Journal of Mathematical Extension
Vol. 15, SI-NTFCA, (2021) (17)1-17
URL: https://doi.org/10.30495/JME.SI.2021.2016
ISSN: 1735-8299
Original Research Paper

Fractional Integral Inequalities Involving Convex Functions via Marichev-Saigo-Maeda Approach

A. Nale*
Dr. Babasaheb Ambedkar Marathwada University

S. Panchal
Dr. Babasaheb Ambedkar Marathwada University

V. Chinchane
Deogiri Institute of Engineering and Management Studies

Z. Dahmani
UMAB University

Abstract. In this paper, we present new integral results on fractional order inequalities involving convex functions using Marichev-Saigo-Maeda integral operators. Using the same integral approach, other fractional integral inequalities for positive and continuous functions are also proved. The obtained results can be seen as a new contribution in inequality theory by means of Marichev-Saigo-Maieda approach.

AMS Subject Classification: 26A99; 26D10.

Keywords and Phrases: Fractional inequalities, Marichev-Saigo-Maeda integral operator, Convex function.

Received: April 2021; Published: July 2021

*Corresponding Author

1 Introduction

The concept fractional calculus is the generalization of traditional calculus into non-integer differential and integral order. It has found very important due to its various applications in the fields of science and engineering such as physical sciences, chemical science and life sciences. Due to the fact that fractional-order derivatives and integrals are capable to characterize the properties of memory effects as an essential aspect in many real-world phenomena, see [3, 4, 5, 6, 26]. Recently, some studies have been conducted on the mathematical analysis of fractional calculus and its applications such as p-Laplacian non periodic nonlinear boundary value problems [20], the discrete fractional three-point BVP for the elastic beam equation [1], existence analysis of quantum integro-difference FBVPs [25], Kuratowski MNC technique [15]. Fractional integral inequalities play a very important role due to their multiple applications in the theory of differential and integral equations, such as continuous dependence solution, uniqueness of solutions, existence and stability of solutions.

During the last few years, many researchers have been concerned with several generalizations, variants and extensions of fractional integral inequalities and their applications by involving the Riemann-Liouville, Erdelyi-Kober, Saigo, Hadamard, k-fractional integral operator and generalized k-fractional integral operators, see [2, 7, 8, 9, 10, 11, 12, 13, 14, 16], and the reference cited therein. In this sense, V.L. Chinchane et al. [9, 10] have presented fractional integral inequalities involving convex functions using the Hadamard and Saigo fractional integral operators. Also, A.B. Nale et al. and G. Rahaman et al. [22, 24] have obtained some fractional integral inequalities involving convex functions by considering generalized Katugampola fractional integral and generalized proportional Hadamard operator. Recently, A. Tassaddiq et al. [32] have established some Minkowski-type fractional integral inequalities using Marichev-Saigo-Maeda fractional operator. Then, S. Joshi et al. [17, 18] have presented new Gruss and Chebyshev type inequalities using Marichev-Saigo-Maeda fractional integral operator. We cite also the paper of O. I. Marichev where it has been introduced new generalizations of the hypergeometric fractional integral including Saigo operator, see [19] (also [30]). In [29], Saigo and Maeda have been concerned with

the hypergeometric fractional integral in terms of any complex order with Appell function in the kernel. In [23], S.D. Purohit et al. have introduced generalized operators of fractional integration involving Appell's function $F_3(.)$ due to Marichev-Saigo-Maeda. In [21], V.N. Mishra has presented a work on the Marichev-Saigo-Maeda fractional calculus operator, Srivastava polynomials and generalized Mittag-Leffler function. In comparison to other researches on the fractional integral inequalities that were published in the literature, we here deal with the Marichev-Saigo-Maeda fractional integral operator. Our aim is to propose some new fractional integral inequalities involving convex functions with the help of Marichev-Saigo-Maeda fractional integral operator.
The paper is organized as follows. In section 2, we give basic definitions and propositions related to Marichev-Saigo-Maeda fractional approach. In section 3, we present some fractional integral inequalities involving convex functions by using the introduced approach. In section 4, we prove other fractional inequalities using Marichev-Saigo-Maeda integral operators. In section 5, some concluding remarks follow. Our results have some relationships with the papers [10, 11, 13]. Some interested inequalities of these papers can be deduced as some special cases of the present work.

