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Existence of Solution for A Class of Fractional Problems with Sign-Changing Functions

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Abstract. In this paper, we study the existence and multiplicity of solutions for the following fractional problem

$$(-\Delta)_p^s u + a(x)|u|^{p-2}u = f(x,u),$$

with the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ where Ω is a bounded domain with smooth boundary, $p \geq 2$, $s \in (0,1)$ and $a(x)$ is a sign-changing function. Moreover, we consider two different assumptions on the function $f(x,u)$, including the cases of nonnegative and sign-changing function.

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1 Introduction and Preliminaries

The topics related to the existence and multiplicity of solutions for fractional elliptic problems have been investigated widely. Also, fractional problems naturally arise in many different branches of science such as optimization [30], conservation laws

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[16], water waves [26, 27], quantum mechanics [32], finance [25], minimal surfaces [20, 21], phase transitions [33, 53], virus transmission [43, 47] and other sciences (see also [1–4, 6–12, 14, 15, 19, 23, 24, 34, 36–42, 49–52, 54]).

In this paper, we investigate the existence and multiplicity of solutions for the following fractional problem

$$\begin{cases} (-\Delta)_p^s u + a(x)|u|^{p-2}u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded subset of $\mathbb{R}^n$, $n > ps$ with $s \in (0,1)$, $p \geq 2$, $a(x) \in L^\infty(\Omega)$ is a sign-changing function and

$$(-\Delta)_p^s u(x) := 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(y)-u(x)|^{p-2}\big(u(y)-u(x)\big)}{|x-y|^{n+ps}} dy.$$

In addition, one of the following assumptions is satisfied:

(f1) $f(x,u) = b(x)|u|^{q-2}u$, where $b(x) \in L^\infty(\Omega)$, $b(x) \geq 0$ a.e. in Ω and $p < q < p_s^*$ where $p_s^* = \frac{np}{n-ps}$ is the fractional Sobolev exponent,

(f2) $f(x,u) = b(x)|u|^{q-2}u + \lambda g(x,u) - h(x)|u|^{r-2}u$, where $\lambda > 0$, $2 \leq r \leq p < q < p_s^*$, $b(x) \in L^\infty(\Omega)$ which may change sign and $h(x) \in C(\bar{\Omega})$ is a nonnegative function.

Recently a great deal of attention has been focused on the study of existence and multiplicity of solutions for fractional differential equations. In particular, in the case of $f(x,u) = \lambda b(x)u^q$, the problem (1) has been studied by some authors and the existence of multiple positive solutions has been established. For instance, Brown and Wu in [18] considered the following problem

$$-\Delta_p u = \lambda a(x)|u|^q + b(x)u^p, \qquad x \in \Omega,$$

with Dirichlet boundary condition, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$, $q < 1 < p < \frac{n+2}{n-2}$ and $a, b : \Omega \to \mathbb{R}$ are smooth functions which may change sign on Ω. They proved the existence of at least two positive solutions by using the Nehari manifold and fibering maps. Also, Barrios *et al.* in [13] obtained the existence and multiplicity of solutions for the following fractional differential equation

$$(-\Delta)^s u = \lambda u^q + u^{2_s^*-1}, \quad x \in \Omega,$$

with Dirichlet boundary condition, where $\Omega \subset \mathbb{R}^n$ is a regular bounded domain, $\lambda > 0$, $0 < s < 1$, $n > 2s$ and $(-\Delta)^s$ denotes the fractional Laplace operator defined by

$$-(-\Delta)^s u(x) = 2 \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \qquad x \in \mathbb{R}^n.$$

Moreover, Ning *et al.* in [46] proved the existence, multiplicity and bifurcation results for the following problem with Dirichlet boundary condition

$$(-\Delta)^s u = \lambda |u|^{q-2}u + \frac{|u|^{p_{s,\alpha}^*-2}u}{|x|^\alpha}, \quad x \in \Omega,$$

where $p \in (1, \infty)$, $0 < s < 1$, Ω is a bounded domain containing the origin in $\mathbb{R}^n$ with Lipschitz boundary, $0 \le \alpha < ps < n$ and $p^*_{s,\alpha} = \frac{(n-\alpha)p}{n-ps}$ is the fractional Hardy-Sobolev exponent. As well as in [48] Saiedinezhad, by using the Nehari manifold with variational arguments, the existence of solutions for the following semilinear elliptic equation was studied

