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Original Research Paper

On Qualitative Behaviors of Nonlinear Singular Systems with Multiple Constant Delays

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Abstract. In this paper, we investigate some qualitative properties of a class of nonlinear singular systems with multiple constant delays. By using the Lyapunov-Krasovskii functional (LKF) method and integral inequalities, we obtain some new sufficient conditions which guarantee that the considered systems are regular, impulse-free and exponentially stable. Two numerical examples are provided to illustrate the application of the obtained results using MATLAB software. By this paper, we extend and improve some results in the literature.

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1 Introduction

When we look at the mathematics literature, it is seen that there are many articles on the stability of solutions of delay differential equations.

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Especially, in recent years, in many books and papers, linear and non-linear singular systems have been discussed and numerous results have been obtained on the stability and admissibility of these systems. In addition, as some recent and interesting works, mathematical modeling of human liver, fractional dynamics, chaotic and non-chaotic behaviors and so on for some different mathematical models have been investigated in the literature [1-6,26-28]

As we know, singular systems, which are also called differential-algebraic, implicit, semi-state or descriptor systems are widely used in various engineering systems such as manufacturing, chemical, economic and circuit systems. The researches on these fields are greatly important. The problems of stability and admissibility behaviors for singular systems with delays have been investigated by many researchers (see, for example [7-25] and references therein).

Some recent and related results can be summarized as the following:

In 2011, Ding et al. [7] considered the following nonlinear singular system with time-varying delay:

$$E\dot{x}(t) = Ax(t) + Bx(t-h(t)) + G_1 f_1(t, x(t)) + G_2 f_2(t, x(t-h(t))).$$

Using Lyapunov-Krasovskii functional and free-weighting method, the authors established some sufficient conditions for the uniformly asymptotic stability of solutions of this singular system.

Later, Liu [10] considered the following linear singular system with constant delay:

$$E\dot{x}(t) = Ax(t) + Bx(t-h).$$

By means of Lyapunov-Krasovskii functional and the integral inequalities, the author obtained asymptotic and exponential stability results for this system.

Further, in 2014, Liu et al. [11] considered the following linear singular system with constant delay:

$$E\dot{x}(t) = Ax(t) + A_d x(t-\tau).$$

By applying Lyapunov-Krasovskii functional and the Wirtinger-based integral inequality method, Liu et al. [11] obtained a new stability criterion for this system in terms of a linear matrix inequality (LMI).

The motivation of this paper has been inspired by the results of Ding et al. [7], Liu [10], Liu et al. ([11], [12]) and those in the literature. Here, we consider the following nonlinear singular system with multiple constant delays:

$$\begin{aligned} E\dot{x}(t) =& Ax(t) + Bx(t-h_1) + Cx(t-h_2) + F_0(t, x(t)) \\ &+ F_1(t, x(t-h_3)) + F_2(t, x(t-h_4)), t > 0, \end{aligned} \tag{1}$$

with

$$x(\theta) = \phi(\theta), \theta \in [-h, 0], h = max\{h_1, h_2, h_3, h_4\} > 0,$$

where $x(t) \in R^n$ is the state vector, $h_1, h_2, h_3, h_4 > 0$ are constant delays, $\phi(t)$ is a continuous initial function defined on $[-h, 0], x_t = x(t+\theta)$ for $-h \leq \theta \leq 0$. $A \in R^{n\times n}$ is a negative definite real symmetric constant matrix, $B, C \in R^{n\times n}$ are real constant matrices, the matrix $E \in R^{n\times n}$ is singular, and it is assumed that $rankE = r \leq n, n \geq 1$. $F_i(t, 0) = 0, F_i \in C^1(R^+ \times R^n, R^n), (i = 0, 1, 2)$, and they satisfy the Lipschitz condition:

$$\|F_i(t, x_0) - F_i(t, y_0)\| \leq \|U_i(x_0 - y_0)\|, \forall t \in R^+, \forall x_0, y_0 \in R^n, \tag{2}$$

where U_0, U_1, U_2 are some known constant matrices.

2 Preliminaries

We now give some definitions and lemmas, which are needed in advance.

Definition 1.1 ([6]). The pair (E, A) is said to be regular if $det(sE - A) \neq 0$. The pair (E, A) is said to be impulse-free if $deg(det(sE - A)) = rank(E)$.
Definition 1.2 ([23]). The singular delay system (1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free.
Definition 1.3 ([23]). The singular delay system (1) is said to be admissible if it is regular, impulse-free and stable.

Lemma 1.1 ([23]).Suppose that the pair (E, A) is regular and impulse-free. Then, the solution of the singular delay system (1) exists and is impulse-free and unique on $[0, \infty)$.
Lemma 1.2 ([10]). For any positive semi-definite matrix

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$$

the following integral inequality holds:

$$-\int_{t-h}^{t} \dot{x}^T(s) X_{33} \dot{x}(s) ds \leq \int_{t-h}^{t} \begin{bmatrix} x^T(t) & x^T(t-h) & \dot{x}^T(s) \end{bmatrix} \times \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} ds.$$

Lemma 1.3 (Schur Complement) ([23]). Given any real matrices P_1, P_2 and P_3 with $P_1 = P_1^T$ and $P_3 > 0$. Then, we have

$$P_1 + P_2 P_3^{-1} P_2^T < 0$$

if and only if

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & -P_3 \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} -P_3 & P_2^T \\ P_2 & P_1 \end{bmatrix} < 0.$$

3 Main Results

A. Assumptions
Throughout this work, we suppose the following condition holds.

(A1) E is a singular matrix, $rankE = r \leq n; P, R_j, Q_j, T_j, (j = 1, 2)$, are positive definite symmetric matrices and $S, U_i, (i = 0, 1, 2)$, matrices with appropriate dimensions and positive semi-definite matrices

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$P^T E = E^T P \geq 0, \tag{3}$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & 0 & 0 & M_{16} & M_{17} & M_{18} & M_{19} \\ * & M_{22} & 0 & 0 & 0 & 0 & 0 & 0 & M_{29} \\ * & * & M_{33} & 0 & 0 & 0 & 0 & 0 & M_{39} \\ * & * & * & M_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & M_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & M_{66} & 0 & 0 & M_{69} \\ * & * & * & * & * & * & M_{77} & 0 & M_{79} \\ * & * & * & * & * & * & * & M_{88} & M_{89} \\ * & * & * & * & * & * & * & * & M_{99} \end{bmatrix} < 0, \tag{4}$$

$$\begin{aligned} &E^T(R_1 - X_{33})E \geq 0, E^T(R_2 - Y_{33})E \geq 0, \\ &E^T(h_1 X_{11} + X_{13} + X_{13}^T + h_2 Y_{11} + Y_{13} + Y_{13}^T)E \geq 0, \end{aligned} \tag{5}$$

where $Z \in R^{n\times(n-r)}$ is any matrix satisfying $E^T Z = 0$ and

$$\begin{aligned} M_{11} =& A^T P + PA + A^T Z S^T + S Z^T A + Q_1 + Q_2 + T_1 + T_2 + \epsilon_0 U_0^T U_0 \\ &+ E^T(h_1 X_{11} + X_{13} + X_{13}^T + h_2 Y_{11} + Y_{13} + Y_{13}^T)E, \\ M_{12} =& P B_{h_1} + S Z^T B_{h_1} + E^T(h_1 X_{12} - X_{13} + X_{23}^T)E, \\ M_{13} =& P B_{h_2} + S Z^T B_{h_2} + E^T(h_2 Y_{12} - Y_{13} + Y_{23}^T)E, \\ M_{16} =& P, M_{17} = P, M_{18} = P, \\ M_{19} =& h_1 A^T R_1 + h_2 A^T R_2, \\ M_{22} =& -Q_1 + E^T(h_1 X_{22} - X_{23} - X_{23}^T)E, \end{aligned}$$

$$
\begin{aligned}
M_{29} =& h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2,\\
M_{33} =& -Q_2 + E^T(h_2 Y_{22} - Y_{23} - Y_{23}^T)E,\\
M_{39} =& h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2,\\
M_{44} =& -T_1 + \epsilon_1 U_1^T U_1, M_{55} = -T_2 + \epsilon_2 U_2^T U_2,\\
M_{66} =& -\epsilon_0 I, M_{69} = h_1 R_1 + h_2 R_2,\\
M_{77} =& -\epsilon_1 I, M_{79} = h_1 R_1 + h_2 R_2,\\
M_{88} =& -\epsilon_2 I, M_{89} = h_1 R_1 + h_2 R_2, M_{99} = -h_1 R_1 - h_2 R_2,
\end{aligned}
$$

where I is $n \times n-$ identity matrix, and $\epsilon_i, (i = 0, 1, 2)$ are positive constants.