2 Preliminaries

Here, we present some basic notation, definitions and lemmas of Marichev-Saigo-Maeda fractional integral operators which are useful later.

Definition 2.1. A real valued function $f(t)\,(t \geq 0)$ is said be in the space $C_\mu, \mu \in \mathbb{R}$ if there exist real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$ and $C[0,\infty)$ is the set of all continuous functions in the interval $[0,\infty)$.

Definition 2.2. [18, 23, 29, 32] Let $v, v^{'}, \xi, \xi^{'}, \vartheta \in \mathbb{C}$, $x > 0$ and $\Re(\vartheta) > 0$, then Marichev-Saigo-Maeda (MSM) fractional integral is defined by

$$(\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta} f)(x) = \frac{x^{-v}}{\Gamma(\vartheta)} \int_0^x (x-t)^{\vartheta-1} t^{-v^{'}} F_3(v, v^{'}, \xi, \xi^{'}; \vartheta; 1-\frac{t}{x}, 1-\frac{x}{t}) f(t) dt, \tag{1}$$

where $F_3(.)$ is the Appel function defined in [31] as follows:

$$ {}_pF_q(v,v^{'},\xi,\xi^{'};\gamma;x;y)=\sum_{m,n=0}^{\infty}\frac{(v)_m(v^{'})_m(\xi)_m(\xi)_n^{'}}{(\gamma)_{m+n}}\frac{x^m y^n}{m!n!}, max(x,y)<1, $$

and $(v)_m = v(v+1)...(v+m-1)$ is Pochhammer symbol.

Lemma 2.3. *Let $v, v^{'}, \xi, \xi^{'}, \vartheta, \rho \in \mathbb{C}$, $x>0$ be such that $\Re(\vartheta)>0$ and $\Re(\tau) > max\{0, \Re(v-v^{'}-\xi-\vartheta), \Re(v^{'}-\xi^{'})\}$. Then there exist the relation*

$$ \begin{aligned} \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta} x^{\rho-1}(x) = & \frac{\Gamma(\rho)\Gamma(\rho+\vartheta-v-v^{'}-\xi)\Gamma(\rho+\xi^{'}-v^{'})}{\Gamma(\rho+\xi^{'})\Gamma(\rho+\vartheta-v-v^{'})\Gamma(\rho+\vartheta-v^{'}-\xi)} \\ & \times x^{\rho-v-v^{'}+\vartheta-1}. \end{aligned} \tag{2} $$

If we consider $\rho=1$ in Lemma 2.3, then we get following relation

$$ (\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[1])(x) = \frac{\Gamma(1+\vartheta-v-v^{'}-\xi)\Gamma(1+\xi^{'}-v^{'})}{\Gamma(1+\xi^{'})\Gamma(1+\vartheta-v-v^{'})\Gamma(1+\vartheta-v^{'}-\xi)} x^{-v-v^{'}+\vartheta}. \tag{3} $$

Consider the following function

$$ \begin{aligned} \mathfrak{J}(x,t) &= \frac{x^{-v}}{\Gamma(\vartheta)}(x-t)^{\vartheta-1}t^{-v^{'}}F_3(v,v^{'},\xi,\xi^{'};\vartheta;1-\frac{t}{x},1-\frac{x}{t}) \\ &= \frac{x^{-v}}{\Gamma(\vartheta)}(x-t)^{\vartheta-1}t^{-v^{'}}\left[(1+\frac{v^{'}(\xi)}{\vartheta})\frac{1-x}{t}+\frac{v(\xi)}{\vartheta}\frac{1-t}{x}+...\right], \end{aligned} \tag{4} $$

Clearly, the function $\mathfrak{J}(x,t)$ is positive.

3 Fractional order inequalities for convex functions

We prove the following theorem.

Theorem 3.1. *Let u, z be two positive continuous functions on $[0,\infty)$ and $u \leq z$ on $[0,\infty)$. Suppose that $\frac{u}{z}$ is decreasing, u is increasing on $[0,\infty)$ and for any convex function ψ, $\psi(0)=0$. Then for all $x>0$, $v, v^{'}, \xi, \xi^{'}, \vartheta \in \mathbb{C}$, $\Re(\vartheta)>0$, we have*

$$\frac{\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[u(x)]}{\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[z(x)]} \geq \frac{\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[\psi(u(x))]}{\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[\psi(z(x))]}, \tag{5}$$

where $v^{'}>-1$, $1> max\left\{0, \Re(v,+v^{'}+\xi-\vartheta), \Re(v^{'}-\xi^{'})\right\}$
$(\vartheta - v^{'}) > max\,(1-\xi, 1-v)$.