$$\Delta^2 u - a(x)\Delta u = b(x)|u|^{p-2}u,$$

with Navier boundary condition $\Delta u = u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $2 < p < 2^* = \frac{2n}{n-2}$. In this paper, the author considered two different assumptions on the potentials $a(x)$ and $b(x)$ including the case of sign-changing weights. Recently in [45] Nhan and Truong studied a class of logarithmic fractional Schrdinger equations with possibly vanishing potentials as follows

$$-\Delta_p^s u + V(x)|u|^{p-2}u = \lambda K(x)|u|^{p-2}u + \mu|u|^{q-2}u\log|u|, \qquad x \in \mathbb{R}^n,$$

where $\lambda, \mu > 0$, $0 < s < 1$ and $n > 2s$. They obtained the existence of at least one nontrivial solution by using the fibrering maps and the Nehari manifold.

In this paper, motivated by the above achievements and due to the widespread use of fractional differential equations we will use variational methods to study of existence and multiplicity of solutions for problem (1) and for this purpose, we consider the fractional Sobolev space $W_0^{s,p}(\Omega)$ with the norm

$$\|u\| := \|u\|_{W_0^{s,p}(\Omega)} = \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}}. \tag{2}$$

For the convenience of the reader we repeat the relevant material from [22] without proofs. Assume

$$X = \left\{u | u : \mathbb{R}^n \to \mathbb{R},\ u|_\Omega \in L^p(\Omega),\ \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy < \infty\right\},$$

with the norm $\|u\|_X = \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}}$, where $Q := \mathbb{R}^{2n}\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega = \mathbb{R}^n\backslash\Omega$.

Also set X_0 denotes the closure of $C_0^\infty(\Omega)$ in X. By the results in [29] the space X_0 is a Hilbert space with the scalar product defined for any $u, v \in X_0$ as

$$\langle u, v\rangle = \int_Q \frac{|u(x)-u(y)|^{p-1}(v(x)-v(y))}{|x-y|^{n+ps}}dxdy,$$

and the norm

$$\|u\|_{X_0} := \left(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}dxdy\right)^{\frac{1}{p}},$$

which is equivalent to the equation defined in (2). Moreover, based on the results found in [29,35], it can be said that the embedding $X_0 \hookrightarrow L^q(\Omega)$ is continous for any $q \in [1, p^*]$ and compact whenever $q \in [1, p^*)$. Thus, there exists a positive constant

S_p such that $\|u\|_{L^p(\Omega} \leq S_p \times \|u\|_{X_0}$. Indeed, the sharp constant S_p is equal to $\frac{1}{\mu_p}$, where

$$\mu_p := \inf\{\frac{\|u\|_{X_0}}{\|u\|_{L^p(\Omega)}} : 0 \neq u \in X_0\}. \tag{3}$$

This paper is organized into 4 sections. Section 2 is devoted to some notations and preliminaries which, will be used in the sequel. In Section 3 we consider and solve problem (1) by assuming condition (f1) . Finally in Section 4, we prove the existence and multiplicity of positive solutions of problem (1) under condition (f2).

2 Main Results

The main results of this paper are in two parts. In the third section through presupposing condition (f1) we did our best to resolve problem (1). Therefore, according to the basic variational arguments, we know that the weak solutions of (1) under assumption (f1), is corresponding to the local minimizer of

$$I(u) = \frac{1}{p}M(u) + \frac{1}{p}A(u) - \frac{1}{q}B(u), \tag{4}$$

where $I : X_0 \to \mathbb{R}$ is the associated Euler-Lagrange functional, $M(u) := \|u\|^p_{X_0}$, $A(u) := \int_\Omega a(x)|u|^p dx$ and $B(u) := \int_\Omega b(x)|u|^q dx$.

