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Barycentric Legendre Interpolation Method for Solving Nonlinear Fractal-Fractional Burgers Equation

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Abstract. In this paper, we formulate a numerical method to approximate the solution of non-linear fractal-fractional Burgers equation. In this model, differential operators are defined in the Atangana-Riemann-Liouville sense with Mittag-Leffler kernel. We first expand the spatial derivatives using barycentric interpolation method, and then we derive an operational matrix (OM) of the fractal-fractional derivative for the Legendre polynomials. To be more precise, two approximation tools are coupled to convert the fractal-fractional Burgers equation into a system of algebraic equations which is technically uncomplicated and can be solved using available mathematical software such as MATLAB. To investigate the agreement between exact and approximate solutions, several examples are examined.

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1 Introduction

The calculus of fractional derivative was proposed more than three centuries ago. This subject has improved by many mathematicians such as Letnikov, Grunwald, Liouvill, Fourier, Euler up to now and furthermore, some researchers could define newer fractional operators [22, 1, 6, 7, 17].

Numerous scientists have successfully investigated the fractional calculus to describe many phenomena in different fields such as the optimal control of tumor invasion, human liver [17], hearing loss [21], the optimal strategy of thermal treatment in cancer therapy, anthrax disease model in animals [36], and the medical image analysis and physics [5, 13, 10, 30].

Many studies have been focused on finding the numerical solutions of fractional partial differential equations (PDEs). For instance, authors in [2] derived an existence criterion for a Caputo conformable hybrid multi-term integro-differential equation equipped with initial conditions, and by applying the lower solution property, the existence and successive approximation of solutions for the mentioned hybrid initial problem are investigated. Rezapour and et.al [35] established some necessary conditions to check the uniqueness-existence of solutions for a general multi-term ψ-fractional differential equation via generalized ψ-integral boundary conditions with respect to the generalized asymmetric operators by employing the fixed-point theory. In [18], the dynamical behaviors of a linear triatomic molecule is studied. Then, a classical Lagrangian approach is followed which produces the classical equations of motion, and the generalized form of the fractional Hamilton equations (FHEs) is formulated in the Caputo sense and in order to solve it, a numerical scheme based on the Euler convolution quadrature rule is introduced. Authors in [11] studied a general form of fractional optimal control problems involving the fractional derivative with singular or non-singular kernel. The necessary conditions for the optimality of the problem are derived and then the solution is generated by using an iterative technique.

Burgers equation or Bateman-Burgers equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow. Many studies have been done on the classical Burgers equation, numerical results have been successfully obtained. Applications of the fractional calculus idea have been developed to study Burgers equation, and many researchers have shown great interest to find an efficient numerical technique for solving this type of problems. For instance, Momani [34] proposed a domain decomposition method to solve the Burgers equation with fractional derivative in space and time. Yildirim [14] used a homotopy analysis method to achieve the numerical solution and analytical solution of Burgers equation with space and time fractional derivative. A perturbation method and generalized differential transform method is used in [31] to solve the coupled Burgers equation and time fractional Burgers equation. Yokus [15] applied a Cole-Hopf transformation method and expansion method to obtain the exact and numerical solution of Burgers equation. Tasbozan used a cubic B-spline finite element method [9] and B-spline Galerkin method [8] to obtain the numerical solution of time fractional Burgers equation.

The Lagrange method is an option for dealing with polynomial interpolations. The main point is that it must manipulate Lagrange polynomials through barycentric interpolation formulas [25]. When the nodes have a uniform distribution, the weight functions become very large, leading to a Runge phenomenon and undermining the advantages of Lagrange interpolation. But if the nodes obey the density proportion $(1-x^2)^{-\frac{1}{2}}$, the interpolation has a good numerical stability, and the easiest collocation points are some types of Chebyshev points [26, 23, 24]. Barycentric method has recently been developed to solve partial differential equations (PDE) and ordinary differential equations such as the problems of high 1D boundary and initial values and nonlinear PDEs [32, 33]. But, there are few reports about the application of barycentric Lagrange interpolation in the literature especially for the fractional order problems.

This paper studies the following one-dimensional non-linear time fractal-fractional Burgers equation

$$
\begin{aligned}
&{}^{FFM}_{\ \ \ \ 0}D_t^{\alpha,\beta} u(x,t) + u(x,t)u_x(x,t) - \gamma u_{xx}(x,t) = f(x,t), \quad \text{in}\, Q,\\
&u(.,t) = \rho(x,t), \qquad \text{on}\, \Sigma,\\
&u(x,0) = g(x), \qquad \text{in}\, \Omega,
\end{aligned}
\tag{1}
$$

where $\Omega = [0,1]$ is the spatial domain and $\partial\Omega$ is the boundary of Ω. Let $Q = \Omega \times (0,1]$, $\Sigma = \partial\Omega \times (0,1]$ and $0 < \alpha < 1$. ${}^{FFM}_{\ \ \ \ 0}D_t^{\alpha,\beta} u(x,t)$ is the fractal fractional derivative of $u(x,t)$ of order (α,β) in the Atangana-Riemann-Liouville concept with Mittag-Leffler non-singular kernel. The interested reader may see more models defined by this new fractional derivative operator in [27, 28, 29] and references therein.