Proof:- If ψ is convex with $\psi(0)=0$, then $\frac{\psi(x)}{x}$ is increasing. Since u is increasing, then $\frac{\psi(u(x))}{z(x)}$ is also increasing. Clearly $\frac{u(x)}{z(x)}$ is decreasing, for all $\tau, \sigma \in [0,\infty)$, and

$$\left(\frac{\psi(u(\tau))}{u(\tau)} - \frac{\psi(u(\sigma))}{u(\sigma)}\right)\left(\frac{u(\sigma)}{z(\sigma)} - \frac{u(\tau)}{z(\tau)}\right) \geq 0, \tag{6}$$

which gives us

$$\frac{\psi(u(\tau))}{u(\tau)}\frac{u(\sigma)}{z(\sigma)} + \frac{\psi(u(\sigma))}{u(\sigma)}\frac{u(\tau)}{z(\tau)} - \frac{\psi(u(\tau))}{u(\tau)}\frac{u(\tau)}{z(\tau)} - \frac{\psi(u(\sigma))}{u(\sigma)}\frac{u(\sigma)}{z(\sigma)} \geq 0. \tag{7}$$

Multiplying (7) by $z(\tau)z(\sigma)$, yields

$$\begin{aligned} &\frac{\psi(u(\tau))}{u(\tau)}u(\sigma)z(\tau) + \frac{\psi(u(\sigma))}{u(\sigma)}u(\tau)z(\sigma) \\ &- \frac{\psi(u(\tau))}{u(\tau)}u(\tau)z(\sigma) - \frac{\psi(u(\sigma))}{u(\sigma)}u(\sigma)z(\tau) \geq 0. \end{aligned} \tag{8}$$

Multiplying both sides of (8) by $\mathfrak{I}(x,\tau)$, $\tau \in (0,x)$, $x>0$, then by integration, one can see that

$$\begin{aligned} &u(\sigma)\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right] + \frac{\psi(u(\sigma))}{u(\sigma)}z(\sigma)\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[u(x)] \\ &- z(\sigma)\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\frac{\psi(u(x))}{u(x)}u(x)\right] - \frac{\psi(u(\sigma))}{u(\sigma)}u(\sigma)\mathfrak{I}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[z(x)] \geq 0. \end{aligned} \tag{9}$$

Also, it is clear that

$$\begin{aligned} &\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right] \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right] + \\ &\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right] \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right] \\ &\geq \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right] \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}u(x)\right] \\ &+ \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}u(x)\right] \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right]. \end{aligned} \tag{10}$$

So, it follows that

$$\begin{aligned} &\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right] \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)z(x)}\right] \\ &\geq \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right] \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}u(x)\right], \end{aligned} \tag{11}$$

$$\frac{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right]}{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right]} \geq \frac{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}u(x)\right]}{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right]}. \tag{12}$$

Now, since $u \leq z$ on $[0,\infty)$ and $\frac{\psi(x)}{x}$ is increasing, then for any $\tau, \sigma \in [0,\infty)$, we can write

$$\frac{\psi(u(\tau))}{u(\tau)} \leq \frac{\psi(z(\tau))}{z(\tau)}. \tag{13}$$

Multiplying (13) by $\mathfrak{J}(x,\tau)z(\tau)$, we can state that

$$\mathfrak{J}(x,\tau)\frac{\psi(u(\tau))}{u(\tau)} \leq \mathfrak{J}(x,\tau)\frac{\psi(z(\tau))}{z(\tau)}. \tag{14}$$

Thus,

$$\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right] \leq \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(z(x))}{z(x)}z(x)\right]. \tag{15}$$

By (12) and (15), we obtain (6).
Let us now prove the result:

Theorem 3.2. *Let u, z be two positive continuous functions on $[0,\infty)$ and $u \leq z$ on $[0,\infty)$. Suppose that $\frac{u}{z}$ is decreasing, u is increasing on $[0,\infty)$ and for any convex function ψ, $\psi(0)=0$. Then for all $x>0$, $v,\alpha,v^{'},\alpha^{'},\xi,\beta,\xi^{'},\beta^{'},\vartheta,\theta \in \mathbb{C}$, $\Re(\vartheta),\Re(\theta)>0$, we have*

$$\frac{\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[u(x)]\,\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[\psi(z(x))]+\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[u(x)]\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[\psi(z(x))]}{\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[z(x)]\,\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[\psi(u(x))]+\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[z(x)]\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[\psi(u(x))]} \geq 1, \tag{16}$$

where $v^{'},\alpha^{'} > -1$, $1 > \max\left\{0,\Re(v+v^{'}+\xi-\vartheta),\Re(v^{'}-\xi^{'})\right\}$, $(\vartheta - v^{'}) > \max(1-\xi,1-v)$, $1 > \max\left\{0,\Re(\alpha+\alpha^{'}+\beta-\theta),\Re(\alpha^{'}-\beta^{'})\right\}$ $(\theta-\alpha^{'}) > \max(1-\beta,1-\alpha)$.

Proof:- Since ψ is convex with $\psi(0)=0$, then $\frac{\psi(x)}{x}$ is increasing. Also, u is increasing, then $\frac{\psi(u(x))}{u(x)}$ is also increasing. Clearly $\frac{u(x)}{z(x)}$ is decreasing, for all $\tau,\sigma \in [0,x)$ $x>0$. So, if we multiply (9) by $\mathfrak{J}(x,\sigma)$ ($\sigma \in (0,x)$, $x>0$), and then, by integration, we can see that

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}\left[u(x)\right]\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right]\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right]\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[u(x)\right]\\
&\geq \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[z(x)\right]\,\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}\left[\psi(u(x))\right]\\
&+\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\psi(u(x))\right]\,\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}\left[z(x)\right].
\end{aligned} \tag{17}$$

Knowing that $u \leq z$ on $[0,\infty)$ and $\frac{\psi(x)}{x}$ is increasing, for $\tau,\sigma \in [0,x)$ $x>0$, then one can observe that

$$\frac{\psi(u(\tau))}{u(\tau)} \leq \frac{\psi(z(\tau))}{z(\tau)}. \tag{18}$$

Therefore,

$$\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)\right] \leq \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\psi(u(x))\right]. \tag{19}$$

Thanks to (17) and (19), we obtain (16).
We have to prove the following result:

Theorem 3.3. *Let u, z and w be positive continuous functions on $[0,\infty)$ and $u \leq z$ on $[0,\infty)$. Under the condition of $\frac{u}{z}$ decreasing, u and w increasing functions on $[0,\infty)$, and for any convex function ψ such that $\psi(0)=0$, then for all $x>0$, $v,v^{'},\xi,\xi^{'},\vartheta \in \mathbb{C}$, $\Re(\vartheta)>0$, we have*

$$\frac{\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[u(x)]}{\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[z(x)]} \geq \frac{\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[\psi(u(x))w(x)]}{\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[\psi(z(x))w(x)]}, \tag{20}$$

where $v^{'}>-1$, $1>max\left\{0,\Re(v,+v^{'}+\xi-\vartheta),\Re(v^{'}-\xi^{'})\right\}$
$(\vartheta-v^{'})>max\,(1-\xi,1-v)$.

Proof: Since $u \leq z$ on $[0,\infty)$ and $\frac{\psi(x)}{x}$ is increasing, thus, for $\tau,\sigma \in [0,x),\ x>0$, we have

$$\frac{\psi(u(\tau))}{u(\tau)} \leq \frac{\psi(z(\tau))}{z(\tau)}. \tag{21}$$

Therefore, we write

$$\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)w(x)\right] \leq \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}\left[\psi(z(x))w(x)\right]. \tag{22}$$

Also, since ψ is convex with $\psi(0)=0$, thus $\frac{\psi(x)}{x}$ is increasing. Also, since u is increasing, then $\frac{\psi(u(x))}{u(x)}$ is also increasing. Clearly, we can say that $\frac{u(x)}{z(x)}$ is decreasing, for all $\tau,\sigma \in [0,x)$ $x>0$

$$\left(\frac{\psi(u(\tau))}{u(\tau)}w(\tau)-\frac{\psi(u(\sigma))}{u(\sigma)}w(\sigma)\right)(u(\sigma)z(\tau)-u(\tau)z(\sigma)) \geq 0, \tag{23}$$

hence,

$$\begin{aligned} &\frac{\psi(u(\tau))w(\tau)}{u(\tau)}u(\sigma)z(\tau)+\frac{\psi(u(\sigma))w(\sigma)}{u(\sigma)}u(\tau)z(\sigma)\\ &-\frac{\psi(u(\tau))w(\tau)}{u(\tau)}u(\tau)z(\sigma)-\frac{\psi(u(\sigma))w(\sigma)}{u(\sigma)}u(\sigma)z(\tau) \geq 0. \end{aligned} \tag{24}$$