Theorem 2.1. If the condition (A1) holds, then system (1) is asymptotically admissible.

Proof. Firstly, we show that the system (1) is regular and impulse free. For this purpose, we choose two invertible matrices $G, H \in R^{n\times n}$ such that

$$
\overline{E} = GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}
$$

Then, the matrix Z can be defined as

$$
Z = G^T \begin{bmatrix} 0 \\ K \end{bmatrix},
$$

where $\overline{K} \in R^{(n-r)\times(n-r)}$ is any invertible matrix.

Next as in (6), we can define

$$
\begin{aligned}
\overline{A} =& GAH = \begin{bmatrix} \overline{A_1} & \overline{A_2} \\ \overline{A_3} & \overline{A_4} \end{bmatrix}, \overline{S} = H^T S = \begin{bmatrix} \overline{S_1} \\ \overline{S_2} \end{bmatrix},\\
\overline{P} =& G^{-T} P G^{-1} = \begin{bmatrix} \overline{P_1} & \overline{P_2} \\ \overline{P_3} & \overline{P_4} \end{bmatrix}, \overline{Z} = G^{-T} Z = \begin{bmatrix} 0 \\ K \end{bmatrix}.
\end{aligned}
$$

From $M_{11} < 0, \epsilon_0 U_0^T U_0 > 0$ and the condition (A1), we can write the following inequality:

$$\Gamma = A^T P + PA + A^T Z S^T + S Z^T A < 0.$$

Pre- and post- multiplying $\Gamma < 0$ by H^T and H, respectively, then we have

$$\begin{aligned}\Gamma =& H^T \Gamma H = \overline{A}^T \overline{P} + \overline{PA} + \overline{A}^T \overline{ZS}^T + \overline{SZ}^T \overline{A} \\ =& \begin{bmatrix} \overline{\Gamma_1} & \overline{\Gamma_2} \\ * & \overline{A_4}^T \overline{P_4} + \overline{P_4 A_4} + \overline{A_4}^T \overline{K S_2}^T + \overline{S_2 K}^T \overline{A_4} \end{bmatrix} < 0. \end{aligned} \tag{7}$$

Since $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ are unrelated to the following discussion, the real expressions for these two variables are omitted here. Next, it follows from the inequality (7) that

$$\overline{A_4}^T \overline{P_4} + \overline{P_4 A_4} + \overline{A_4}^T \overline{K S_2}^T + \overline{S_2 K}^T \overline{A_4} < 0, \tag{8}$$

and thus $\overline{A_4}$ is nonsingular. For this reason, the pair (E, A) is regular and impulse-free (see [23]). In the light of Definition 1.2, the system (1) is also regular and impulse-free.

We now prove the asymptotic stability of the system (1). For this aim, we define a new LKF as follows:

$$V(t, x_t) = \sum_{i=1}^{4} V_i(t, x_t), \tag{9}$$

where

$$
\begin{aligned}
V_1(t,x_t) =& x^T(t)PEx(t),\\
V_2(t,x_t) =& \int_{t-h_1}^{t} x^T(s)Q_1x(s)ds + \int_{t-h_2}^{t} x^T(s)Q_2x(s)ds,\\
V_3(t,x_t) =& \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}(\alpha)E^TR_1E\dot{x}(\alpha)d\alpha d\theta\\
&+ \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}(\alpha)E^TR_2E\dot{x}(\alpha)d\alpha d\theta,\\
V_4(t,x_t) =& \int_{t-h_3}^{t} x^T(s)T_1x(s)ds + \int_{t-h_4}^{t} x^T(s)T_2x(s)ds.
\end{aligned}
$$

It is clear that the LKF in (9) is positive definite. In view of the Newton-Leibnitz formula, calculating the time derivative of the LKF $V(t,x_t)$ in (9) along the system (1), we obtain

$$
\dot{V}(t,x_t) = \sum_{i=1}^{4} \dot{V}_i(t,x_t), \tag{10}
$$

where

$$
\begin{aligned}
\dot{V}_1(t,x_t) =& x^T(t)A^TPx(t) + x^T(t)PAx(t)\\
&+ x^T(t-h_1)B_{h_1}^TPx(t) + x^T(t-h_2)B_{h_2}^TPx(t)\\
&+ x^T(t)PB_{h_1}x(t-h_1) + x^T(t)PB_{h_2}x(t-h_2)\\
&+ F_0^T(t,x(t))Px(t)\\
&+ F_1^T(t,x(t-h_3))Px(t) + F_2^T(t,x(t-h_4))Px(t)\\
&+ x^T(t)PF_0(t,x(t))\\
&+ x^T(t)PF_1(t,x(t-h_3)) + x^T(t)PF_2(t,x(t-h_4)), \qquad (11)\\
\dot{V}_2(t,x_t) =& x^T(t)Q_1x(t) - x^T(t-h_1)Q_1x(t-h_1)\\
&+ x^T(t)Q_2x(t) - x^T(t-h_2)Q_2x(t-h_2), \qquad (12)\\
\dot{V}_3(t,x_t) =& h_1\dot{x}^T(t)E^TR_1E\dot{x}(t)\\
&- \int_{t-h_1}^{t} \dot{x}^T(s)E^TR_1E\dot{x}(s)ds + h_2\dot{x}^T(t)E^TR_2E\dot{x}(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{t-h_2}^{t} \dot{x}^T(s)E^T R_2 E\dot{x}(s)ds \\
=&h_1\dot{x}^T(t)E^T R_1 E\dot{x}(t) - \int_{t-h_1}^{t} \dot{x}^T(s)E^T(R_1 - X_{33})E\dot{x}(s)ds \\
&- \int_{t-h_1}^{t} \dot{x}^T(s)E^T X_{33}E\dot{x}(s)ds \\
&+ h_2\dot{x}^T(t)E^T R_2 E\dot{x}(t) - \int_{t-h_2}^{t} \dot{x}^T(s)E^T(R_2 - Y_{33})E\dot{x}(s)ds \\
&- \int_{t-h_2}^{t} \dot{x}^T(s)E^T Y_{33}E\dot{x}(s)ds, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_4(t, x_t) =&x^T(t)T_1x(t) - x^T(t-h_3)T_1x(t-h_3) + x^T(t)T_2x(t) \\
&- x^T(t-h_4)T_2x(t-h_4). \qquad (14)
\end{aligned}
$$