Also in Section 4, where condition (f2) is satisfied, we have the following problem

$$\begin{cases} (-\Delta)^s_p u + a(x)|u|^{p-2}u = b(x)|u|^{q-2}u + \lambda g(x,u) - h(x)|u|^{r-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

The Euler-Lagrange functional associated with this problem is $\tilde{I} : X_0 \to \mathbb{R}$ such that

$$\tilde{I}(u) = \frac{1}{p}\big(M(u) + A(u)\big) - \frac{1}{q}B(u) - \lambda \int_\Omega G(x,|u|)dx + \frac{1}{r}H(u), \tag{6}$$

where $G(x,u) := \int_0^u g(x,s)ds$ and $H(u) := \int_\Omega h(x)|u|^r dx$. Also we know that if $\tilde{I}(u)$ denotes the energy functional corresponding to a problem, then all of the critical points of $\tilde{I}(u)$ must lie in the set $N := \{u; \langle \tilde{I}'(u), u\rangle = 0\}$, which is known as the Nehari manifold (see [44,55]). On the other hand, the fibering map

$$\varphi_u(t) := \tilde{I}(tu) = \frac{t^p}{p}\big(M(u) + A(u)\big) - \frac{t^q}{q}B(u) - \lambda \int_\Omega G(x,t|u|)dx + \frac{t^r}{r}H(u), \tag{7}$$

is closely linked to the Nehari manifold, i.e., $\varphi'_u(1) = 0$ if and only if $u \in N$, (see [17, 28]). So it is reasonable to divide the Nehari manifold into three parts corresponding to local minima, local maxima and inflction points of the critical points of $\varphi'_u(t)$, and hence we define $N^+ := \{u \in N, \varphi''_u(1) > 0\}$, $N^- := \{u \in N, \varphi''_u(1) < 0\}$ and $N^0 := \{u \in N, \varphi''_u(1) = 0\}$.

Besides, we set

$$a^+ := \operatorname{ess\,sup}\{a(x), x \in \Omega\},$$

and

$$\tilde{A}(u) := M(u) + A(u) = \|u\|_{X_0}^p + \int_\Omega a(x)|u|^p dx.$$

In the sequel we need the following lemma:

Lemma 2.1. *If $a^+ < \mu_p$, then there exists $\delta_1 > 0$ such that for every $u \in X_0$, $\tilde{A}(u) \geq \delta_1 \|u\|_{X_0}^p$.*

Proof. If $\int_\Omega a(x)|u|^p dx \geq 0$, then the lemma is obvious; otherwise, it would be proved by contradiction, supposing that $\int_\Omega a(x)|u|^p dx < 0$. If for every $\delta > 0$, there exists $u \in X_0$ such that $\tilde{A}(u) < \delta \|u\|_{X_0}^p$, it can be deduced that

$$(1-\delta)\|u\|_{X_0}^p < \int_\Omega -a(x)|u|^p dx < a^+ \int_\Omega |u|^p dx.$$

Now, by taking $\delta < 1 - \frac{a^+}{\mu_p}$ we get $\frac{a^+}{1-\delta} < \mu_p$ which leads to a contradiction with (3).□

Remark 2.2. According to lemma 2.1 we know that if $a^+ < \mu_p$ then $\tilde{A}(u) \geq \delta_1 \|u\|_{X_0}^p$ for $\delta_1 > 0$; on the other hand $\tilde{A}(u) \leq \|u\|_{X_0}^p + \|a\|_\infty S_p^p \|u\|_{X_0}^p$. Therefore, in the sequel for $a^+ < \mu_p$, we consider X_0 with the following norm:

$$\|u\|_{\tilde{A}} := \left(\tilde{A}(u)\right)^{\frac{1}{p}} = \left(\|u\|_{X_0}^p + \int_\Omega a(x)|u|^p dx\right)^{\frac{1}{p}}. \tag{8}$$

The main results in this paper are the following theorems.

Theorem 2.3. *Suppose that $f(x,u)$ satisfies condition* (f1) *and $a^+ < \mu_p$, then problem* (1) *admits at least one weak solution in X_0.*

Theorem 2.4. *Assume $a^+ < \mu_p$, then:*
(i). If $f(x,u)$ satisfies condition (f2)*, then there exists λ^* such that for $0 < \lambda < \lambda^*$, $\tilde{I}$ admits a minimizer on N^+ which is a nontrivial weak solution of problem (5).*
(ii). If $f(x,u)$ satisfies condition (f2)*, then there exists λ^{**} such that for $0 < \lambda < \lambda^{**}$, there exists a minimizer of $\tilde{I}$ on N^- which is a nontrivial weak solution of problem (5).*