So, it yields that

$$u(\sigma)\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)w(x)\right]+\frac{\psi(u(\sigma))}{u(\sigma)}z(\sigma)w(\sigma)\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right]$$
$$-\,z(\sigma)\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\psi(u(x))w(x)\right]-\frac{\psi(u(\sigma))}{u(\sigma)}u(\sigma)w(\sigma)\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right]\geq 0. \tag{25}$$

We have also

$$\frac{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right]}{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right]}\geq\frac{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\psi(u(x))w(x)\right]}{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))}{u(x)}z(x)w(x)\right]}. \tag{26}$$

By (22) and (26), we obtain (20).
Now, we pass to prove a generalization for Theorem 3.3.

Theorem 3.4. *Let u, z and w be three positive continuous functions on $[0,\infty)$ and $u\leq z$ on $[0,\infty)$. Suppose that $\frac{u}{z}$ is decreasing, u and w are increasing functions on $[0,\infty)$, and for any convex function ψ such that $\psi(0)=0$, $x>0$, $v,\alpha,v',\alpha',\xi,\beta,\xi',\beta',\vartheta,\theta\in\mathbb{C}$, $\mathfrak{R}(\vartheta),\mathfrak{R}(\theta)>0$. Then we have*

$$\frac{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[u(x)]\,\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[\psi(z(x))w(x)]+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[u(x)]\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[\psi(z(x))w(x)]}{\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[z(x)]\,\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[\psi(u(x))w(x)]+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[z(x)]\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[\psi(u(x))w(x)]}$$
$$\geq 1, \tag{27}$$

where $v',\alpha'>-1$, $1>\max\left\{0,\mathfrak{R}(v+v'+\xi-\vartheta),\mathfrak{R}(v'-\xi')\right\}$, $(\vartheta-v')>\max\left(1-\xi,1-v\right)$, $1>\max\left\{0,\mathfrak{R}(\alpha+\alpha'+\beta-\theta),\mathfrak{R}(\alpha'-\beta')\right\}$ $(\theta-\alpha')>\max\left(1-\beta,1-\alpha\right)$.

Proof:- Multiplying (25) by $\mathfrak{J}(x,\sigma)$ $(\sigma\in(0,x),\ x>0)$, we have

$$\begin{aligned}&\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}\left[u(x)\right]\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))w(x)}{u(x)}z(t)\right]\\&+\,\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}\left[\frac{\psi(u(x))w(x)}{u(x)}z(x)\right]\times\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[u(x)\right]\\&\geq\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[z(x)\right]\,\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}\left[\psi(u(x))w(x)\right]\\&+\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\psi(u(x))w(x)\right]\,\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}\left[z(x)\right],\end{aligned} \tag{28}$$

and knowing that $u \leq z$ on $[0,\infty)$ and using the fact that $\frac{\psi(x)w(x)}{x}$ is increasing, we state that

$$\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\frac{\psi(u(x))w(x)}{u(x)}z(x)\right] \leq \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}\left[\psi(z(x))w(x)\right], \tag{29}$$

and

$$\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}\left[\frac{\psi(u(x))w(x)}{u(x)}z(x)\right] \leq \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}\left[\psi(z(x))w(x)\right]. \tag{30}$$

Hence, from equation (28), (29) and (30), we obtain (27).

4 Other fractional integral inequalities

In [13], the authors have proved the inequalities using Riemann-Liouville fractional integral. Here, we investigate some other inequalities using Marichev-Saigo-Maeda integral operators.

Theorem 4.1. *Let u, w be two positive and continuous functions on $[0,\infty)$ such that u is decreasing and w is increasing on $[0,\infty)$, then for all $x>0$, $v,v',\xi,\xi',\vartheta \in \mathbb{C}$, $\Re(\vartheta)>0$, $l \geq m > 0$, and $n>0$, we have*

$$\frac{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^l(x)]}{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^m(t)]} \geq \frac{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^l(t)]}{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^m(t)]}, \tag{31}$$

where $v' > -1$, $1 > max\left\{0, \Re(v,+v'+\xi-\vartheta), \Re(v'-\xi')\right\}$ $(\vartheta - v') > max\,(1-\xi, 1-v)$.