Our idea of finding the numerical solution is based on using a new barycentric Lagrange interpolation method for expanding the spatial derivatives of one-dimensional generalized Burgers equation, and then apply Legendre polynomials to simplify the time derivatives. Consequently, we obtain the operational matrix of Legendre polynomial for time fractal-fractional derivative. The main advantage of Legendre polynomials is that by using only a few Legendre foundations, we achieve satisfactory results. Also, the advantages of the barycentric interpolation method are accurate, fast, simple and easy to implement boundary conditions in order to prevent singularity. Moreover, the barycentric interpolation method requires $O(n)$ operations while the classical Lagrange interpolation method needs $O(n^2)$ operations. These benefits make the presented method effective for solving the identified problem.

The paper is organized as follows: Some definitions and explanation of fractal-fractional derivative are provided in Section 2. In Section 3, some properties of Legendre polynomials are explained. Barycentric interpolation method is given in Section 4. Section 5 is devoted to reduce the problem (1) by barycentric interpolation method and Legendre polynomials. Finally, multiple numerical examples are provided to show the effectiveness of the suggested method in Section 6.

2 Fractal-fractional Derivative

In this section, we summarize the important issues in the field of fractal-fractional calculus.

Definition 2.1. [22] *The Mittag-Leffler function $\boldsymbol{E}_{\alpha,\beta}(t)$ is a function which depends on two parameters α and β and can be defined as*

$$\boldsymbol{E}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}, \qquad \alpha \in \mathbb{R}^+, t \in \mathbb{R},$$

where $\Gamma(\cdot)$ is the Gamma function. If $\beta = 1$, $\boldsymbol{E}_{\alpha,1}(t)$ is called one parameter Mittag-Leffler function and can be written as $\boldsymbol{E}_{\alpha}(t)$. This *type of function is an essential function in fractional calculus.*

Definition 2.2. [3, 4]*Let $h(t)$ is a continuous function on open interval (a, b), and $(\alpha, \beta) \in (0, 1)$. The fractal-fractional differentiation of order (α, β) in the type of Atangana-Riemann-Liouville sense with the generalized Mittag-Leffler kernel is given as follows*

$$ {}^{FFM}_{\quad 0}D_t^{\alpha,\beta} h(t) = \frac{C(\alpha) t^{(1-\beta)}}{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{dt}} \int_0^t \boldsymbol{E}_\alpha \left(\frac{-\alpha(t-s)^\alpha}{1-\alpha} \right) h(s) \mathrm{ds}, $$

where $C(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Lemma 2.3. *Suppose that $\alpha, \beta \in (0, 1)$ are real constants. Then, for $j \in \mathbb{N} \cup \{0\}$ we have*

$$ {}^{FFM}_{\quad 0}D_t^{\alpha,\beta} t^j = \frac{C(\alpha) j! t^{j-\beta+1}}{\beta(1-\alpha)} \boldsymbol{E}_{\alpha,j+1} \left(\frac{-\alpha t^\alpha}{1-\alpha} \right). $$

3 Legendre Polynomials

Orthogonal Legendre polynomials on interval $[-1, 1]$ are defined as follows

$$ L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), \qquad m = 1, 2, \cdots, $$

where $L_0(x) = 1$ and $L_1(x) = x$. By change of variable $x = 2t - 1$, we have the shifted Legendre variable on interval $[0, 1]$ as

$$ p'_{m+1}(x) = \frac{2m+1}{m+1} (2t-1) L_m(t) - \frac{m}{m+1} L_{m-1}(t), \qquad m = 1, 2, \cdots, $$

where $p'_0(t) = 1$ and $p'_1(t) = 2t - 1$. Set $p_i(t) = \sqrt{2i+1} p'_i(t)$. Then for $p_i(t)$ we have

$$ \int_0^1 p_i(t) p_j(t) \mathrm{dt} = \begin{cases} 1, & i = j, \\ 0 & i \neq j. \end{cases} $$

$p_i(t)$ can be rewritten as the series of power functions as:

$$ p_i(t) = \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)! t^k}{(i-k)!(k!)^2}. \tag{2} $$

Let $T_m(t) = [1, t, \cdots, t^m]^T$ and $\Psi_m(t) = [p_0, p_1, \cdots, p_m]^T$. Then, $\Psi_m(t)$ can be written as

$$\Psi_m(t) = AT_m(t), \tag{3}$$

where

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\sqrt{3} & 2\sqrt{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^m\sqrt{2m+1} & (-1)^{m+1}\sqrt{2m+1}\frac{(m+1)!}{(m-1)!} & \cdots & (-1)^{2m}\sqrt{2m+1}\frac{(2m)!}{(m!)^2} \end{bmatrix}.$$

Theorem 3.1. *[19] Suppose H is an inner product space and $Y \subset H$ a complete subspace. Let $\{e_0, e_1, \cdots, e_n\}$ is an orthogonal basis for H . Then, for every $f \in H$ the best approximation f_0 of f in Y is given by*

$$f_0 = \sum_{i=0}^{n} < f, e_i > e_i,$$

such that

$$\forall y \in Y \quad \|f - f_0\|_2 \leqslant \|f - y\|_2,$$

where $\|f\|_2 = \sqrt{< f, f >}$.