Next, we calculate the derivative of the terms in (13) as follows, respectively:

$$
\begin{aligned}
h_1\dot{x}^T(t)E^T R_1 E\dot{x}(t) =&h_1[x^T(t)A^T + x^T(t-h_1)B_{h_1}^T + x^T(t-h_2)B_{h_2}^T \\
&+ F_0^T(t, x(t)) + F_1^T(t, x(t-h_3)) \\
&+ F_2^T(t, x(t-h_4))]R_1[Ax(t) + B_{h_1}x(t-h_1) \\
&+ B_{h_2}x(t-h_2) + F_0(t, x(t)) + F_1(t, x(t-h_3)) \\
&+ F_2(t, x(t-h_4))] \\
=&h_1x^T(t)A^T R_1 Ax(t) + h_1x^T(t)A^T R_1 B_{h_1}x(t-h_1) \\
&+ h_1x^T(t)A^T R_1 B_{h_2}x(t-h_2) \\
&+ h_1x^T(t)A^T R_1 F_0(t, x(t)) \\
&+ h_1x^T(t)A^T R_1 F_1(t, x(t-h_3)) \\
&+ h_1x^T(t)A^T R_1 F_2(t, x(t-h_4)) \\
&+ h_1x^T(t-h_1)B_{h_1}^T R_1 Ax(t) \\
&+ h_1x^T(t-h_1)B_{h_1}^T R_1 B_{h_1}x(t-h_1) \\
&+ h_1x^T(t-h_1)B_{h_1}^T R_1 B_{h_2}x(t-h_2)
\end{aligned}
$$

$$
\begin{aligned}
&+ h_1 x^T(t-h_1) B_{h_1}^T R_1 F_0(t,x(t)) \\
&+ h_1 x^T(t-h_1) B_{h_1}^T R_1 F_1(t,x(t-h_3)) \\
&+ h_1 x^T(t-h_1) B_{h_1}^T R_1 F_2(t,x(t-h_4)) \qquad (15)\\
&+ h_1 x^T(t-h_2) B_{h_2}^T R_1 A x(t) \\
&+ h_1 x^T(t-h_2) B_{h_2}^T R_1 B_{h_1} x(t-h_1) \\
&+ h_1 x^T(t-h_2) B_{h_2}^T R_1 B_{h_2} x(t-h_2) \\
&+ h_1 x^T(t-h_2) B_{h_2}^T R_1 F_0(t,x(t)) \\
&+ h_1 x^T(t-h_2) B_{h_2}^T R_1 F_1(t,x(t-h_3)) \\
&+ h_1 x^T(t-h_2) B_{h_2}^T R_1 F_2(t,x(t-h_4)) \\
&+ h_1 F_0^T(t,x(t)) R_1 A x(t) \\
&+ h_1 F_0^T(t,x(t)) R_1 B_{h_1} x(t-h_1) \\
&+ h_1 F_0^T(t,x(t)) R_1 B_{h_2} x(t-h_2) \\
&+ h_1 F_0^T(t,x(t)) R_1 F_0(t,x(t)) \\
&+ h_1 F_0^T(t,x(t)) R_1 F_1(t,x(t-h_3)) \\
&+ h_1 F_0^T(t,x(t)) R_1 F_2(t,x(t-h_4)) \\
&+ h_1 F_1^T(t,x(t-h_3)) R_1 A x(t) \\
&+ h_1 F_1^T(t,x(t-h_3)) R_1 B_{h_1} x(t-h_1) \\
&+ h_1 F_1^T(t,x(t-h_3)) R_1 B_{h_2} x(t-h_2) \\
&+ h_1 F_1^T(t,x(t-h_3)) R_1 F_0(t,x(t)) \\
&+ h_1 F_1^T(t,x(t-h_3)) R_1 F_1(t,x(t-h_3)) \\
&+ h_1 F_1^T(t,x(t-h_3)) R_1 F_2(t,x(t-h_4)) \\
&+ h_1 F_2^T(t,x(t-h_4)) R_1 A x(t) \\
&+ h_1 F_2^T(t,x(t-h_4)) R_1 B_{h_1} x(t-h_1) \\
&+ h_1 F_2^T(t,x(t-h_4)) R_1 B_{h_2} x(t-h_2) \\
&+ h_1 F_2^T(t,x(t-h_4)) R_1 F_0(t,x(t)) \\
&+ h_1 F_2^T(t,x(t-h_4)) R_1 F_1(t,x(t-h_3)) \\
&+ h_1 F_2^T(t,x(t-h_4)) R_1 F_2(t,x(t-h_4)), \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
h_2 \dot{x}^T(t) E^T R_2 E \dot{x}(t) = {}& h_2[x^T(t)A^T + x^T(t-h_1)B_{h_1}^T \\
&+ x^T(t-h_2)B_{h_2}^T + F_0^T(t,x(t)) \\
&+ F_1^T(t,x(t-h_3)) + F_2^T(t,x(t-h_4))]R_2[Ax(t) \\
&+ B_{h_1}x(t-h_1) + B_{h_2}x(t-h_2) \\
&+ F_0(t,x(t)) + F_1(t,x(t-h_3)) + F_2(t,x(t-h_4))]
\end{aligned}
$$

$$
\begin{aligned}
=&h_2x^T(t)A^TR_2Ax(t) + h_2x^T(t)A^TR_2B_{h_1}x(t-h_1)\\
&+ h_2x^T(t)A^TR_2B_{h_2}x(t-h_2) + h_2x^T(t)A^TR_2F_0(t,x(t))\\
&+ h_2x^T(t)A^TR_2F_1(t,x(t-h_3)) + h_2x^T(t)A^TR_2F_2(t,x(t-h_4))\\
&+ h_2x^T(t-h_1)B_{h_1}^TR_2Ax(t) + h_2x^T(t-h_1)B_{h_1}^TR_2B_{h_1}x(t-h_1)\\
&+ h_2x^T(t-h_1)B_{h_1}^TR_2B_{h_2}x(t-h_2)\\
&+ h_2x^T(t-h_1)B_{h_1}^TR_2F_0(t,x(t))\\
&+ h_2x^T(t-h_1)B_{h_1}^TR_2F_1(t,x(t-h_3))\\
&+ h_2x^T(t-h_1)B_{h_1}^TR_2F_2(t,x(t-h_4))\\
&+ h_2x^T(t-h_2)B_{h_2}^TR_2Ax(t)\\
&+ h_2x^T(t-h_2)B_{h_2}^TR_2B_{h_1}x(t-h_1)\\
&+ h_2x^T(t-h_2)B_{h_2}^TR_2B_{h_2}x(t-h_2)\\
&+ h_2x^T(t-h_2)B_{h_2}^TR_2F_0(t,x(t))\\
&+ h_2x^T(t-h_2)B_{h_2}^TR_2F_1(t,x(t-h_3))\\
&+ h_2x^T(t-h_2)B_{h_2}^TR_2F_2(t,x(t-h_4))\\
&+ h_2F_0^T(t,x(t))R_2Ax(t)\\
&+ h_2F_0^T(t,x(t))R_2B_{h_1}x(t-h_1)\\
&+ h_2F_0^T(t,x(t))R_2B_{h_2}x(t-h_2)\\
&+ h_2F_0^T(t,x(t))R_2F_0(t,x(t))\\
&+ h_2F_0^T(t,x(t))R_2F_1(t,x(t-h_3))\\
&+ h_2F_0^T(t,x(t))R_2F_2(t,x(t-h_4))\\
&+ h_2F_1^T(t,x(t-h_3))R_2Ax(t)\\
&+ h_2F_1^T(t,x(t-h_3))R_2B_{h_1}x(t-h_1)\\
&+ h_2F_1^T(t,x(t-h_3))R_2B_{h_2}x(t-h_2)\\
&+ h_2F_1^T(t,x(t-h_3))R_2F_0(t,x(t))\\
&+ h_2F_1^T(t,x(t-h_3))R_2F_1(t,x(t-h_3))\\
&+ h_2F_1^T(t,x(t-h_3))R_2F_2(t,x(t-h_4))\\
&+ h_2F_2^T(t,x(t-h_4))R_2Ax(t)
\end{aligned}
$$

$$
\begin{aligned}
&+ h_2F_2^T(t, x(t-h_4))R_2B_{h_1}x(t-h_1) \\
&+ h_2F_2^T(t, x(t-h_4))R_2B_{h_2}x(t-h_2) \\
&+ h_2F_2^T(t, x(t-h_4))R_2F_0(t, x(t)) \\
&+ h_2F_2^T(t, x(t-h_4))R_2F_1(t, x(t-h_3)) \\
&+ h_2F_2^T(t, x(t-h_4))R_2F_2(t, x(t-h_4)). \qquad (17)
\end{aligned}
$$