Proof:- By considering $\sigma,\tau \in (0,x)$, we can observe that

$$(w^n(\sigma)-w^n(\tau))\left(u^l(\tau)u^m(\sigma)-u^m(\tau)u^l(\sigma)\right) \geq 0,$$

which means that

$$w^n(\sigma)u^l(\tau)u^m(\sigma)+w^m(\tau)u^m(\tau)u^l(\sigma) \geq w^m(\sigma)u^m(\tau)u^l(\sigma)+w^m(\tau)u^m(\sigma)u^l(\tau). \tag{32}$$

Also, we have

$$\begin{aligned} &w^n(\sigma)u^l(\sigma)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^l(x)] + u^l(\sigma)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^m(x)] \\ &\geq w^m(\sigma)u^l(\sigma)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^m(x)] + u^m(\sigma)\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^m u^l(x)]. \end{aligned} \tag{33}$$

It is clear also that

$$\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^m(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^l(x)] \geq \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^l(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^m(x)], \tag{34}$$

which gives (31).

Now, we present the following result:

Theorem 4.2. *Let u, w be two positive and continuous functions on $[0,\infty)$ such that u is decreasing and w is increasing on $[0,\infty)$, then for all $x > 0$, $v, v', \xi, \xi', \vartheta \in \mathbb{C}$, $\Re(\vartheta) > 0$, $l \geq m > 0$, and $n > 0$, we have*

$$\frac{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^l(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w^n u^m(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[u^l(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^m(x)]}{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^m(x)]\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w^n u^m(x)] + \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[u^m(x)]\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^m(x)]} \geq 1, \tag{35}$$

where $v', \alpha' > -1$, $1 > \max\left\{0, \Re(v + v' + \xi - \vartheta), \Re(v' - \xi')\right\}$, $(\vartheta - v') > \max(1 - \xi, 1 - v)$, $1 > \max\left\{0, \Re(\alpha + \alpha' + \beta - \theta), \Re(\alpha' - \beta')\right\}$ $(\theta - \alpha') > \max(1 - \beta, 1 - \alpha)$.

Proof:- Multiplying inequality (33) by $\mathfrak{I}(x,\sigma)$ $(\sigma \in (0,x),\ x > 0)$ which remains positive. Then integrate the resulting identity with respect to σ from 0 to x, we obtain the result (35).

Theorem 4.3. *Let u, w be two positive and continuous functions on $[0,\infty)$ such that u is decreasing and w is increasing on $[0,\infty)$, then for all $x > 0$, $v, v', \xi, \xi', \vartheta \in \mathbb{C}$, $\Re(\vartheta) > 0$, $l \geq m > 0$, and $n > 0$, we have*

$$(u^n(\tau)w^n(\sigma) - u^n(\sigma)w^n(\tau))\left(u^m(\sigma)u^l(\tau) - u^m(\tau)u^l(\sigma)\right) \geq 0.$$

Then we have

$$\frac{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^{n+l}(x)]}{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[u^{n+m}(x)]} \geq \frac{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^l(x)]}{\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w^n u^m(x)]}, \tag{36}$$

where $v^{'} > -1,\ 1 > max\left\{0, \Re(v, +v^{'} + \xi - \vartheta), \Re(v^{'} - \xi^{'})\right\}$
$(\vartheta - v^{'}) > max\,(1 - \xi, 1 - v)$.

Proof:- Consider $\tau, \sigma \in (0, t)$, we get

$$(u^n(\tau)w^n(\sigma) - u^n(\sigma)w^n(\tau))\left(u^m(\sigma)u^l(\tau) - u^m(\tau)u^l(\sigma)\right) \geq 0,$$

and using the same arguments as the proof of Theorem 4.1.