So, we can approximate f as

$$f(t) \simeq \sum_{j=0}^{m} c_j p_j(t) = C^T \Psi_m(t),$$

where c_j can be calculated as follows

$$c_j =< f(t), p_j(t) >,$$

and we have

$$C^T = [c_0, \cdots, c_m], \qquad \Psi_m^T = [p_0, \cdots, p_m].$$

For more detail about Legendre polynomials, see [12]

3.1 Operational matrix

In this section, the operational matrix of Legendre polynomial for the fractal-fractional derivative will be obtained.

Theorem 3.2. *Suppose that $(\alpha, \beta) \in (0,1)$, and $\Psi_m(t)$ is the vector of Legendre polynomials up to m degree. The fractal-fractional derivative of $\Psi_m(t)$ of order (α, β) can be written as*

$$ {}^{FFM}_{\ \ \ \ 0}D_t^{\alpha,\beta}\Psi_m(t) \simeq D^{\alpha,\beta}\Psi_m(t), $$

where $D^{\alpha,\beta}$ is $(m+1)\times(m+1)$ matrix whose elements are given by

$$ D_{ij}^{\alpha,\beta} = \frac{C(\alpha)}{\beta(1-\alpha)} \sum_{k=0}^{m}\sum_{r=0}^{m} k! a_{ik} a_{jr} w_{kr}, \quad i,j = 0,1,\cdots,m. $$

Proof. Using equations (2) and (3), $p_i(t)$'s can be written as

$$ p_i(t) = \sum_{k=0}^{m} a_{ik} t^k. $$

Then

$$ {}^{FFM}_{\ \ \ \ 0}D_t^{\alpha,\beta} p_i(t) = \frac{C(\alpha)}{\beta(1-\alpha)} \sum_{k=0}^{m} k! a_{ik} t^{k-\beta+1} \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^\alpha}{1-\alpha}\right). $$

Using Legendre polynomials and Theorem 3.1, we can approximate $t^{k-\beta+1}\mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^\alpha}{1-\alpha}\right)$ as follows

$$ t^{k-\beta+1}\mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^\alpha}{1-\alpha}\right) \simeq \sum_{j=0}^{m} b_{kj} p_j(t). $$

where

$$ b_{kj} = \int_0^1 t^{k-\beta+1}\mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^\alpha}{1-\alpha}\right) p_j(t)\mathrm{dt}, $$

or

$$ b_{kj} = \sum_{r=0}^{m} a_{jr} \int_0^1 \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^\alpha}{1-\alpha}\right) t^{k+r-\beta+1}\mathrm{dt} = \sum_{r=0}^{m} a_{jr} w_{kr}, $$

where

$$w_{kr} = \int_0^1 \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^{\alpha}}{1-\alpha}\right) t^{k+r-\beta+1}\mathrm{dt}.$$

Then

$$D_{ij}^{\alpha,\beta} = \frac{C(\alpha)}{\beta(1-\alpha)}\sum_{k=0}^{m}\sum_{r=0}^{m} k! a_{ik} a_{jr} w_{kr}.$$

□

We can approximate w_{kr} as

$$\begin{aligned} w_{kr} &= \int_0^1 \mathbf{E}_{\alpha,k+1}\left(\frac{-\alpha t^{\alpha}}{1-\alpha}\right) t^{k+r-\beta+1}\mathrm{dt} \\ &= \sum_{z=0}^{\infty} \frac{(-\alpha)^z}{(1-\alpha)^z \Gamma(z\alpha+k+1)(\alpha z+k+r-\beta+2)}. \end{aligned}$$

By taking $(n+1)$-terms, we can approximate w_{kr} as follows

$$w_{kr} = \sum_{z=0}^{n} \frac{(-\alpha)^z}{(1-\alpha)^z \Gamma(z\alpha+k+1)(\alpha z+k+r-\beta+2)},$$

