omega_2) = (\frac{1}{3} - \omega_2)^2$. Hence for $\kappa = \frac{1}{4} \in (0, 1)$

$$d_\alpha(T^{2n}\omega_1, T\omega_2) \leq \kappa d_\alpha(T^{2n-1}\omega_1, \omega_2).$$

Subcase B: If $\frac{1}{4} < \omega_2 \leq \frac{1}{3}$, $T\omega_2 = \frac{1}{2}(1 - \omega_2)$, then $d_\alpha(T^{2n}\omega_1, T\omega_2) = d_\alpha(\frac{1}{3}, \frac{1}{2}(1 - \omega_2)) = (\frac{1}{3} - \frac{1}{2} + \frac{\omega_2}{2})^2$. This implies that $d_\alpha(T^{2n}\omega_1, T\omega_2) = (\frac{-1}{6} + \frac{\omega_2}{2})^2 = \frac{1}{4}(\omega_2 - \frac{1}{3})^2$, and $d_\alpha(T^{2n-1}\omega_1, \omega_2) = d_\alpha(\frac{1}{3}, \omega_2) = (\frac{1}{3} - \omega_2)^2$. Hence for $\kappa = \frac{1}{4} \in (0, 1)$

$$d_\alpha(T^{2n}\omega_1, T\omega_2) \leq \kappa d_\alpha(T^{2n-1}\omega_1, \omega_2).$$

Hence, in all cases, the conditions of Theorem 2.2 are satisfied, and $\frac{1}{3}$ is the unique fixed point of T.

Definition 2.6. Let S_1, S_2 be nonempty subsets of a controlled metric space (W, d_α). Then a cyclic map $T : S_1 \cup S_2 \to S_1 \cup S_2$ is called a controlled cyclic orbital F-contraction, if for some $\omega_1 \in S_1$, there exists $\Omega > 0$ such that for all $\omega_1, \omega_2 \in W$ with $d_\alpha(T\omega_1, T\omega_2) > 0$, the following inequality holds:

$$\Omega + F(\kappa d_\alpha(T^{2n}\omega_1, T\omega_2)) \leq F(d_\alpha(T^{2n-1}\omega_1, \omega_2)), \tag{10}$$

where $\kappa > 1$, $\omega_2 \in S_1$ and $n \in \mathbb{N}$.

Theorem 2.7. *Let S_1 and S_2 be two nonempty subsets of a complete controlled metric space (W, d_α).Let $T : S_1 \cup S_2 \to S_1 \cup S_2$ be a continuous controlled cyclic orbital F-contraction such that for any $\omega_0 \in W$,*

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\alpha(\omega_{i+1}, \omega_{i+2})}{\alpha(\omega_i, \omega_{i+1})} \alpha(\omega_{i+1}, \omega_m) < \kappa, \tag{11}$$

where $\kappa > 1$ and $\omega_n = T^n\omega_0$, $n = 1, 2, \ldots$. Then $S_1 \cap S_2$ is nonempty and T has a fixed point.

Proof. let us consider an arbitrary $\omega_0 \in S_1$ satisfying equation (10). As for any $n \in \mathbb{N}$, either n or $n+1$ is even, so we have

$$F(\kappa^n d_\alpha(T^{n+1}\omega, T^n\omega)) \leq F(d_\alpha(T\omega, \omega)) - n\Omega. \tag{12}$$

By taking limit $n \to \infty$ in equation (12), we get $\lim_{n\to\infty} F(\kappa^n d_\alpha(\omega_n, \omega_{n+1})) = -\infty$. Thus from condition (II) of Definition 1.4, we have

$$\lim_{n\to\infty} \kappa^n d_\alpha(\omega_n, \omega_{n+1}) = 0.$$

Also from condition (III), there exists $l \in (0, 1)$ such that

$$\lim_{n\to\infty} (\kappa^n d_\alpha(\omega_n, \omega_{n+1}))^l F(\kappa^n d_\alpha(\omega_n, \omega_{n+1})) = 0.$$

From equation (12), for all $n \in \mathbb{N}$, the following holds:

$$\lim_{n\to\infty} (\kappa^n d_\alpha(\omega_n, \omega_{n+1}))^l F(\kappa^n d_\alpha(\omega_n, \omega_{n+1})) - \lim_{n\to\infty} (\kappa^n d_\alpha(\omega_n, \omega_{n+1}))^l F(\mathcal{A}_0) \leq$$

$$\lim_{n\to\infty} -(\kappa^n d_\alpha(\omega_n, \omega_{n+1}))^l n\Omega \leq 0. \tag{13}$$