Using Lemma 1.2 and the Newton-Leibnitz formula for the terms

$$
-\int_{t-h_1}^{t} \dot{x}^T(s)E^TX_{33}E\dot{x}(s)ds
$$

and

$$
-\int_{t-h_2}^{t} \dot{x}^T(s)E^TY_{33}E\dot{x}(s)ds
$$

in (13), respectively, we obtain the following inequalities:

$$
\begin{aligned}
&-\int_{t-h_1}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)ds \qquad (18) \\
\leq &\int_{t-h_1}^{t} \begin{bmatrix} x^T(t) & x^T(t-h_1) & \dot{x}^T(s) \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_1) \\ \dot{x}(s) \end{bmatrix} ds \\
\leq &x^T(t)[h_1X_{11} + X_{13}^T + X_{13}]x(t) + x^T(t)[h_1X_{12} - X_{13} + X_{23}^T]x(t-h_1) \\
&+ x^T(t-h_1)[h_1X_{12}^T - X_{13}^T + X_{23}]x(t) \\
&+ x^T(t-h_1)[h_1X_{22} - X_{23} - X_{23}^T]x(t-h_1), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{t-h_2}^{t} \dot{x}^T(s)Y_{33}\dot{x}(s)ds \\
\leq &\int_{t-h_2}^{t} \begin{bmatrix} x^T(t) & x^T(t-h_2) & \dot{x}^T(s) \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_2) \\ \dot{x}(s) \end{bmatrix} ds \\
\leq &x^T(t)[h_2Y_{11} + Y_{13}^T + Y_{13}]x(t) + x^T(t)[h_2Y_{12} - Y_{13} + Y_{23}^T]x(t-h_2) \\
&+ x^T(t-h_2)[h_2Y_{12}^T - Y_{13}^T + Y_{23}]x(t) \\
&+ x^T(t-h_2)[h_2Y_{22} - Y_{23} - Y_{23}^T]x(t-h_2). \qquad (20)
\end{aligned}
$$

In addition, nothing $E^T Z = 0$, we can deduce

$$\begin{aligned}0 =& 2\dot{x}^T(t)E^T Z S^T x(t),\\ 0 =& x^T(t)A^T Z S^T x(t) + x^T(t-h_1)B_{h_1}^T Z S^T x(t) + x^T(t-h_2)B_{h_2}^T Z S^T x(t)\\ &+ x^T(t)SZ^T A x(t) + x^T(t)SZ^T B_{h_1} x(t-h_1) + x^T(t)SZ^T B_{h_2} x(t-h_2).\end{aligned} \tag{21}$$

For nonlinear functions $F_i(t, x(t))$ endowed with $\epsilon_i > 0, (i = 0, 1, 2)$, we can obtain

$$\begin{aligned}0 \le& -\epsilon_0 F_0^T(t, x(t))F_0(t, x(t)) + \epsilon_0 x^T(t)U_0^T U_0 x(t),\\ 0 \le& -\epsilon_1 F_1^T(t, x(t-h_3))F_1(t, x(t-h_3)) + \epsilon_1 x^T(t-h_3)U_1^T U_1 x(t-h_3),\\ 0 \le& -\epsilon_2 F_2^T(t, x(t-h_4))F_2(t, x(t-h_4)) + \epsilon_2 x^T(t-h_4)U_2^T U_2 x(t-h_4).\end{aligned} \tag{22}$$

On the gathering the estimates (10)-(22), we have

$$\begin{aligned}\dot{V}(t, x_t) <& \xi^T(t)\Xi\xi(t) - \int_{t-h_1}^{t} \dot{x}^T(s)E^T(R_1 - X_{33})E\dot{x}(s)ds\\ &- \int_{t-h_2}^{t} \dot{x}^T(s)E^T(R_2 - Y_{33})E\dot{x}(s)ds,\end{aligned} \tag{23}$$

where

$$\begin{aligned}\xi^T(t) =& x^T(t)\ x^T(t-h_1)\ x^T(t-h_2)\ x^T(t-h_3)\ x^T(t-h_4)\\ &F_0^T(t, x(t))\ F_1^T(t, x(t-h_3))\ F_2^T(t, x(t-h_4))\end{aligned}$$

and

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & 0 & \Xi_{16} & \Xi_{17} & \Xi_{18} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} & \Xi_{27} & \Xi_{28} \\ * & * & \Xi_{33} & 0 & 0 & \Xi_{36} & \Xi_{37} & \Xi_{38} \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & \Xi_{67} & \Xi_{68} \\ * & * & * & * & * & * & \Xi_{77} & \Xi_{78} \\ * & * & * & * & * & * & * & \Xi_{88} \end{bmatrix}$$

with

$$\begin{aligned}
\Xi_{11} =& A^T P + PA + A^T Z S^T + S Z^T A + Q_1 + Q_2 + \epsilon_0 U_0^T U_0 \\
&+ E^T (h_1 X_{11} + X_{13}^T + X_{13}) E \\
&+ T_1 + T_2 + E^T (h_2 Y_{11} + Y_{13}^T + Y_{13}) E \\
&+ h_1 A^T R_1 A + h_2 A^T R_2 A, \\
\Xi_{12} =& P B_{h_1} + S Z^T B_{h_1} + E^T (h_1 X_{12} - X_{13} + X_{23}^T) E \\
&+ h_1 A^T R_1 B_{h_1} + h_2 A^T R_2 B_{h_1}, \\
\Xi_{13} =& P B_{h_2} + S Z^T B_{h_2} + E^T (h_2 Y_{12} - Y_{13} + Y_{23}^T) E \\
&+ h_1 A^T R_1 B_{h_2} + h_2 A^T R_2 B_{h_2}, \\
\Xi_{16} =& P + h_1 A^T R_1 + h_2 A^T R_2, \Xi_{17} = P + h_1 A^T R_1 + h_2 A^T R_2, \\
\Xi_{18} =& P + h_1 A^T R_1 + h_2 A^T R_2, \\
\Xi_{22} =& -Q_1 + E^T (h_1 X_{22} - X_{23} - X_{23}^T) E + h_1 B_{h_1}^T R_1 B_{h_1} \\
&+ h_2 B_{h_1}^T R_2 B_{h_1}, \\
\Xi_{23} =& h_1 B_{h_1}^T R_1 B_{h_2} + h_2 B_{h_1}^T R_2 B_{h_2}, \\
\Xi_{26} =& h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2, \Xi_{27} = h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2, \\
\Xi_{28} =& h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2, \\
\Xi_{33} =& -Q_2 + E^T (h_2 Y_{22} - Y_{23} - Y_{23}^T) E + h_1 B_{h_2}^T R_1 B_{h_2} \\
&+ h_2 B_{h_2}^T R_2 B_{h_2}, \\
\Xi_{36} =& h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2, \Xi_{37} = h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2,
\end{aligned}$$

$$
\begin{aligned}
\Xi_{38} =& h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2,\\
\Xi_{44} =& -T_1 + \epsilon_1 U_1^T U_1, \Xi_{55} = -T_2 + \epsilon_2 U_2^T U_2,\\
\Xi_{66} =& h_1 R_1 + h_2 R_2 - \epsilon_0 I, \Xi_{67} = h_1 R_1 + h_2 R_2,\\
\Xi_{68} =& h_1 R_1 + h_2 R_2,\\
\Xi_{77} =& h_1 R_1 + h_2 R_2 - \epsilon_1 I, \Xi_{78} = h_1 R_1 + h_2 R_2,\\
\Xi_{88} =& h_1 R_1 + h_2 R_2 - \epsilon_2 I.
\end{aligned}
$$