If we define

$$M = \begin{bmatrix} 1! & 0 & \cdots & 0 \\ 0 & 2! & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m! \end{bmatrix}, \quad W = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0m} \\ w_{10} & w_{11} & \cdots & w_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{mm} \end{bmatrix},$$

we can express $D^{\alpha,\beta}$ in a matrix form as follows

$$D^{\alpha,\beta} = \frac{C(\alpha)}{\beta(1-\alpha)} AMW^T A^T.$$

Setting $n=4$, and $(\alpha,\beta) = \left(\frac{1}{2}, \frac{3}{4}\right)$, one can obtain the following form of $D^{\alpha,\beta}$

$$D^{(\frac{1}{2},\frac{3}{4})} = \begin{bmatrix} 0.8893050 & 0.0272249 & -0.0376262 & 0.0267639 & -0.0187959 \\ 0.4189647 & 1.1353605 & 0.0768211 & -0.0561239 & 0.0371724 \\ 0.0130113 & 0.4451404 & 1.2249472 & 0.1049496 & -0.0644275 \\ 0.0713947 & 0.0011382 & 0.4388556 & 1.2758877 & 0.1240585 \\ -0.0034177 & 0.0845659 & -0.0090849 & 0.4314818 & 1.3099623 \end{bmatrix}.$$

4 Barycentric Interpolation Method

Suppose the data $f_0, f_1, \cdots, f_N$ corresponding to nodes $x_0, x_1, \cdots, x_N$, and they sampled from a function f, i.e., $f = f(x_i)$, $i = 0, 1, \cdots, N$. Suppose Π_N is the space of polynomial of degree at most N. If f_i, $i = 0, 1, \cdots, N$ are sampled from a linear polynomial $f(x) \in \Pi_N$, then barycentric interpolant of this data can be defined as

$$f(x) = \sum_{i=0}^{N} f_i \varphi_i(x).$$

According to the Lagrangian interpolation method, we can define $\varphi_i(x)$ in the following form

$$\varphi_i(x) = \frac{\prod_{k=0, i\neq k}^{N}(x - x_k)}{\prod_{k=0, i\neq k}^{N}(x_i - x_k)}. \tag{4}$$

The barycentric weights can be defined as

$$w_i = \frac{1}{\prod_{k=0,\, k\neq i}^{N}(x_i - x_k)} = \frac{1}{\varphi'(x_i)}, \qquad i = 0, 1, \cdots, N.$$

If we define $l(x) = (x - x_0)(x - x_1)\cdots(x - x_N)$, then

$$\varphi_i(x) = l(x)\frac{w_i}{x - x_i}, \qquad i = 0, 1, \cdots, N. \tag{5}$$

Inserting equation (5) into equation (4), we have

$$f(x) = l(x)\sum_{i=0}^{N} \frac{w_i}{x - x_i} f_i. \tag{6}$$

Through interpolation function $f(x) = 1$, it can be obtained

$$1 = \sum_{i=0}^{N} \varphi_i(x) = l(x)\sum_{i=0}^{N} \frac{w_i}{x - x_i}. \tag{7}$$

Dividing equation (6) by equation (7) results in

$$f(x) = \frac{\sum_{i=0}^{N} \frac{w_i}{x - x_i} f_i}{\sum_{i=0}^{N} \frac{w_i}{x - x_i}},$$

which is known as *Barycenteric Formula*.

4.1 Differentiation of Polynomial Interpolates

Suppose smooth function $f(x)$ is in the form of barycentric interpolation method as

$$f(x) = \sum_{i=0}^{N} f_i \varphi_i(x),$$

where

$$\varphi_i(x) = \frac{\frac{w_i}{x-x_i}}{\sum_{k=0}^{N} \frac{w_k}{x-x_k}}.$$

To approximate the s-order derivative of a smooth function f at point $x = x_j$, we have the following relation

$$f^{(s)}(x_j) = \sum_{i=0}^{N} \varphi_i^{(s)}(x_j) f_i = \sum_{i=0}^{N} \varphi_{ji}^{(s)} f_i,$$

where $\varphi_{ji}^{(s)}$ can be achieved as follows

$$\varphi_{ji}^{(1)} = \begin{cases} \frac{w_j}{w_i} \frac{1}{x_j - x_i}, & i \neq j, \\ -\sum_{l \neq j} \varphi_{jl}^{(1)}, & i = j, \end{cases}$$

$$\varphi_{ji}^{(s)} = \begin{cases} k(\varphi_{ji}^{(1)} \varphi_{jj}^{(k-1)} - \frac{\varphi_{ji}^{(k-1)}}{x_j - x_i}), & i \neq j, \\ -\sum_{l \neq i} \varphi_{jl}^{(k)}, & i = j. \end{cases}$$

For more details, see [20].

5 Expansion Using Barycenteric Interpolation Method and Legendre Polynomials

In this section, spatial derivative of equation (1) is reduced by barycentric interpolation method, and the time derivative is expanded in terms of Legendre polynomials. Based on barycentric interpolation method, we choose $N+1$ interpolation points as $\{x_i\}_{i=0}^N$ to build basis functions $\{\varphi_i(x)\}_{i=0}^N$. Then, we can approximate $u(x,t)$ in equation (1) as

$$\begin{cases} u(x,t) = \sum_{i=0}^N \varphi_i(x) u_i(t), \\ u_x(x,t) = \sum_{i=0}^N \varphi_i^{(1)}(x) u_i(t). \\ u_{xx}(x,t) = \sum_{i=0}^N \varphi_i^{(2)}(x) u_i(t). \end{cases} \tag{8}$$