5 Concluding Remarks

In this paper, we have applied Marichev-Saigo-Maeda integral operators to establish new integral inequalities for convex functions. Some other inequalities for positive monotone functions have also been studied. It is to note that:
If we set $v^{'} = 0$ in (1), then it will be reduced to Saigo operators [17, 23, 27, 28] and then we have:

$$\left(\mathfrak{J}_{0,x}^{v,0,\xi,\xi^{'},\vartheta} f\right)(x) = \left(\mathfrak{J}_{0,x}^{\vartheta,v-\vartheta,-\xi} f\right)(x), \tag{37}$$

where the hypergeometric operator that appears in the right hand side is defined as

$$\mathfrak{J}_{0,x}^{v,v^{'},\vartheta} f(x) = \frac{x^{-v-\xi}}{\Gamma(\vartheta)} \int_0^x (x-t)_2^{v-1} F_1(v+\xi; v; 1-\frac{t}{x}) f(t)dt, (\vartheta > 0, v, \xi \in \mathbb{C}). \tag{38}$$

Further, we can reduce (1) to Erdelyi-Kober and Riemann-Liouville type operators which are special cases of Saigo fractional operators (38). Thus, the obtained fractional inequalities in this paper present to the reader new contribution for Marichev-Saigo-Maeda integral operator applications.
We propose to interested researchers to study fractional differential equations using the inverse operator of the present MSM integral approach.

Acknowledgements
We would like to thank the reviewers and editor for their valuable comments and suggestions towards the improvement of the manuscript.

References

[1] J. Alzabut, A. G. M. Selvam, R. Dhineshbabu and M. K. A Kaabar, The Existence, Uniqueness, and Stability Analysis of the Discrete Fractional Three-Point Boundary Value Problem for the Elastic Beam Equation, *Symmetry* 2021, 13, 789. https://doi.org/10.3390/sym13050789.

[2] D. Baleanu, S. D. Purohit and J. C. Prajapati, Integral inequalities involving generalized Erdélyi-Kober fractional integral operators, *Open. Math.* 14, (2016), 89-99.

[3] D. Baleanu, A. Jajarmi, H. Mohammadi and S. Rezapour, A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative, *Chaos, Solitons and Fractals*, 134, 2020, 109705, https://doi.org/10.1016/j.chaos.2020.109705.

[4] D. Baleanu, S. S. Sajjadi, A. Jajarmi and O. Defterli, On a nonlinear dynamical system with both chaotic and nonchaotic behaviors: a new fractional analysis and control, *Adv. Differ. Equ.*, 2021, 234 (2021). https://doi.org/10.1186/s13662-021-03393-x.

[5] D. Baleanu, S. S. Sajjadi, J. H Asad, A. Jajarmi and E. Estiri, Hyperchaotic behaviors, optimal control, and synchronization of a nonautonomous cardiac conduction system, *Adv. Differ, Equ.*, 2021, 157 (2021). https://doi.org/10.1186/s13662-021-03320-0.

[6] D. Baleanu, S. S. Sajjadi, A. Jajarmi, O.Defterli and J. H Asad, The fractional dynamic of a linear triatomic molecule, *Romanian Reports in Physics* 73, 105 (2021).

[7] V. L. Chinchane, On Chebyshev type inqualities using generalized K-fractional integral operator, *Progr. Fract. Differ. Appl.*, 3(3),(2017) 1-8.

[8] V. L. Chinchane and D. B. Pachpatte, On some Grüss-type fractional inequalities using Saigo fractional integral operator, *Journal of Mathematics*, Article ID 527910, Vol.2014 (2014), 1-9.

[9] V. L. Chinchane and D. B. Pachpatte, On new fractional integral inequalities involving convex functions usinh Hadamard fractional integral, *Bulletin of Allahabad Mathematical Society*, 2(31),(2016) 183-192.

[10] V. L. Chinchane and D. B. Pachpatte, Note on fractional integral inequality involving convex function using Saigo fractional integral, *Indian Journal of Mathematics*, Vol.1(61),(2019),27-39.

[11] Z. Dahmani, A note on some new fractional results involving convex functions, *Acta Math. Univ. Comenianae*, 81(2),(2012), 241-246.

[12] Z. Dahmani, L. Tabharit and S. Taf, New generalisation of Grüss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.*, 2(3), (2012), 92-99.

[13] Z. Dahmani and N. Bedjaoui, Some generalized integral inequalities, *J. Advan. Res. Appl. Math*, 3(4),(2011), 58-66.

[14] Z. Dahmani and H. Metakkel El Ard, Generalization of some integral inequalities using Riemann-Liouville operator, *Int. J. Open Problem Compt. Math.*, 4(4), (2011), 40-46.