By letting $n \to \infty$ in (13), we obtain

$$\lim_{n\to\infty} n(\kappa^n d_\alpha(\omega_n, \omega_{n+1}))^l = 0. \tag{14}$$

From equation (14), there exists $n_1 \in \mathbb{N}$ such that $n(\kappa^n d_\alpha(\omega_n, \omega_{n+1}))^l \leq 1$ for all $n \geq n_1$. Thus for all $n \geq n_1$, we have

$$d_\alpha(\omega_n, \omega_{n+1}) \leq \frac{1}{\kappa^n n^{\frac{1}{l}}}. \tag{15}$$

From the triangle inequality and equation (15) for $m > n \geq n_1$, we have

$$\begin{aligned}
d_\alpha(T^n\omega, T^m\omega) &= d_\alpha(\omega_n, \omega_m) \\
&\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \alpha(\omega_{n+1}, \omega_m) d_\alpha(\omega_{n+1}, \omega_m) \\
&\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \alpha(\omega_n, \omega_m)\alpha(\omega_{n+1}, \omega_{n+2}) d_\alpha(\omega_{n+1}, \omega_{n+2}) \\
&+ \alpha(\omega_n, \omega_m)\alpha(\omega_{n+2}, \omega_m) d_\alpha(\omega_{n+2}, \omega_m) \\
&\leq \dots \\
&\leq \alpha(\omega_n, \omega_{n+1}) d_\alpha(\omega_n, \omega_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1}) d_\alpha(\omega_i, \omega_{i+1}) \\
&+ \prod_{j=n+1}^{m-1} \alpha(\omega_j, \omega_m)\alpha(\omega_{m-1}, \omega_m) d_\alpha(\omega_{m-1}, \omega_m) \\
&\leq \alpha(\omega_n, \omega_{n+1}) \frac{1}{\kappa^n n^{\frac{1}{l}}} + \sum_{i=n+1}^{m-1} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1}) \frac{1}{\kappa^i i^{\frac{1}{l}}} \\
&\leq \alpha(\omega_n, \omega_{n+1}) \frac{1}{\kappa^n n^{\frac{1}{l}}} + \sum_{i=n+1}^{\infty} \left(\prod_{j=1}^{i} \alpha(\omega_j, \omega_m) \right) \alpha(\omega_i, \omega_{i+1}) \frac{1}{\kappa^i i^{\frac{1}{l}}}.
\end{aligned}$$

By (11), the series $\sum_{n=1}^{\infty} \frac{1}{\kappa^n n^{\frac{1}{l}}} \prod_{i=1}^{n} \alpha(\omega_i, \omega_m)\alpha(\omega_i, \omega_{i+1})$ converges by the ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{i=1}^{\infty} \frac{1}{\kappa^i i^{\frac{1}{l}}} \prod_{j=1}^{i} \alpha(\omega_j, \omega_m)\alpha(\omega_i, \omega_{i+1}), \quad S_n \quad = \quad \sum_{p=1}^{n} \frac{1}{\kappa^p p^{\frac{1}{l}}} \prod_{q=1}^{p} \alpha(\omega_q, \omega_m)\alpha(\omega_p, \omega_{p+1}).$$

Therefore for $m > n$, we have

$$d_\alpha(\omega_n, \omega_m) \leq \alpha(\omega_n, \omega_{n+1}) \frac{1}{n^{\frac{1}{l}}} + S_{m-1} - S_n. \tag{16}$$

Condition (11) yields that $\lim_{n\to\infty} S_n$ exists and hence the real sequence $\{S_n\}$ is Cauchy. Taking $n, m \to \infty$, in the inequality (16), we get that

$$\lim_{n,m\to\infty} d_\alpha(\omega_n, \omega_m) = 0. \tag{17}$$

We conclude that $\{\omega_n\}$ is a Cauchy sequence. Hence, as a result, there exists $\rho \in S_1 \cup S_2$ such that $\omega_n \to \rho$. Now, note that $\{\omega_{2n}\} = \{T^{2n}\omega_0\}$ is a sequence in S_1 and $\{\omega_{2n-1}\} = \{T^{2n-1}\omega_0\}$ is a sequence in S_2 and both converge to ρ. As the sets S_1 and S_2 are closed in W and $\rho \in S_1 \cap S_2$. Hence, $S_1 \cap S_2$ is nonempty. Next, we prove that ρ is a fixed point of T. Let $\rho \neq T\rho$, then from the triangle inequality

$$d_\alpha(\rho, T\rho) \leq \alpha(\rho, T^{2n}\omega_0) d_\alpha(\rho, T^{2n}\omega_0) + \alpha(T^{2n}\omega_0, T\rho) d_\alpha(T^{2n}\omega_0, T\rho). \tag{18}$$

Since $T^{2n-1}\omega_0 \to \rho$ as $n \to \infty$, and from the continuity of T, we have $\lim_{n\to\infty} d_\alpha(T^{2n}\omega_0, T\rho) = 0$. Hence, from equation (18), we get $d_\alpha(\rho, T\rho) = 0$ as $n \to \infty$. Hence, $\rho = T\rho$ and ρ is the fixed point of T.