By substituting equation (8) into equation (1), we get

$$\begin{aligned} {}^{FFM}_{0}D_t^{\alpha,\beta}\left(\sum_{i=0}^N \varphi_i(x)u_i(t)\right) + \left(\sum_{i=0}^N \varphi_i(x)u_i(t)\right)\left(\sum_{i=0}^N \varphi_i^{(1)}(x)u_i(t)\right) \\ - \gamma \sum_{i=0}^N \varphi_i^{(2)}(x)u_i(t) = f(x,t), \quad \text{in}\, Q, \end{aligned}$$

with initial and boundary conditions

$$\begin{aligned} &\sum_{i=0}^N \varphi_i(x)u_i(t_0) = g(x), \qquad \text{in}\,\Omega, \\ &\sum_{i=0}^N \varphi_i(x_0)u_i(t) = \rho(x_0,t), \\ &\sum_{i=0}^N \varphi_i(x_N)u_i(t) = \rho(x_N,t). \end{aligned}$$

At $x = x_j$ we have the following equations

$$\begin{aligned} {}^{FFM}_{0}D_t^{\alpha,\beta}\left(\sum_{i=0}^N \varphi_i(x_j)u_i(t)\right) + \left(\sum_{i=0}^N \varphi_i(x_j)u_i(t)\right)\left(\sum_{i=0}^N \varphi_i^{(1)}(x_j)u_i(t)\right) \\ - \gamma \sum_{i=0}^N \varphi_i^{(2)}(x_j)u_i(t) = f(x_j,t), \quad j = 1, \cdots, N-1, \end{aligned}$$

$$\sum_{i=0}^{N}\varphi_i(x_j)u_i(t_0)=g(x_j),\quad j=1,\cdots,N-1,$$
$$\sum_{i=0}^{N}\varphi_i(x_0)u_i(t)=\rho(x_0,t),$$
$$\sum_{i=0}^{N}\varphi_i(x_N)u_i(t)=\rho(x_N,t).$$

Since

$$\varphi_i(x_j)=\varphi_{ji}=\delta_{ji},\qquad \varphi_i^{(2)}(x_j)=\varphi_{ji}^{(2)}.$$

Then

$$ {}^{FFM}_{\quad 0}D_t^{\alpha,\beta}u_j(t)+u_j(t)\left(\sum_{i=0}^{N}\varphi_{ji}^{(1)}u_i(t)\right)-\gamma\sum_{i=0}^{N}\varphi_{ji}^{(2)}u_i(t)=f(x_j,t), \quad (9)$$

where $j=1,\cdots,N-1$, and

$$u_j(t_0)=g(x_j),\quad j=1,\cdots,N-1,$$
$$u_0(t)=\rho(x_0,t),$$
$$u_N(t)=\rho(x_N,t).$$

Suppose that $p_r(t)$ is the Legendre polynomial of degree r. Then, $u(x_j,t)$, $j=1,\cdots,N-1$, can be expanded in terms of Legendre polynomials as follows

$$u_j(t)=\sum_{r=0}^{m}u_r^j p_r(t)=[u_0^j,u_1^j,\cdots,u_m^j]\Psi_m(t)=\mathrm{u}^j\Psi_m(t).$$

Therefore, the time derivatives cab be computed as

$$ {}^{FFM}_{\quad 0}D_t^{\alpha,\beta}u_j(t)=[u_0^j,u_1^j,\cdots,u_m^j]D^{\alpha,\beta}\Psi_m(t)=$$
$$\mathrm{u}^jD^{\alpha,\beta}\Psi_m(t)=\sum_{r=0}^{m}\sum_{k=0}^{m}u_r^jD_{rk}^{\alpha,\beta}p_k(t).$$

where

$$\mathrm{u}^j=[u_0^j,u_1^j,\cdots,u_m^j].$$

It is worth noting that the collocation points can be selected randomly or uniformly. Here, we choose m nodal points as $t_k = k \triangle t, k = 0, 1, \cdots, m$ and $\triangle t = \frac{1}{m}$. Then, we substitute points $t = t_z$ and $x = x_j$ at the expanded form of equation (9) as follows

$$\sum_{r=0}^{m}\sum_{k=0}^{m} u_r^j D_{rk}^{\alpha,\beta} p_k(t_z) + \left(\sum_{r=0}^{m} u_j^k p_r(t_z)\right)\left(\sum_{i=0}^{N}\sum_{r=0}^{m} \varphi_{ji}^{(1)} p_r(t_z) u_r^i\right)$$
$$-\gamma \sum_{i=0}^{N}\sum_{r=0}^{m} \varphi_{ji}^{(2)} p_r(t_z) u_r^i = f(x_j, t_m). \quad j = 1, \cdots, N-1, \quad z = 1, \cdots, m,$$

with initial and boundary conditions

$$\sum_{r=0}^{m} u_r^j p_r(t_0) = g(x_j), \quad j = 1, \cdots, N-1, z = 0,$$
$$u_0(t_z) = \rho(x_0, t_z), \qquad z = 0, \cdots, m,$$
$$u_N(t_z) = \rho(x_N, t_z), \quad z = 0, \cdots, m.$$

The final system only includes algebraic equations which is technically uncomplicated and can be solved using available mathematical software such as MATLAB.