[15] S. Etemad, M. S. Souid, B. Telli and et al., Investigation of the neutral fractional differential inclusions of Katugampola-type involving both retarded and advanced arguments via Kuratowski MNC technique, *Adv. Differ. Equ.* 2021, 214 (2021), https://doi.org/10.1186/s13662-021-03377-x.

[16] A. A. George, *Fractional differentiation inequalities,* Springer Publishing Company, New York, 2009.

[17] S. Joshi and E. Mittal, R. Pandey and S. Purohit, Some Grüss type inequalities involviing generalized fractional integral operator, *Bulletin of the Transilvania University of Brasov* 12(61),(2019), 41-52.

[18] S. Joshi and E. Mittal and R. Pandey, some fractional integral inequalities involving appell hypergeometric function, *J.Sci. Arts*, 1(2016), 23-30.

[19] O. I. Marichev, Volterra equation of Mellin convolution type with a Horn function in the kernal,*Izv.AN BSSR Ser. Fiz. Mat. Nauk* 1(1974) 128-129.

[20] M. M. Matar, M. I. Abbas, J. Alzabut, et al. Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives, *Adv. Differ. Equ.* 2021, 68 (2021). https://doi.org/10.1186/s13662-021-03228-9.

[21] V. N. Mishra, D. L. Suthar and S. D. Purohit Marichev-Saigo-Maeda fractional calcus operators, Srivastava polynomials and generalized Mittag-Leffler function,*Cogent Mathematics*,(2017), 4: 1320830, 1-11.

[22] A. B. Nale, S. K. Panchal and V. L. Chinchane, Certain fractional integral inequalities using generalized Katugampola fractional integral operator, *Malaya J. Math.*, 3(8), (2020), 791-797.

[23] S. D. Purohit, D. L. Sutar and S. K. Kalla, Marichev-Saigo- Maeda fractional integration opreators of the Bessel functions, *Le. Matematiche*, LXVII(2012), Fasc. I, 21-32.

[24] G. Rahman, A. Tassaddiq, F. Jarad, A. Khan, and K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Diff. Equ.*, 2019:454, (2019) 1-10.

[25] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad and M. K. A. Kaabar, Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs, *Symmetry* 2021, 13, 469. https://doi.org/10.3390/sym13030469.

[26] S. Rezapour, H. Mohammadi and A. Jajarmi, A new mathematical model for Zika virus transmission, *Adv. Differ. Equ.*, 2020, 589(2020). https://doi.org/10.1186/s13662-020-03044-7.

[27] R. K. Saxena and M. Saigo, Generalized fractional calculus of the H- function associated with the Appell function, *J. Frac. Calc.*, 19(2001), 89-104.

[28] M. Saigo, A remark on integral operators involving the Grüss hypergeometric functions, *Math. Rep. Kyushu Univ.*, 11(1978), 135-143.

[29] M. Saigo and N. Maeda, More generalization of fractional calculus, In: Rusev. P. Dimovski, I, Kiryakova, V(eds.) Transform methods and special Functions, Varna, 1996 *Proc. 2nd intern. Workshop*, pp.386-400.IMI-BAS, Sofia (1998).

[30] S. G. Somko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivative theory and application,* Gordon and Breach, Yverdon, Switzerland, 1993.

[31] H. M. Srivastava and P. W. Karlson, *Multiple gaussion hypergeometric series,* Ellis Horwood Limited, New York, 1985.

[32] A. Tassaddiq, A. Khan, G. Rahman, K. S. Nisar, M. S. Abouzaid and I. Khan, Fractional integral inequalities involving Marichev-Saigo- Maeda fractional integral operator, *J. Inequal. Appl.*, 2020, 185 (2020), 1-14, https://doi.org/10.1186/s13660-020-02451-4.

Asha B. Nale
Department of Mathematics,
Research Student of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad-431 004, INDIA.
E-mail: ashabnale@gmail.com

Satish K. Panchal
Department of Mathematics
Professor of Mathematics
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad-431 004, INDIA.
E-mail: drskpanchal@gmail.com

Vaijanath L. Chinchane
Department of Mathematics
Assistant Professor of Mathematics
Deogiri Institute of Engineering and Management Studies,

Aurangabad-431 004, INDIA.
E-mail: chinchane85@gmail.com

Zoubir Dahmani
Department of Mathematics
Professor of Mathematics
Laboratory, LPAM, Faculty SEI, UMAB University, 27000 ,
Mostaganem-RP, Algeria.
E-mail: zzdahmani@yahoo.fr