□

Remark 2.8. As controlled metric space is a generalization of b-metric space, which is different from extended b-metric space. Therefore Theorem 2.7 is different from Theorem 2 of Karapinar et al. [9].

Remark 2.9. (a) For $\alpha(\omega_1, \omega_2) = s \geq 1$, Theorem 2.7 reduces to the b-metric space.

(b) For $\alpha(\omega_1, \omega_2) = 1$, Theorem 2.7 reduces to the metric space.

Example 2.10. Let $W = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$. Define $d_\alpha : W \times W \to [0, \infty)$ by $d_\alpha(\omega_1, \omega_2) = (\omega_1 - \omega_2)^2$. Then, clearly (W, d_α) is a complete controlled metric space with $\alpha : W \times W \to [1, \infty)$ defined as $\alpha(\omega_1, \omega_2) = 3\omega_1 + 2\omega_2 + 1$. Let $S_1 = \{\frac{1}{2^{2n-1}} : n \in \mathbb{N}\} \cup \{0\}$ and $S_2 = \{\frac{1}{2^{2n}} : n \in \mathbb{N}\} \cup \{0\}$. Define a mapping $T : S_1 \cup S_2 \to S_1 \cup S_2$ as

$$T\omega_1 = \begin{cases} \frac{1}{2^{n+1}}, & \text{if } \omega_1 \in \{\frac{1}{2^n} : n \in \mathbb{N}\}; \\ 0, & \text{if } \omega_1 = 0. \end{cases}$$

Clearly $T(S_1) \subseteq S_2$, $T(S_2) \subseteq S_1$ and T is a cyclic map. Next, fix any $\omega_1 \in S_1$. Let $\omega_1 = \frac{1}{2^{2n-1}}$, then we have

$$T\omega_1 = \frac{1}{2^{2n}},\ T^2\omega_1 = T(T\omega_1) = \frac{1}{2^{2n+1}},\ \ldots.$$

Thus, $T^{2n}\omega_1 = \frac{1}{2^{2n+2n-1}} = \frac{1}{2^{4n-1}}$ and $T^{2n-1}\omega_1 = \frac{1}{2^{4n-2}}$. For ω_2, we will take the following cases:

Case 1: If $\omega_2 \in S_1/\{0, 1\}$, let

$$\omega_2 = \frac{1}{2^{2m-1}}, \quad (m > n \geq 1).$$

We have $T\omega_2 = \frac{1}{2^{2m}}$. Also,

$$\begin{aligned} d_\alpha(T^{2n}\omega_1, T\omega_2) &= d_\alpha\left(\frac{1}{2^{4n-1}}, \frac{1}{2^{2m}}\right) \\ &= \left(\frac{1}{2^{4n-1}} - \frac{1}{2^{2m}}\right)^2 \\ &= \left(\frac{2^{2m} - 2^{4n-1}}{2^{4n+2m-1}}\right)^2, \end{aligned}$$

and

$$\begin{aligned} d_\alpha(T^{2n-1}\omega_1, \omega_2) &= d_\alpha\left(\frac{1}{2^{4n-2}}, \frac{1}{2^{2m-1}}\right) \\ &= \left(\frac{1}{2^{4n-2}} - \frac{1}{2^{2m-1}}\right)^2 \\ &= \left(\frac{2^{2m-1} - 2^{4n-2}}{2^{4n+2m-3}}\right)^2. \end{aligned}$$

Define the function $F:[0,\infty)\to\mathbb{R}$ by $F(t)=\ln t$, for all $t\in[0,\infty)$ and $\Omega>0$.