6 Numerical Experiments

In this section, some comparative examples are provided to show the strength of the proposed method in approximating the solution of one-dimensional fractal fractional Burgers equations with a nonlocal boundary condition. The numerical results are performed in MATLAB 2016b on an Intel core i5 (6G RAM) Windows Win10 system.

Example 6.1. Consider the nonlinear time fractal-fractional Burgers equation (1) in $[0, 1]$, and $f(x, t)$ is given as

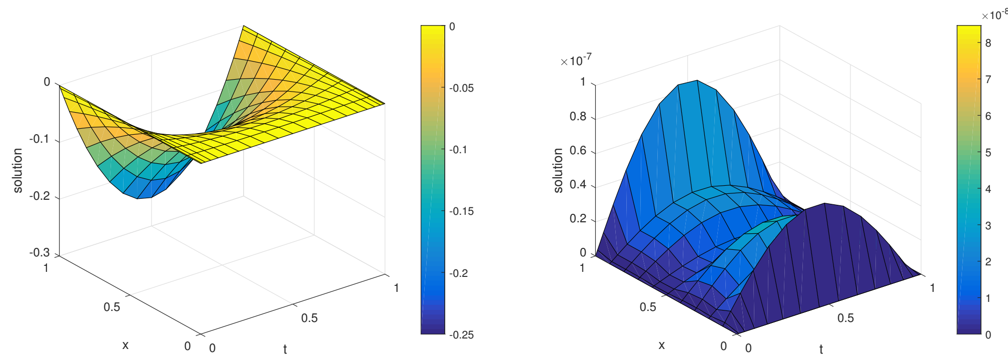
$$f = \frac{C(\alpha)2!t^{2-\beta+1}}{\beta(1-\alpha)} \mathbf{E}_{\alpha,3}(x^2 - x) + t^4(x^2 - x)(2x - 1) - 2\gamma t^2,$$

such that the exact solution is

$$u = t^2(x^2 - x).$$

The initial and boundary condition can be obtained using the exact solution.

The proposed method is applied to solve this example. Fig. 1 shows the numerical solutions obtained by the presented method and the absolute errors between exact and approximate solutions at $\beta = 0.25$ and $\alpha = 0.25$ respectively. Tables 1 and 2, display the absolute errors with respect to spatial points N when $m = 10$ for some values of α and β. Moreover, we reported the absolute error of the solution for some values of m, α and β in Table 3 and 4. The reported results illustrate that one can obtain excellent solution by applying a few number of barycentric basis and Legendre polynomials.



(a) (b)

Figure 1: The approximate solutions (a) and the absolute error (b) for $\alpha = 0.35, \beta = 0.25$ with $N = 15, m = 10$ and $\gamma = 1$, in Example 6.1.

Table 1: Absolute error with $\beta = 0.75$, $m = 10$, $\gamma = 1$ and various values of N for Example 6.1.

N	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
5	7.6333e-8	7.5973e-8	9.8691e-8	1.7226e-7	7.6549e-7
10	5.1388e-8	5.2932e-8	6.6592e-8	1.1272e-7	5.0343e-7
15	5.1027e-8	5.2560e-8	6.6126e-8	1.1193e-7	4.9991e-7
20	2.2227e-8	2.0725e-8	2.4510e-8	7.0863e-8	2.1523e-7

Table 2: Absolute error with $\alpha = 0.75$, $m = 10$, $\gamma = 1$ and various values of N for Example 6.1.

N	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
5	3.3341e-7	2.4689e-7	2.7782e-7	2.3804e-7	6.3024e-8
10	1.7455e-7	1.6826e-7	2.2806e-7	1.6362e-7	5.6285e-8
15	1.7307e-7	1.6700e-7	2.2641e-7	1.6246e-7	5.5889e-8
20	1.0857e-7	7.7842e-8	1.6780e-7	1.0129e-7	4.8236e-8

Table 3: Absolute error with $\beta = 0.75$, $N = 15$, $\gamma = 1$ and various values of m for Example 6.1.

m	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
2	2.6187e-4	1.6680e-4	1.5567e-5	2.9822e-4	6.4759e-4
4	7.8026e-6	6.7678e-6	6.1771e-6	2.2074e-6	6.2143e-5
6	1.0703e-6	9.9998e-7	1.1559e-6	1.5901e-6	5.2159e-6
8	2.5217e-7	2.4954e-7	3.1258e-7	5.2400e-7	1.4720e-6

Table 4: Absolute error with $\alpha = 0.75$, $N = 15$, $\gamma = 1$ and various values of m for Example 6.1.