3 Proof of Theorem 2.3

In this section, we consider problem (1) such that $f(x,u)$ satisfies condition (f1), using (4) for every $u \neq 0$, $I(tu) \hookrightarrow -\infty$ as $t \hookrightarrow \infty$. Thus, $I(u)$ is not bounded below and so the minimizing process on the hole space X_0 may not be possible. If for every $\alpha \in \mathbb{R}$, we let

$$B_\alpha := \{u \in X_0 : \int_\Omega b(x)|u|^q dx = \alpha\},$$

then, for every $u \in B_\alpha$, by using Remark 2.2 we have $I(u) = \frac{1}{p}\|u\|_{\tilde{A}}^p - \frac{1}{q}\alpha$. Thus, $I|_{B_\alpha}$ is certainly bounded below and the minimizing approach of $I(u)$ on B_α is equivalent to the minimizing approach of $\|u\|^p_{\tilde{A}}$ on B_α. Set $\inf_{u\in B_\alpha}\|u\|^p_{\tilde{A}} =: m_\alpha$, we will show that m_α is obtained by a function, and a multiple of this function is a minimizer of $I(u)$ and as a result, weak solution of (1).

Lemma 3.1. *For every $\alpha > 0$, there exists a nonnegative function $u_\alpha \in B_\alpha$ such that $\|u\|^p_{\tilde{A}} = m_\alpha$.*

Proof. By the coercivity of $I(u)$ on B_α, there exists a bounded minimizer sequence $\{u_n^{(\alpha)}\}$ for $\chi(u) := \|u\|^p_{\tilde{A}}$ on B_α. Therefore, $\{|u_n^{(\alpha)}|\}$ is a minimizer sequence in B_α, so we can suppose that $u_n^{(\alpha)}(x) \geq 0$ a.e. on Ω. By the reflexivity of X_0, there exists a subsequence of $\{u_n^{(\alpha)}\}$, for simplicity is denoted by $\{u_n^{(\alpha)}\}$, which is weakly convergent to $u_\alpha \in X_0$ ($u_n^{(\alpha)} \rightharpoonup u_\alpha$). So by compact embedding $X_0 \hookrightarrow L^q(\Omega)$, $\{u_n^{(\alpha)}\}$ is strongly convergent in $L^q(\Omega)$, and hence

$$lim_{n\to\infty}\int_\Omega b(x)|u_n^{(\alpha)}|^q dx = \int_\Omega b(x)|u_\alpha|^q dx,$$

which means $u_\alpha \in B_\alpha$. If $u_n^{(\alpha)} \not\to u_\alpha$ in X_0, then $\|u_\alpha\|^p_{\tilde{A}} < \liminf \|u_n\|^p_{\tilde{A}} = m_\alpha$ which is a contradiction with $u_\alpha \in B_\alpha$. So $u_n \to u_\alpha$ in X_0 and $m_\alpha = \inf_{u\in S_\alpha}\|u\|^p_{\tilde{A}} = \|u_\alpha\|^p_{\tilde{A}}$. $\square$

Proof of Theorem 2.3. Let $\chi(u) = \|u\|^p_{\tilde{A}}$, now if u_α is a minimizer of $\chi(u)$ under the condition $B(u) = \alpha$, then by Lagrange multiplier theorem, there exists $\lambda \in \mathbb{R}$ such that $\chi'(u_\alpha) = \lambda B'(u_\alpha)$, and hence for every $v \in X_0$ we have

$$\langle \chi'(u_\alpha), v\rangle = q\lambda \int_\Omega b(x)|u_\alpha|^{q-2}u_\alpha v dx.$$

By taking $u_\alpha = Cw_\alpha$ we get

$$C^{p-1}\int_Q \frac{|w_\alpha(x) - w_\alpha(y)|^{p-2}(w_\alpha(x) - w_\alpha(y))(v(x) - v(y))}{|x-y|^{n+ps}}dxdy$$
$$+ C^{p-1}\int_\Omega a(x)|w_\alpha|^{p-2}w_\alpha v dx = \frac{q\lambda}{p}C^{q-1}\int_\Omega b(x)|w_\alpha|^{q-2}w_\alpha v dx.$$