$$\begin{aligned}
F(2d_\alpha(T^{2n}\omega_1,\omega_2))-F(d_\alpha(T^{2n-1}\omega_1,\omega_2)) &= \ln 2+2\left(\ln\frac{2^{2m}-2^{4n-1}}{2^{4n+2m-1}}-\ln\frac{2^{2m-1}-2^{4n-2}}{2^{4n+2m-3}}\right)\\
&= \ln 2+2\ln\left(\frac{2^{2m}-2^{4n-1}}{2^{4n+2m-1}}\times\frac{2^{4n+2m-3}}{2^{2m-1}-2^{4n-2}}\right)\\
&= \ln 2+2\ln\left(\frac{2^{2m}-2^{4n-1}}{2^{2m-1}-2^{4n-2}}\times 2^{-2}\right)\\
&= 2\ln\left(\frac{2^{2m}-2^{4n-1}}{2^{2m+1}-2^{4n}}\right)\\
&= \ln 2+2\ln\left(\frac{2^{2m}-2^{4n-1}}{2(2^{2m}-2^{4n-1})}\right)\\
&= \ln 2+2\ln\frac{1}{2}\\
&< -\frac{1}{3}.
\end{aligned}$$

Case 2: If $\omega_2=0,\ T0=0$, then

$$\begin{aligned}
d_\alpha(T^{2n}\omega_1,T\omega_2) &= d_\alpha\left(\frac{1}{2^{4n-1}},0\right)\\
&= \left(\frac{1}{2^{4n-1}}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
d_\alpha(T^{2n-1}\omega_1,\omega_2) &= d_\alpha\left(\frac{1}{2^{4n-2}},0\right)\\
&= \left(\frac{1}{2^{4n-2}}\right)^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
F(2d_\alpha(T^{2n}\omega_1,\omega_2))-F(d_\alpha(T^{2n-1}\omega_1,\omega_2)) &= \ln 2+2\left(\ln\frac{1}{2^{4n-1}}-\ln\frac{1}{2^{4n-2}}\right)\\
&= 2\ln\left(\frac{1}{2^{4n-1}}\times\frac{2^{4n-2}}{1}\right)\\
&= \ln 2+2\ln\left(2^{-1}\right)\\
&= \ln 2+2\ln\frac{1}{2}\\
&< -\frac{1}{3}.
\end{aligned}$$

Hence, for $\Omega = \frac{1}{3}$ and $\kappa = \frac{1}{3}$, T is a controlled cyclic orbital F-contraction. Thus, all the conditions of Theorem 2.7 are satisfied, and 0 is the fixed point of T.

3 Applications to the Existence of a Solution of a Fredholm Integral Equation

In this section, we ensure the existence of solution of a Fredholm integral equation.

Theorem 3.1. *Let $W = \mathbb{C}([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on closed interval $[a,b]$ with a controlled metric given as*

$$d_\alpha(\omega_1,\omega_2) = \sup_{t\in[a,b]} |\omega_1(t) - \omega_2(t)|^2. \tag{19}$$

Clearly, (W, d_α) is a complete controlled metric space with $\alpha : W \times W \to [1,\infty)$ defined as $\alpha(\omega_1,\omega_2) = 3|\omega_1(t)| + 2|\omega_2(t)| + 2$. Let us consider the Fredholm integral equation:

$$\omega(t) = \int_a^b \mathcal{M}(t,r,\omega(r))\,\mathrm{d}r + \mathbf{v}(t), \tag{20}$$

where $t, r \in [a,b]$, $\mathbf{v} : [a,b] \to \mathbb{R}$ and $\mathcal{M} : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ both are continuous functions. Let $S_1 = S_2 = W = (\mathbb{C}([a,b]),\mathbb{R})$. Clearly, S_1, S_2 are closed subsets of W. Define $T : S_1 \cup S_2 \to S_1 \cup S_2$ by

$$\omega(t) = \int_a^b \mathcal{M}(t,r,\omega(r))\,\mathrm{d}r + \mathbf{v}(t), \quad \textit{for} \quad t,\ r \in [a,b], \tag{21}$$

where both $\mathbf{v}$ and $\mathcal{M}$ are continuous functions. Clearly, $T(S_1) \subset S_2$ and $T(S_2) \subset S_1$. Thus, T is a cyclic map on $S_1 \cup S_2$. Moreover, let us consider that for $\omega \in W$ and $t, r \in [a,b]$, the following condition holds:

$$|\mathcal{M}(t,r,\omega(r)) - \mathcal{M}(t,r,T\omega(r))| \leq \frac{1}{2}|\omega(r) - T\omega(r)|. \tag{22}$$

Then, the integral equation (20) has a solution. Now, we will show that T satisfies all the conditions of Theorem 2.2. For $\omega \in W$, consider

$$\begin{aligned}
|T^2\omega(t) - T\omega(t)|^2 &= |T(T\omega(t)) - T\omega(t)|^2 \\
&= |\int_a^b [\mathcal{M}(t,r,T\omega(r)) - \mathcal{M}(t,r,\omega(r))\,\mathrm{d}r|^2 \\
&\leq \int_a^b \frac{1}{4}|T\omega(r) - \omega(r)|^2\,\mathrm{d}r \\
&\leq \frac{1}{4} d_\alpha(T\omega,\omega).
\end{aligned}$$