m	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
2	0.0019	3.8498e-4	1.5988e-4	3.8638e-4	4.5771e-4
4	6.6572e-5	4.2616e-5	2.1335e-5	3.3767e-6	1.9089e-5
6	8.9298e-6	5.5234e-6	4.4299e-6	2.3373e-6	8.6532e-7
8	1.7483e-6	1.2081e-6	1.0407e-6	7.5148e-7	9.8263e-8

Example 6.2. In this example, we consider the fractal-fractional Burgers equation (1) in $[0,1]$, and $f(x,t)$ is given by

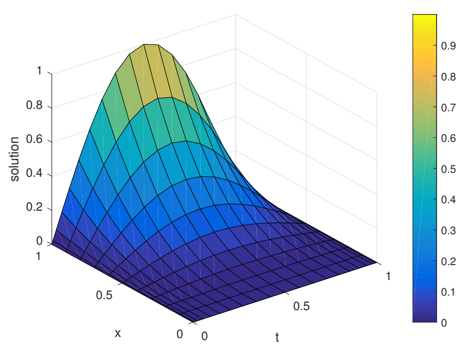
$$f = \frac{C(\alpha)3!t^{3-\beta+1}}{\beta(1-\alpha)}\mathbf{E}_{\alpha,4}\sin(\pi x) + \pi t^6 \sin(\pi x)\cos(\pi x) + \pi^2\gamma t^3 \sin(\pi x).$$

such that the exact solution is

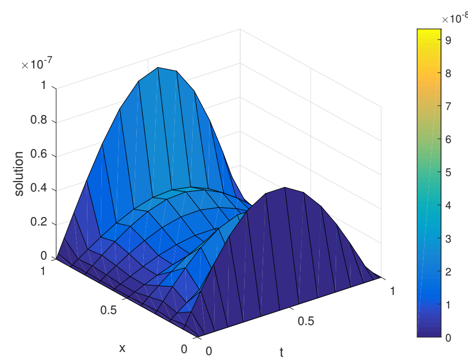
$$u = t^3 \sin(\pi x).$$

The initial and boundary conditions can be obtained according to exact solution.

The established method with some values of N and m is used to solve this example. Fig. 2 shows the numerical solution and absolute errors at $\alpha = 0.65, \beta = 0.5$ with $N = 15$, $m = 10$ and $\gamma = 1$. The absolute errors of the solutions for some values of α, β and N are summarized in Tables 5 and 6 with $m = 10$. Furthermore, Tables 7 and 8 reports the absolute values for $N = 15$ and some different values of m, α and β. Based on the reports obtained from this example, it is concluded that increasing the number of Legendre basis and barycentric basis, the approximated solutions tend to the exact solutions.



(a)



(b)

Figure 2: The approximate solution (a) and absolute error (b) $\alpha = 0.65, \beta = 0.50$ with $N = 15, m = 10$ in Example 6.2.

Table 5: Absolute error with $\beta = 0.5$, $m = 10$, $\gamma = 1$ and various values of N for Example 6.2.

N	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
5	1.0510e-7	1.0517e-6	1.0463e-6	1.0302e-6	9.7861e-7
10	1.5528e-8	1.6254e-8	2.2557e-8	4.2154e-8	2.2121e-7
15	1.5528e-8	1.6254e-8	2.2557e-8	4.2154e-8	1.6729e-7
20	1.1483e-8	7.1838e-9	1.2300e-8	1.8696e-8	1.6729e-7

Table 6: Absolute error with $\alpha = 0.5$, $m = 10$, $\gamma = 1$ and various values of N for Example 6.2.

N	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
5	9.8762e-7	1.0314e-6	1.0463e-6	1.0563e-6	1.0659e-6
10	2.7379e-8	2.7376e-8	2.2557e-8	1.4697e-8	4.1775e-9
15	1.4638e-8	2.7376e-8	2.2557e-8	1.4697e-8	4.1775e-9
20	1.4638e-8	1.6602e-8	1.2300e-8	6.5502e-9	2.9158e-9

Table 7: Absolute error with $\beta = 0.5$, $N = 15$, $\gamma = 1$ and various values of m for Example 6.2.

m	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
2	0.0133	0.0138	0.0174	0.0284	0.0821
4	3.8392e-4	3.0450e-4	2.1106e-4	1.7216e-5	2.8874e-4
6	2.4782e-6	2.5160e-6	3.1605e-6	4.1750e-6	6.1371e-6
8	1.3152e-7	1.3878e-7	1.8948e-7	3.4125e-7	6.4796e-7

Table 8: Absolute error with $\alpha = 0.5$, $N = 15$, $\gamma = 1$ and various values of m for Example 6.2.

m	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
2	0.0874	0.0345	0.0174	0.0089	0.0051
4	0.0022	6.6576e-4	2.1106e-4	3.1594e-5	4.0335e-5
6	8.1609e-6	5.2489e-6	3.1605e-6	1.3377e-6	1.8409e-7
8	3.1363e-7	2.4858e-7	1.8948e-7	1.0602e-7	5.5803e-9