Now, by assuming $C = (\frac{p}{q\lambda})^{\frac{1}{q-p}}$ we have

$$\int_Q \frac{|w_\alpha(x) - w_\alpha(y)|^{p-2}(w_\alpha(x) - w_\alpha(y))(v(x) - v(y))}{|x-y|^{n+ps}}dxdy$$
$$+ \int_\Omega a(x)|w_\alpha|^{p-2}w_\alpha v dx = \int_\Omega b(x)|w_\alpha|^{q-2}w_\alpha v dx,$$

consequently, w_α is a weak solution of (1) under assumption (f1). $\square$

Lemma 3.2. *For $\alpha \neq \beta$ the minimizers of $\chi(u)$ on B_α and B_β give the same weak solution of (1).*

Proof. For $\alpha \neq \beta$, we have

$$B_\alpha = \{u \in X_0 : \int_\Omega b(x)|u|^q dx = \alpha\} = \{(\frac{\alpha}{\beta})^{1/q} v : v \in X_0, \int_\Omega b(x)|v|^q dx = \beta\}.$$

Therefore,

$$m_\alpha = \inf_{u \in B_\alpha} \|u\|_{\tilde{A}}^p = (\frac{\alpha}{\beta})^{p/q} m_\beta.$$

So u_α minimizes $\|u\|_{\tilde{A}}^p$ on B_α if and only if $(\frac{\beta}{\alpha})^{1/q} u_\alpha$ minimizes $\|u\|_{\tilde{A}}^p$ on B_β. Indeed, by substituting $C_\alpha = (\frac{\alpha}{m_\alpha})^{\frac{1}{q-p}}$ we have

$$w_\alpha = \frac{1}{C_\alpha} u_\alpha = (\frac{m_\alpha}{\alpha})^{\frac{1}{q-p}} (\frac{\alpha}{\beta})^{1/q} u_\beta = (\frac{m_\beta}{\beta})^{\frac{1}{q-p}} u_\beta = \frac{u_\beta}{C_\beta} = w_\beta. \square$$

4 Proof of Theorem 2.4

In this section, where condition (f2) is satisfied, we study the existence and multiplicity results for problem (5). One of the main difficulties in this problem will be the nonlinearity of $g(x,u)$. To overcome this difficulty we need to restrict $g(x,u)$ to the following conditions:

(g1) $g(x,u) \in C^1(\Omega \times \mathbb{R})$ such that $g(x,0) \geq 0$, $g(x,0) \not\equiv 0$ and there exists $\bar{g_1}(x) \in L^\infty(\Omega)$ such that, $|g_u(x,u)| \leq \bar{g_1}(x) u^{p-2}$ where $(x,u) \in \Omega \times \mathbb{R}^+$.

(g2) For $u \in L^p(\Omega)$, $\int_\Omega g_u(x,t|u|)u^2 dx$ has the same sign for every $t \in (0,\infty)$.

A typical example of $g(x,u)$ is given by $g(x,u) = \sqrt[4]{(1+u^2)^p}$, for other examples, please refer to [5].

Remark 4.1. If $g(x,u)$ satisfies (g1), then there exists $\bar{g_2}(x) \in L^p(\Omega)$ such that $g(x,u) \leq \bar{g_2}(x)(1+u^{p-1})$ and $G(x,u) \leq 2\bar{g_2}(x)(1+u^p)$, for all $(x,u) \in \Omega \times \mathbb{R}^+$. Moreover, based on the compactness of the embedding $X_0 \hookrightarrow \mathrm{L}^r(\Omega)$ for $1 \leq r < p^*$ and the fact that the operator $u \longmapsto g(x,u)$ is continuous, we conclude that the functional $J(u) = \int_\Omega G(x,u)dx$ is weakly continuous, i.e., if $u_n \rightharpoonup u$, then $J(u_n) \to J(u)$ and moreover the operator $J'(u) = \int_\Omega g(x,u)u dx$ is weak to strong continuous, i.e., if $u_n \rightharpoonup u$, then $J'(u_n) \to J'(u)$.