In order to guarantee $\dot{V}(t, x_t) < 0$, we need to ensure that $\Xi < 0, E^T(R_1 - X_{33})E \geq 0$ and $E^T(R_2 - Y_{33})E \geq 0$. It follows from the Lyapunov-Krasovskii stability theorem and Lemma 1.3 and the Schur complement (see [23]) that the conditions (4) and (5) are satisfied. Therefore, the zero solution of the system (1) is asymptotically stable. Consequently, since the system (1) is regular, impulse free and asymptotically stable, it is asymptotically admissible.

Now, we present a new assumption for the next theorem.

B. Assumption

(A2) Assume that there exist a singular matrix E with $rankE = r \leq n$, positive definite symmetric matrices $P, R_j, Q_j, T_j, (j = 1, 2)$, the matrices $S, U_i, (i = 0, 1, 2)$ with appropriate dimensions and positive semi-definite matrices

$$
X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0, Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0
$$

such that the following LMIs hold:

$$
P^T E = E^T P \geq 0, \tag{24}
$$

$$N=\begin{bmatrix} N_{11} & N_{12} & N_{13} & 0 & 0 & N_{16} & N_{17} & N_{18} & N_{19} \\ * & N_{22} & 0 & 0 & 0 & 0 & 0 & 0 & N_{29} \\ * & * & N_{33} & 0 & 0 & 0 & 0 & 0 & N_{39} \\ * & * & * & N_{44} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & N_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & N_{66} & 0 & 0 & N_{69} \\ * & * & * & * & * & * & N_{77} & 0 & N_{79} \\ * & * & * & * & * & * & * & N_{88} & N_{89} \\ * & * & * & * & * & * & * & * & N_{99} \end{bmatrix} < 0, \tag{25}$$

$$\begin{aligned} &E^T(R_1-X_{33})E \geq 0, E^T(R_2-Y_{33})E \geq 0, \\ &E^T(h_1X_{11}+X_{13}+X_{13}^T+h_2Y_{11}+Y_{13}+Y_{13}^T)E \geq 0, \end{aligned} \tag{26}$$

where $Z \in R^{n\times(n-r)}$ is any matrix satisfying $E^TZ=0$ and

$$\begin{aligned}
N_{11} =&(A+0.5\alpha I)^TP+P(A+0.5\alpha I)+A^TZS^T+SZ^TA \\
&+Q_1+Q_2+T_1+T_2+\epsilon_0U_0^TU_0 \\
&+e^{-\alpha h_1}E^T(h_1X_{11}+X_{13}+X_{13}^T)E \\
&+e^{-\alpha h_2}E^T(h_2Y_{11}+Y_{13}+Y_{13}^T)E, \\
N_{12} =&PB_{h_1}+SZ^TB_{h_1}+e^{-\alpha h_1}E^T(h_1X_{12}-X_{13}+X_{23}^T)E, \\
N_{13} =&PB_{h_2}+SZ^TB_{h_2}+e^{-\alpha h_2}E^T(h_2Y_{12}-Y_{13}+Y_{23}^T)E, \\
N_{16} =&P, N_{17}=P, N_{18}=P, N_{19}=h_1A^TR_1+h_2A^TR_2, \\
N_{22} =&e^{-\alpha h_1}[E^T(h_1X_{22}-X_{23}-X_{23}^T)E-Q_1], \\
N_{29} =&h_1B_{h_1}^TR_1+h_2B_{h_1}^TR_2, \\
N_{33} =&e^{-\alpha h_2}[E^T(h_2Y_{22}-Y_{23}-Y_{23}^T)E-Q_2], \\
N_{39} =&h_1B_{h_2}^TR_1+h_2B_{h_2}^TR_2, N_{44}=-e^{-\alpha h_3}T_1+\epsilon_1U_1^TU_1, \\
N_{55} =&-e^{-\alpha h_4}T_2+\epsilon_2U_2^TU_2, N_{66}=-\epsilon_0I, N_{69}=h_1R_1+h_2R_2, \\
N_{77} =&-\epsilon_1I, N_{79}=h_1R_1+h_2R_2, N_{88}=-\epsilon_2I, N_{89}=h_1R_1+h_2R_2, \\
N_{99} =&-h_1R_1-h_2R_2,
\end{aligned}$$

where I is $n \times n-$ identity matrix, and $\epsilon_i, (i = 0, 1, 2)$ are positive constants.

Theorem 2.2. If the assumption (A2) holds, then system (1) is exponentially admissible.

Proof. In Theorem 2.1, it has been shown that the system (1) is regular and impulse free. We now show that the zero solution of system (1) is exponentially stable. From this point of view, we define as a new LKF by

$$V(t, x_t) = \sum_{i=1}^{4} V_i(t, x_t), \tag{27}$$

where

$$\begin{aligned}
V_1(t, x_t) =& e^{\alpha t} x^T(t) P E x(t),\\
V_2(t, x_t) =& \int_{t-h_1}^{t} e^{\alpha s} x^T(s) Q_1 x(s) ds\\
&+ \int_{t-h_2}^{t} e^{\alpha s} x^T(s) Q_2 x(s) ds,\\
V_3(t, x_t) =& \int_{-h_1}^{0} \int_{t+\theta}^{t} e^{\alpha s} \dot{x}(\alpha) E^T R_1 E \dot{x}(\alpha) d\alpha d\theta\\
&+ \int_{-h_2}^{0} \int_{t+\theta}^{t} e^{\alpha s} \dot{x}(\alpha) E^T R_2 E \dot{x}(\alpha) d\alpha d\theta,\\
V_4(t, x_t) =& \int_{t-h_3}^{t} e^{\alpha s} x^T(s) T_1 x(s) ds + \int_{t-h_4}^{t} e^{\alpha s} x^T(s) T_2 x(s) ds.
\end{aligned}$$

It is clear that the LKF (25) is positive definite. Calculating the time derivative $V(t, x_t)$ along the system (1) and using the Newton-Leibnitz formula, we get