This implies that

$$\mathrm{d}_\alpha(T^2\omega, T\omega) \leq \frac{1}{4}\mathrm{d}_\alpha(T\omega, \omega),$$

where $\kappa = \frac{1}{4} \in (0,1)$. *Thus, all the conditions of Theorem 2.2 follow by the hypothesis. Hence,* T *has a fixed point, that is, the Fredholm integral equation* (20) *has a solution.*

Theorem 3.2. *Let* $W = \mathbb{C}([a,b],\mathbb{R})$ *be the space of all continuous real valued functions defined on the closed interval* $[a,b]$ *with a metric given by*

$$\mathrm{d}_\alpha(\omega_1,\omega_2) = \sup_{t\in[a,b]} |\omega_1(t) - \omega_2(t)|^2. \tag{23}$$

Clearly, (W, d_α) *is a complete controlled metric space with* $\alpha : W \times W \to [1,\infty)$ *defined as* $\alpha(\omega_1,\omega_2) = 3|\omega_1(t)| + 2|\omega_2(t)| + 2$. *Let us consider the Fredholm integral equation*

$$\omega(t) = \int_a^b \mathcal{M}(t,r,\omega(r))\,\mathrm{d}r + \mathbf{v}(t), \tag{24}$$

where $t, r \in [a,b]$, $\mathbf{v} : [a,b] \to \mathbb{R}$ *and* $\mathcal{M} : [a,b]\times[a,b]\times\mathbb{R} \to \mathbb{R}$ *both are continuous functions. Let* $S_1 = S_2 = W = (\mathbb{C}([a,b]),\mathbb{R})$. *clearly* S_1, S_2 *are closed subsets of* W. *Define* $T : S_1 \cup S_2 \to S_1 \cup S_2$ *by*

$$\omega(t) = \int_a^b \mathcal{M}(t,r,\omega(r))\,\mathrm{d}r + \mathbf{v}(t), \quad \textit{for} \quad t,\ r \in [a,b], \tag{25}$$

where both $\mathbf{v}$ *and* $\mathcal{M}$ *are continuous functions. Clearly,* $T(S_1) \subset S_2$ *and* $T(S_2) \subset S_1$. *Thus,* T *is a cyclic map on* $S_1 \cup S_2$. *Moreover, let us consider that for* $\omega \in W$, $\Omega > 0$ *and* $t, r \in [a,b]$, *the following condition holds:*

$$|\mathcal{M}(t,r,\omega(r)) - \mathcal{M}(t,r,T\omega(r))| \leq |\frac{1}{2}e^{-\frac{\Omega}{2}}(\omega(r) - T\omega(r))|. \tag{26}$$

Then, the integral equation (24) *has a solution. Now, we will show that* T *satisfies all the conditions of Theorem 2.7. For* $\omega \in W$, *consider*

$$\begin{aligned}
|T^2\omega(t) - T\omega(t)|^2 &= |T(T\omega(t)) - T\omega(t)|^2 \\
&= |\int_a^b [\mathcal{M}(t,r,T\omega(r)) - \mathcal{M}(t,r,\omega(r))\,\mathrm{d}r|^2 \\
&\leq \int_a^b |\frac{1}{2}e^{-\frac{\Omega}{2}}(T\omega(r) - \omega(r))|^2\,\mathrm{d}r \\
&\leq \frac{1}{4}e^{-\Omega}\mathrm{d}_\alpha(T\omega,\omega).
\end{aligned}$$

Which implies that

$$4\mathrm{d}_\alpha(T^2\omega, T\omega) \leq e^{-\Omega}\mathrm{d}_\alpha(T\omega,\omega).$$

Thus,

$$\ln(4d_\alpha(T^2\omega, T\omega)) \leq \ln(e^{-\Omega}) + \ln(d_\alpha(T\omega, \omega)).$$

That is,

$$\Omega + \ln(4d_\alpha(T^2\omega, T\omega)) \leq \ln(d_\alpha(T\omega, \omega)).$$

Hence, all the conditions of Theorem 2.7 are satisfied for $F(t_1) = \ln t_1$, $t_1 > 0$ *and* $\kappa = 4$. *Thus, T has a fixed point, that is, the Fredholm integral equation (24) has a solution.*

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