The following lemma shows that minimizers for $\tilde{I}$ on N are usually critical points for $\tilde{I}$, as proved by Brown and Zhang in [17]

Lemma 4.2. *Let u_0 be a local minimizer for $\tilde{I}(u)$ on N such that $u_0 \notin N^0$, then u_0 is a critical point of $\tilde{I}(u)$.*

Motivated by the above lemma, we will get conditions for $N^0 = \emptyset$

Lemma 4.3. *If $a^+ < \mu_p$ then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, we have $N^0 = \emptyset$.*

Proof. Suppose the other way round, then for $u \in N^0$, using (g1), (7) and the relation of $\varphi_u''(1) = 0$, we have $(p-1)\|u\|_{\tilde{A}}^p \leq (q-1)\tilde{S}_q^q\|b\|_\infty\|u\|_{\tilde{A}}^q + \|\bar{g_1}\|_\infty \tilde{S}_p^p \lambda \|u\|_{\tilde{A}}^p$, $\big($where $\tilde{S}_r$ denotes the best Sobolev constant for the embedding of X_0 with the norm $\|u\|_{\tilde{A}}$ into $\mathrm{L}^r(\Omega)\big)$ and hence

$$\|u\|_{\tilde{A}} \geq \Big(\frac{p-1-\|\bar{g_1}\|_\infty \tilde{S}_p^p\lambda}{(q-1)\tilde{S}_q^q\|b\|_\infty}\Big)^{\frac{1}{q-p}}. \tag{9}$$

On the other hand, from (7), (g1), Remark 4.1 and using the fact that $(q-1)\varphi_u'(1) - \varphi_u''(1) = 0$, we obtain

$$\begin{aligned}(q-p)\|u\|_{\tilde{A}}^p &\leq \lambda\big(\int_\Omega (q-1)g(x,|u|)|u| - g_u(x,|u|)u^2\big)dx + (r-q)H(u)\\ &\leq 2\lambda(q-1)\|\bar{g_2}\|_\infty|\Omega| + \lambda(2(q-1)\|\bar{g_2}\|_\infty + \|\bar{g_1}\|_\infty)\tilde{S}_p^p\|u\|_{\tilde{A}}^p,\end{aligned}$$

which concludes

$$\|u\|_{\tilde{A}} \leq \big(\frac{2(q-1)\lambda\|\bar{g_2}\|_\infty|\Omega|}{q-p-\lambda(2(q-1)\|\bar{g_2}\|_\infty + \|\bar{g_1}\|_\infty)\tilde{S}_p^p}\big)^{\frac{1}{p}}.$$

Therefore, using (9) we must have

$$\big(\frac{p-1-\|\bar{g_1}\|_\infty S\tilde{S}_p^p\lambda}{(q-1)\tilde{S}_q^q\|b\|_\infty}\big)^{\frac{1}{q-p}} \leq \big(\frac{2(q-1)\lambda\|\bar{g_2}\|_\infty|\Omega|}{q-p-\lambda(2(q-1)\|\bar{g_2}\|_\infty + \|\bar{g_1}\|_\infty)\tilde{S}_p^p}\big)^{\frac{1}{p}},$$

which is a contradiction for λ sufficiently small. So there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0, \;\; N^0 = \emptyset$. □

Lemma 4.4. *If $a^+ < \mu_p$ then there exists $\lambda_1 > 0$ such that for $\lambda < \lambda_1$, $\tilde{I}(u)$ is coercive and bounded below on N.*

Proof. For $u \in N$, using (6) and remark 4.1 we have

$$\begin{aligned}\tilde{I}(u) &= (\frac{1}{p} - \frac{1}{q})\|u\|_{\tilde{A}}^p - \lambda\int_\Omega (G(x,|u|) - \frac{1}{q}g(x,|u|)|u|)dx + (\frac{1}{r} - \frac{1}{q})H(u)\\ &\geq (\frac{1}{p} - \frac{1}{q})\|u\|_{\tilde{A}}^p - 2\lambda\|\bar{g_2}\|_\infty(1 + \frac{1}{q})|\Omega| - 2\lambda\|\bar{g_2}\|_\infty\tilde{S}_p^p(1+\frac{1}{q})\|u\|_{\tilde{A}}^p.\end{aligned}$$

As a result, $\tilde{I}$ is coercive and bounded below on N for $0 < \lambda < \lambda_1 = \frac{q-p}{2p(q+1)\|\bar{g_2}\|_\infty\tilde{S}_p^p}$.
□