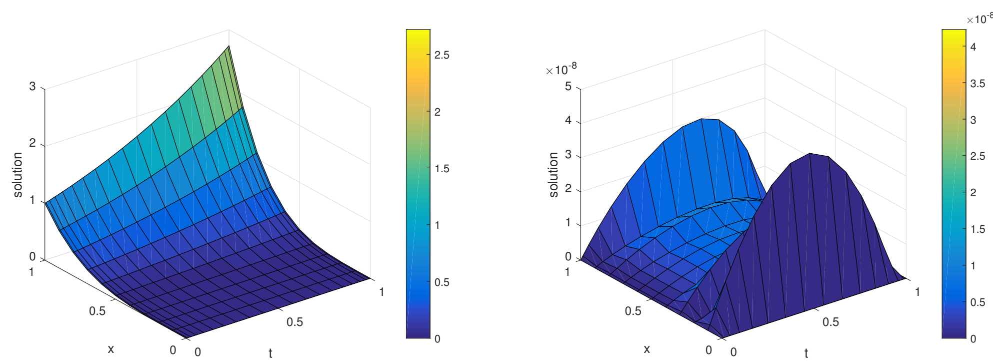
Example 6.3. As the last test problem, we consider problem (1) where

$$f = \frac{C(\alpha)4!t^{4-\beta+1}}{\beta(1-\alpha)}\mathbf{E}_{\alpha,5}\exp(x) + t^8\exp(2x) - \gamma t^4\exp(x).$$

such that the exact solution is given by

$$u = t^4\exp(x).$$

The proposed method is used to solve this example with some values of N and m. Fig. 3 displays the approximated solution and the absolute error in the case $N = 15, m = 0, \alpha = 0.85, \beta = 0.75$ and $\gamma = 1$. We summarized the absolute error of the solution for some values of N, α and β with $m = 10$ in tables 9 and 10, while Tables 11 and 12 provides the absolute errors for $N = 15$ and distinct values of m, β and α. The achieved results show that this method is efficient for this example with high order of accuracy.



(c) (d)

Figure 3: The approximate solution (a) and absolute error (b) for $N = 15, m = 10$ and $\alpha = 0.85, \beta = 0.75$, in Example 6.3.

Table 9: Absolute error with $\beta = 0.25$, $m = 10$, $\gamma = 1$ and various values of N for Example 6.3.

N	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
5	1.0344e-6	1.0374e-6	1.0273e-6	1.0031e-6	9.1667e-7
10	1.3795e-8	1.7002e-8	2.4839e-8	5.3223e-8	4.9209e-7
15	1.3795e-8	1.7002e-8	2.4839e-8	5.3223e-8	1.9364e-7
20	2.3281e-9	5.3124e-9	8.4444e-9	3.1851e-8	9.7347e-8

Table 10: Absolute error with $\alpha = 0.25$, $m = 10$, $\gamma = 1$ and various values of N for Example 6.3.

N	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
5	9.9852e-7	1.0395e-6	1.0519e-6	1.0597e-6	1.0668e-6
10	1.5927e-8	1.7693e-8	1.5980e-8	1.1397e-8	3.8757e-9
15	1.5927e-8	1.7693e-8	1.5980e-8	1.1397e-8	3.8757e-9
20	8.7100e-9	5.0482e-9	7.7647e-9	6.6976e-9	2.3472e-9

Table 11: Absolute error with $\beta = 0.25$, $N = 15$, $\gamma = 1$ and various values of m for Example 6.3.

m	$\alpha = 0.1$	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$
2	0.0347	0.0353	0.0419	0.0592	0.1303
4	0.0013	0.0011	8.7845e-4	5.1882e-4	0.1361
6	3.3109e-6	3.8797e-6	5.8401e-6	1.0090e-5	8.5796e-6
8	1.4346e-7	1.6983e-7	2.5989e-7	5.2499e-7	2.6318e-6

Table 12: Absolute error with $\alpha = 0.25$, $N = 15$, $\gamma = 1$ and various values of m for Example 6.3.

m	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
2	0.0770	0.0281	0.0134	0.0063	0.0021
4	0.0028	8.5547e-4	3.2420e-4	1.0793e-4	1.1475e-5
6	3.9808e-6	3.4365e-6	2.4554e-6	1.3449e-6	2.8104e-7
8	1.5170e-7	1.5714e-7	1.3377e-7	8.6969e-8	2.2879e-8

7 Conclusion

In this paper, we presented a technique based on the barycentric interpolation method and Legendre polynomials to obtain the numerical solution of non-linear time fractal-fractional Burgers equation. The barycentric interpolation method is accurate, fast, simple and easy to implement boundary conditions in order to prevent singularity, and also the main advantage of Legendre polynomials is that by using only a few Legendre basis functions, we achieve satisfactory results. The fractal-fractional derivative is defined in the Atangana-Riemann-Liouville sense with Mittag-Leffler kernel. The matrices obtained from these two methods converts the given problem to a system of nonlinear algebraic equations which can be solved numerically. Numerical experiments are given to show the efficiency and accuracy of the proposed technique.

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