$$\dot{V}(t, x_t) = \sum_{i=1}^{4} \dot{V}_i(t, x_t), \tag{28}$$

where

$$\begin{aligned}
\dot{V}_1(t,x_t) =& e^{\alpha t}\{x^T(t)(A+0.5\alpha I)^T Px(t) + x^T(t)P(A+0.5\alpha I)x(t) \\
&+ x^T(t-h_1)B_{h_1}^T Px(t) + x^T(t-h_2)B_{h_2}^T Px(t) \\
&+ x^T(t)PB_{h_1}x(t-h_1) + x^T(t)PB_{h_2}x(t-h_2) \\
&+ F_0^T(t,x(t))Px(t) + F_1^T(t,x(t-h_3))Px(t) \\
&+ F_2^T(t,x(t-h_4))Px(t) \\
&+ x^T(t)PF_0(t,x(t)) + x^T(t)PF_1(t,x(t-h_3)) \\
&+ x^T(t)PF_2(t,x(t-h_4))\}, \qquad (29)\\
\dot{V}_2(t,x_t) =& e^{\alpha t}\{x^T(t)Q_1x(t) - e^{-\alpha h_1}x^T(t-h_1)Q_1x(t-h_1) + x^T(t)Q_2x(t) \\
&- e^{-\alpha h_2}x^T(t-h_2)Q_2x(t-h_2)\}, \qquad (30)\\
\dot{V}_3(t,x_t) =& e^{\alpha t}h_1\dot{x}^T(t)E^TR_1E\dot{x}(t) - \int_{t-h_1}^{t} e^{\alpha s}\dot{x}^T(s)E^TR_1E\dot{x}(s)ds \\
&+ e^{\alpha t}h_2\dot{x}^T(t)E^TR_2E\dot{x}(t) - \int_{t-h_2}^{t} e^{\alpha s}\dot{x}^T(s)E^TR_2E\dot{x}(s)ds \\
=& e^{\alpha t}\{h_1\dot{x}^T(t)E^TR_1E\dot{x}(t) \\
&- \int_{t-h_1}^{t} e^{\alpha(s-t)}\dot{x}^T(s)E^T(R_1-X_{33})E\dot{x}(s)ds \\
&- \int_{t-h_1}^{t} e^{\alpha(s-t)}\dot{x}(s)E^TX_{33}E\dot{x}(s)ds + h_2\dot{x}^T(t)E^TR_2E\dot{x}(t) \\
&- \int_{t-h_2}^{t} e^{\alpha(s-t)}\dot{x}^T(s)E^T(R_2-Y_{33})E\dot{x}(s)ds \\
&- \int_{t-h_2}^{t} e^{\alpha(s-t)}\dot{x}(s)E^TY_{33}E\dot{x}(s)ds\}, \qquad (31)\\
\dot{V}_4(t,x_t) =& e^{\alpha t}\{x^T(t)T_1x(t) - e^{-\alpha h_3}x^T(t-h_3)T_1x(t-h_3) + x^T(t)T_2x(t) \\
&- e^{-\alpha h_4}x^T(t-h_4)T_2x(t-h_4)\}. \qquad (32)
\end{aligned}$$

It is clear that, for any a scalar $s \in [t-h_1,t], e^{-\alpha h_1} \le e^{\alpha(s-t)} \le 1$ and

$$-\int_{t-h_1}^{t} e^{\alpha(s-t)}\dot{x}^T(s)E^TX_{33}E\dot{x}(s)ds \le -e^{-\alpha h_1}\int_{t-h_1}^{t} \dot{x}^T(s)E^TX_{33}E\dot{x}(s)ds.$$

Similarly, for any a scalar $s \in [t-h_2, t]$, we have $e^{-\alpha h_2} \le e^{\alpha(s-t)} \le 1$ and

$$-\int_{t-h_2}^{t} e^{\alpha(s-t)} \dot{x}^T(s) E^T Y_{33} E \dot{x}(s) ds \le -e^{-\alpha h_2} \int_{t-h_2}^{t} \dot{x}^T(s) E^T Y_{33} E \dot{x}(s) ds.$$

For nonlinear functions $F_i(t, x(t))$ endowed with $\epsilon_i > 0, (i = 0, 1, 2)$, it follows that

$$\begin{aligned}
0 \le & -\epsilon_0 F_0^T(t, x(t)) F_0(t, x(t)) + \epsilon_0 x^T(t) U_0^T U_0 x(t), \\
0 \le & -\epsilon_1 F_1^T(t, x(t-h_3)) F_1(t, x(t-h_3)) + \epsilon_1 x^T(t-h_3) U_1^T U_1 x(t-h_3), \\
0 \le & -\epsilon_2 F_2^T(t, x(t-h_4)) F_2(t, x(t-h_4)) + \epsilon_2 x^T(t-h_4) U_2^T U_2 x(t-h_4).
\end{aligned}$$

From this point of view, it can be followed from the combination of the equalities (26)-(30) and the above inequalities that

$$\begin{aligned}
\dot{V}(t, x_t) < & e^{\alpha t} \{ \xi^T(t) \Psi \xi(t) - \int_{t-h_1}^{t} e^{-\alpha h_1} \dot{x}^T(s) E^T (R_1 - X_{33}) E \dot{x}(s) ds \\
& - \int_{t-h_2}^{t} e^{-\alpha h_2} \dot{x}^T(s) E^T (R_1 - Y_{33}) E \dot{x}(s) ds \},
\end{aligned}$$

where

$$\begin{aligned}
\xi^T(t) = & [x^T(t) \; x^T(t-h_1) \; x^T(t-h_2) \; x^T(t-h_3) \\
& x^T(t-h_4) F_0^T(t, x(t)) F_1^T(t, x(t-h_3)) \; F_2^T(t, x(t-h_4))]
\end{aligned}$$

and

$$\Psi = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & 0 & \Psi_{16} & \Psi_{17} & \Psi_{18} \\
* & \Psi_{22} & \Psi_{23} & 0 & 0 & \Psi_{26} & \Psi_{27} & \Psi_{28} \\
* & * & \Psi_{33} & 0 & 0 & \Psi_{36} & \Psi_{37} & \Psi_{38} \\
* & * & * & \Psi_{44} & 0 & 0 & 0 & 0 \\
* & * & * & * & \Psi_{55} & 0 & 0 & 0 \\
* & * & * & * & * & \Psi_{66} & \Psi_{67} & \Psi_{68} \\
* & * & * & * & * & * & \Psi_{77} & \Psi_{78} \\
* & * & * & * & * & * & * & \Psi_{88}
\end{bmatrix}$$

with

$$
\begin{aligned}
\Psi_{11} =&(A+0.5\alpha I)^T P + P(A+0.5\alpha I) + A^T Z S^T + S Z^T A \\
&+ Q_1 + Q_2 + T_1 + T_2 \\
&+ e^{-\alpha h_1} E^T (h_1 X_{11} + X_{13}^T + X_{13}) E \\
&+ \epsilon_0 U_0^T U_0 + e^{-\alpha h_2} E^T (h_2 Y_{11} + Y_{13}^T + Y_{13}) E \\
&+ h_1 A^T R_1 A + h_2 A^T R_2 A, \\
\Psi_{12} =&P B_{h_1} + S Z^T B_{h_1} + e^{-\alpha h_1} E^T (h_1 X_{12} - X_{13} + X_{23}^T) E \\
&+ h_1 A^T R_1 B_{h_1} + h_2 A^T R_2 B_{h_1}, \\
\Psi_{13} =&P B_{h_2} + S Z^T B_{h_2} + e^{-\alpha h_2} E^T (h_2 Y_{12} - Y_{13} + Y_{23}^T) E \\
&+ h_1 A^T R_1 B_{h_2} + h_2 A^T R_2 B_{h_2}, \\
\Psi_{16} =&P + h_1 A^T R_1 + h_2 A^T R_2, \\
\Psi_{17} =&P + h_1 A^T R_1 + h_2 A^T R_2, \\
\Psi_{18} =&P + h_1 A^T R_1 + h_2 A^T R_2, \\
\Psi_{22} =&e^{-\alpha h_1} [-Q_1 + E^T (h_1 X_{22} - X_{23} - X_{23}^T) E] \\
&+ h_1 B_{h_1}^T R_1 B_{h_1} + h_2 B_{h_1}^T R_2 B_{h_1}, \\
\Psi_{23} =&h_1 B_{h_1}^T R_1 B_{h_2} + h_2 B_{h_1}^T R_2 B_{h_2}, \\
\Psi_{26} =&h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2, \Psi_{27} = h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2, \\
\Psi_{28} =&h_1 B_{h_1}^T R_1 + h_2 B_{h_1}^T R_2, \\
\Psi_{33} =&e^{-\alpha h_2} [-Q_2 + E^T (h_2 Y_{22} - Y_{23} - Y_{23}^T) E] \\
&+ h_1 B_{h_2}^T R_1 B_{h_2} + h_2 B_{h_2}^T R_2 B_{h_2}, \\
\Psi_{36} =&h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2, \Psi_{37} = h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2, \\
\Psi_{38} =&h_1 B_{h_2}^T R_1 + h_2 B_{h_2}^T R_2, \\
\Psi_{44} =&- e^{-\alpha h_3} T_1 + \epsilon_1 U_1^T U_1, \Psi_{55} = -e^{-\alpha h_4} T_2 + \epsilon_2 U_2^T U_2, \\
\Psi_{66} =&h_1 R_1 + h_2 R_2 - \epsilon_0 I, \Psi_{67} = h_1 R_1 + h_2 R_2, \\
\Psi_{68} =&h_1 R_1 + h_2 R_2, \\
\Psi_{77} =&h_1 R_1 + h_2 R_2 - \epsilon_1 I, \Psi_{78} = h_1 R_1 + h_2 R_2, \\
\Psi_{88} =&h_1 R_1 + h_2 R_2 - \epsilon_2 I.
\end{aligned}
$$