Lemma 4.5. *If $a^+ < \mu_p$ then there exists $\lambda_2 > 0$ such that, for $0 < \lambda < \lambda_2$, $\varphi_u(t)$ takes on positive values for all non-zero $u \in X_0$.*

Proof. If $B(u) \leq 0$, then using (7) and by elementary calculus we can show that $\varphi_u(t) > 0$ for sufficiently large t. Suppose there exists $u \in X_0$ such that $B(u) > 0$, through using (7), (8) and by elementary calculus, $\psi_u(t) := \frac{t^p}{p}\|u\|^p_{\tilde{A}} - \frac{t^q}{q}B(u)$ takes on a maximum at $t_{max} = \big(\frac{\|u\|^p}{B(u)}\big)^{\frac{1}{q-p}}$ and so

$$\psi_u(t_{max}) = (\frac{1}{p} - \frac{1}{q})\Big\{\frac{(\|u\|^p_{\tilde{A}})^q}{(B(u))^p}\Big\}^{\frac{1}{q-p}} \geq (\frac{1}{p} - \frac{1}{q})\Big\{\frac{1}{|b^+\|^p_\infty \tilde{S}^{pq}_q}\Big\}^{\frac{1}{q-p}} := \delta_2 > 0, \tag{10}$$

where δ_2 is independent of u. Moreover, for $1 \leq \alpha < p^*$ we have

$$\begin{aligned}(t_{max})^\alpha \int_\Omega |u|^\alpha dx &\leq \tilde{S}^\alpha_\alpha \Big(\frac{\|u\|^p_{\tilde{A}}}{B(u)}\Big)^{\frac{\alpha}{q-p}} (\|u\|^p_{\tilde{A}})^{\frac{\alpha}{p}} = \tilde{S}^\alpha_\alpha \Big\{\frac{\|u\|^{pq}_{\tilde{A}}}{(B(u))^p}\Big\}^{\frac{\alpha}{p(q-p)}} \\ &= \tilde{S}^\alpha_\alpha (\frac{qp)}{q-p})^{\frac{\alpha}{p}} (\psi_u(t_{max}))^{\frac{\alpha}{p}} = c(\psi_u(t_{max}))^{\frac{\alpha}{p}},\end{aligned}$$

hence using remark 4.1, we conclude that

$$\begin{aligned}\lambda \int_\Omega G(x, t_{max}|u|)dx - \frac{1}{r}H(t_{max}|u|) &\leq 2\lambda\|\bar{g}_2\|_\infty \int_\Omega \big(1 + |t_{max}u|^p\big)dx \\ &\leq 2\lambda\|\bar{g}_2\|_\infty|\Omega| + c_1\lambda\|\bar{g}_2\|_\infty \psi_u(t_{max}),\end{aligned}$$

where c_1 is independent of u. So from (10) we get

$$\begin{aligned}\varphi_u(t_{max}) &= \psi_u(t_{max}) - \lambda \int_\Omega G(x, t_{\max}|u|)dx + \frac{1}{r}H(t_{\max}|u|) \\ &\geq \psi_u(t_{max})\Big(1 - \lambda 2\|\bar{g}_2\|_\infty|\Omega|\big(\psi_u(t_{max})\big)^{-1} - \lambda c_1\|\bar{g}_2\|_\infty\Big) \\ &\geq \delta_2\Big(1 - \delta_2^{-1}\lambda 2\|\bar{g}_2\|_\infty|\Omega| - \lambda c_1\|\bar{g}_2\|_\infty\Big).\end{aligned}$$

Clearly $\varphi_u(t_{max}) > \epsilon > 0$, for all nonzero u, provided that

$$\lambda < \lambda_2 := \frac{1}{2\|\bar{g}_2\|_\infty|\Omega|\delta_2^{-1} + c_1\|\bar{g}_2\|_\infty}.\square$$

Since $\varphi_u(0) \leq 0$, so it is clear that if $a^+ < \mu_p$ and $0 < \lambda < \lambda_2$ then there exists $0 < \tau < t_{max}$ such that $\varphi'_u(\tau) > 0$, and so we have the following corollary:

Corollary 4.6. *(i). If $a^+ < \mu_p$, $0 < \lambda < \lambda_1$ and $B(u) \leq 0$ for $u \in X_0 \setminus \{0\}$, then there exists t_1 such that $t_1 u \in N^+$ and $\tilde{I}(t_1 u) < 0$.*
(ii). If $a^+ < \mu_p$, $0 < \lambda < \min\{\lambda_1, \lambda_2\}$ and $B(u) > 0$ for $u \in X_0 \setminus \{0\}$, then there exist $t_1 < t_2$ such that $t_1 u \in N^+$, $t_2 u \in N^-$ and $\tilde{I}(t_1 u) < 0$.