In order to guarantee $\dot{V}(t, x_t) < 0$, it is needed to show that $\Psi < 0$ and $E^T(R_1 - X_{33})E \geq 0, E^T(R_2 - Y_{33})E \geq 0$ From the above discussion, it follows that all of those conditions, i.e., (23) and (24), hold. In this case, we can conclude the zero solution of the system (1) is exponentially stable. Consequently, since the system (1) is regular, impulse free and exponentially stable, and it is exponentially admissible. This result completes the proof of Theorem 2.2.

4 Numerical Applications

In this section, for the particular cases of the considered equations, we give two examples to show the satisfaction of our assumptions
Example 3.1. For the particular case of the system (1), we consider the following nonlinear singular system with two constant delays:

$$\begin{aligned}
\frac{d}{dt}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right) &= \begin{bmatrix} -7.885 & 0 \\ -5.95 & -6.125 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
&+ \begin{bmatrix} -1.2085 & 0 \\ -1.245 & -1.2095 \end{bmatrix} \times \begin{bmatrix} x_1(t-0.105) \\ x_2(t-0.105) \end{bmatrix} \\
&+ \begin{bmatrix} -1.1025 & -1.135 \\ 0 & -1.1035 \end{bmatrix} \begin{bmatrix} x_1(t-0.125) \\ x_2(t-0.125) \end{bmatrix} \\
&+ \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix} \\
&+ \begin{bmatrix} x_1(t-0.105)e^{-x_1^2(t-0.105)} \\ x_2(t-0.105)e^{-x_2^2(t-0.105)} \end{bmatrix} \\
&+ \begin{bmatrix} x_1(t-0.125)e^{-x_1^2(t-0.125)} \\ x_2(t-0.125)e^{-x_2^2(t-0.125)} \end{bmatrix}
\end{aligned}$$

where

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -7.885 & 0 \\ -5.95 & -6.125 \end{bmatrix},$$
$$B_h = \begin{bmatrix} -1.2085 & 0 \\ -1.245 & -1.2095 \end{bmatrix},$$
$$B_d = \begin{bmatrix} -1.1025 & -1.135 \\ 0 & -1.1035 \end{bmatrix},$$
$$F_0(t, x(t)) = \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix}$$
$$F_1(t, x(t-h)) = \begin{bmatrix} x_1(t-0.105)e^{-x_1^2(t-0.105)} \\ x_2(t-0.105)e^{-x_2^2(t-0.105)} \end{bmatrix},$$
$$F_2(t, x(t-d)) = \begin{bmatrix} x_1(t-0.125)e^{-x_1^2(t-0.125)} \\ x_2(t-0.125)e^{-x_2^2(t-0.125)} \end{bmatrix}$$

and

$$h_1 = h_3 = 0.105, h_2 = h_4 = 0.125.$$

It is clear that the system in Example 3.1 is regular and impulse free. Let $\epsilon_0 = 8.25, \epsilon_1 = 8.45, \epsilon_2 = 8.65$ and choose

$$X_{11} = \begin{bmatrix} 9.305 & -0.125 \\ -0.125 & 9.365 \end{bmatrix}, X_{12} = \begin{bmatrix} 1.225 & 0 \\ 0 & 1.252 \end{bmatrix},$$
$$X_{13} = \begin{bmatrix} 1.215 & 0.65 \\ 0.65 & 1.485 \end{bmatrix},$$
$$X_{22} = \begin{bmatrix} 4.625 & 1.25 \\ 1.25 & 5.105 \end{bmatrix}, X_{23} = \begin{bmatrix} 1.125 & 1.225 \\ 1.225 & 1.985 \end{bmatrix},$$
$$X_{33} = \begin{bmatrix} 1.235 & 1.265 \\ 1.265 & 1.505 \end{bmatrix},$$

$$Y_{11} = \begin{bmatrix} 8.105 & -0.165 \\ -0.165 & 8.125 \end{bmatrix}, Y_{12} = \begin{bmatrix} 0.125 & 0 \\ 0 & 0.235 \end{bmatrix},$$
$$Y_{13} = \begin{bmatrix} 1.125 & 0.5 \\ 0.5 & 1.265 \end{bmatrix},$$
$$Y_{22} = \begin{bmatrix} 3.325 & 0.75 \\ 0.75 & 3.845 \end{bmatrix}, Y_{23} = \begin{bmatrix} 1.627 & 1.325 \\ 1.325 & 2.895 \end{bmatrix},$$
$$Y_{33} = \begin{bmatrix} 2.685 & 1.215 \\ 1.215 & 2.578 \end{bmatrix},$$
$$P = \begin{bmatrix} 8.65 & 0 \\ 0 & 7.85 \end{bmatrix}, Q_1 = \begin{bmatrix} 4.1 & 3.85 \\ 3.85 & 4.2 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 3.2 & 3.063 \\ 3.063 & 3.5 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 1.245 & 0 \\ 0 & 0.106 \end{bmatrix}, R_2 = \begin{bmatrix} 2.695 & 0 \\ 0 & 0.103 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0 \\ 2.45 \end{bmatrix}, S = \begin{bmatrix} -1.25 \\ -1.35 \end{bmatrix},$$
$$U_0 = \begin{bmatrix} -0.05 & 0 \\ 0 & -0.05 \end{bmatrix}, U_1 = \begin{bmatrix} -0.03 & 0 \\ 0 & -0.03 \end{bmatrix},$$
$$U_2 = \begin{bmatrix} -0.04 & 0 \\ 0 & -0.04 \end{bmatrix}.$$

Under the above assumptions, all eigenvalues in the special case of the LMI defined by Ξ satisfy $\lambda_{max}(\Xi) \leq -0.4722$. Consequently, it is clear that all conditions of Theorem 2.1 can be satisfied. Thus, the system (1) is asymptotically admissible.

Example 3.2. As a special case of the system (1), we consider the following nonlinear singular system with two constant delays:

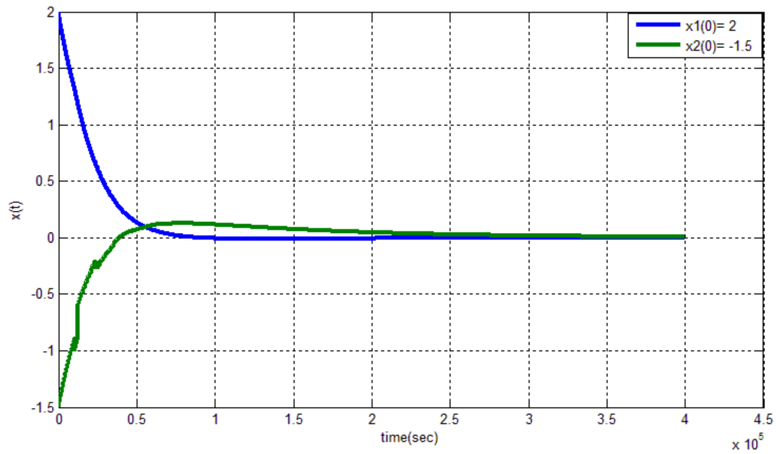

Figure 1: Trajectories of the solution of $x(t)$ of the system in Example 3.1, when $h = h_{max} = 0.125$.

$$
\begin{aligned}
\frac{d}{dt}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) &= \begin{bmatrix} -7.885 & 0 \\ -3.95 & -8.125 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
&+ \begin{bmatrix} -1.2085 & 0 \\ -1.245 & -1.2095 \end{bmatrix} \\
&\times \begin{bmatrix} x_1(t-0.105) \\ x_2(t-0.105) \end{bmatrix} + \begin{bmatrix} -1.1025 & -1.135 \\ 0 & -1.1035 \end{bmatrix} \\
&\times \begin{bmatrix} x_1(t-0.125) \\ x_2(t-0.125) \end{bmatrix} + \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix} \\
&+ \begin{bmatrix} x_1(t-0.105)e^{-x_1^2(t-0.105)} \\ x_2(t-0.105)e^{-x_2^2(t-0.105)} \end{bmatrix} \\
&+ \begin{bmatrix} x_1(t-0.125)e^{-x_1^2(t-0.125)} \\ x_2(t-0.125)e^{-x_2^2(t-0.125)} \end{bmatrix}
\end{aligned}
$$

where

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -7.885 & 0 \\ -3.95 & -8.125 \end{bmatrix},$$
$$B_h = \begin{bmatrix} -1.2085 & 0 \\ -1.245 & -1.2095 \end{bmatrix},$$
$$B_d = \begin{bmatrix} -1.1025 & -1.135 \\ 0 & -1.1035 \end{bmatrix},$$
$$F_0(t, x(t)) = \begin{bmatrix} x_1(t)e^{-x_1^2(t)} \\ x_2(t)e^{-x_2^2(t)} \end{bmatrix},$$
$$F_1(t, x(t-h)) = \begin{bmatrix} x_1(t-0.105)e^{-x_1^2(t-0.105)} \\ x_2(t-0.105)e^{-x_2^2(t-0.105)} \end{bmatrix},$$
$$F_2(t, x(t-d)) = \begin{bmatrix} x_1(t-0.125)e^{-x_1^2(t-0.125)} \\ x_2(t-0.125)e^{-x_2^2(t-0.125)} \end{bmatrix}$$

and

$$h_1 = h_3 = 0.105, h_2 = h_4 = 0.125.$$

It is clear that the system given in Example 3.2 is regular and impulse free. For the considered special case, let $\alpha = 0.2, \epsilon_0 = 8.25, \epsilon_1 = 8.45, \epsilon_2 = 8.65$,

$$X_{11} = \begin{bmatrix} 9.305 & -0.125 \\ -0.125 & 9.365 \end{bmatrix}, X_{12} = \begin{bmatrix} 1.225 & 0 \\ 0 & 1.252 \end{bmatrix},$$
$$X_{13} = \begin{bmatrix} 1.215 & 0.65 \\ 0.65 & 1.485 \end{bmatrix},$$
$$X_{22} = \begin{bmatrix} 4.625 & 1.25 \\ 1.25 & 5.105 \end{bmatrix}, X_{23} = \begin{bmatrix} 1.125 & 1.225 \\ 1.225 & 1.985 \end{bmatrix},$$
$$X_{33} = \begin{bmatrix} 1.235 & 1.265 \\ 1.265 & 1.505 \end{bmatrix},$$

$$Y_{11} = \begin{bmatrix} 8.105 & -0.165 \\ -0.165 & 8.125 \end{bmatrix}, Y_{12} = \begin{bmatrix} 0.125 & 0 \\ 0 & 0.235 \end{bmatrix},$$
$$Y_{13} = \begin{bmatrix} 1.125 & 0.5 \\ 0.5 & 1.265 \end{bmatrix},$$
$$Y_{22} = \begin{bmatrix} 3.325 & 0.75 \\ 0.75 & 3.845 \end{bmatrix}, Y_{23} = \begin{bmatrix} 1.627 & 1.325 \\ 1.325 & 2.895 \end{bmatrix},$$
$$Y_{33} = \begin{bmatrix} 2.685 & 1.215 \\ 1.215 & 2.578 \end{bmatrix},$$
$$P = \begin{bmatrix} 8.5 & 0 \\ 0 & 7 \end{bmatrix}, Q_1 = \begin{bmatrix} 4.1 & 3.85 \\ 3.85 & 4.2 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 3.2 & 3.063 \\ 3.063 & 3.5 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 1.245 & 0 \\ 0 & 0.106 \end{bmatrix}, R_2 = \begin{bmatrix} 2.695 & 0 \\ 0 & 0.103 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0 \\ 2.45 \end{bmatrix}, S = \begin{bmatrix} -1.25 \\ -1.35 \end{bmatrix},$$
$$U_0 = \begin{bmatrix} -0.05 & 0 \\ 0 & -0.05 \end{bmatrix}, U_1 = \begin{bmatrix} -0.03 & 0 \\ 0 & -0.03 \end{bmatrix},$$
$$U_2 = \begin{bmatrix} -0.04 & 0 \\ 0 & -0.04 \end{bmatrix}.$$

Hence, it can be shown that the all eigenvalues of the LMI defined by Ψ satisfies $\lambda_{max}(\Psi) \leq -0.05$.

Consequently, it is clear that all conditions of Theorem 3.2 hold. Thus, the system (1) is exponentially admissible.

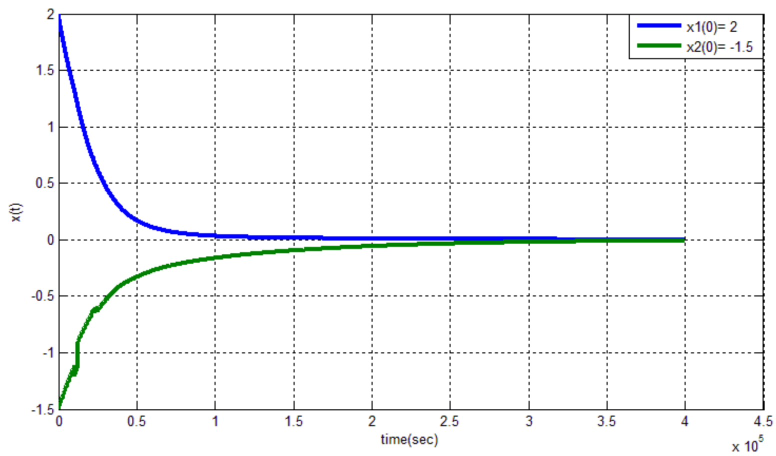

Figure 2: Trajectories of the solution of $x(t)$ of the system in Example 3.2, when $h = h_{max} = 0.125$ and $\alpha = 0.20$.

5 Conclusion

In this paper, we consider a class of nonlinear singular systems with multiple constant delays. Defined a new Lyapunov-Krasovskii functional, using LMI and integral inequality matrix, we investigate asymptotic admissibility and exponential admissibility of the considered system. Two numerical examples are also given with their simulations to demonstrate the applicability of the main results of this paper. The obtained results include and generalize some recent results in the literature. As for proper future studies, instead of considered systems, their fractional model can be investigated for the problems of this paper.

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