Proof.(i). From the (7) and (g1), for a fixed u, we know $\varphi'_u(0) < 0$ and using lemma 4.4, $\lim_{t\to\infty} \varphi'_u(t) = +\infty$, so by the intermediate value theorem, there exists $t_1 > 0$ such that $\varphi'_u(t_1) = 0$. Now since $\varphi'_u(t) < 0$ for $0 < t < t_1$ and $\varphi'_u(t) > 0$ for $t_1 < t$, thus $t_1 u \in N^+$ and $\tilde{I}(t_1 u) < \tilde{I}(0) = 0$. $\square$

(ii). As in the proof of (i), we obtain $\varphi'_u(0) < 0$, $\lim_{t\to\infty} \varphi'_u(t) = -\infty$ and by using lemma 4.5 we get $\varphi'_u(\tau) > 0$ for a suitable τ, so the intermediate value theorem concludes that there exist t_1, t_2 such that $0 < t_1 < \tau < t_2$, $\varphi'_u(t_1) = \varphi'_u(t_2) = 0$, $t_1 u \in N^+$, $t_2 u \in N^-$ and $\tilde{I}(t_1 u) < \tilde{I}(0) = 0$. $\square$

Lemma 4.7. *There exists $\lambda_3 > 0$ such that if $0 < \lambda < \lambda_3$, then $B(u) > 0$ provided that $u \in N^-$.*

Proof. Suppose otherwise, that is, $-(q-1)B(u) \geq 0$ and by (7)

$$\varphi_u''(1) = (p-1)\|u\|_{\tilde{A}}^p - (q-1)B(u) - \lambda \int_\Omega g_u(x,|u|)u^2 dx + (r-1)H(u) < 0,$$

so using (g1), $(p-1)\|u\|_{\tilde{A}}^p < \lambda \|\bar{g_2}\|_\infty \tilde{S}_p^p \|u\|_{\tilde{A}}^p$, which is a contradiction for $\lambda < \lambda_3 := \frac{p-1}{\|\bar{g_2}\|_\infty S_p^p}$. □

Proof of Theorem 2.4(i). Assume $\lambda^* := \min\{\lambda_0, \lambda_1, \lambda_2\}$, as in lemma 4.4, $\tilde{I}$ is bounded below on N and so on N^+. Let $\{u_n\}$ be a minimizing sequence for $\tilde{I}$ on N^+, i.e., $\lim_{n\to\infty} \tilde{I}(u_n) = \inf_{u\in N^+} \tilde{I}(u) = c$, and by Ekeland's variational principle [31] we may assume $\langle \tilde{I}'(u_n), u_n\rangle \to 0$.

On the other hand, similar to lemma 4.4, $\tilde{I}(u_n) - \frac{1}{q}\langle \tilde{I}'(u_n), u_n\rangle \geq C\|u_n\|_{\tilde{A}} - K$, so $\{u_n\}$ is bounded in X_0 and without loss of generality, we may assume that $u_n \rightharpoonup u_1$ in X_0 and $u_n \to u_1$ in $L^r(\Omega)$ for $1 \leq r < p^*$ and $u_n(x) \to u_1(x)$, a.e.

By corollary 4.6 for $u_1 \in X_0 \setminus \{0\}$ there exists t_1 such that $t_1 u_1 \in N^+$ and so $\varphi'_{u_1}(t_1) = 0$. Now we show that $u_n \to u_1$ in X_0. Suppose this is false, then $\|u_1\|_{\tilde{A}}^p < \liminf_{n\to\infty} \|u_n\|_{\tilde{A}}^p$. So from (6) and remark 4.1, $\varphi'_{u_n}(t_1) > \varphi'_{u_1}(t_1) = 0$, for sufficiently large n. Since $\{u_n\} \subseteq N^+$, by considering possible maps it is easy to see that $\varphi'_{u_n}(t) < 0$ for $0 < t < 1$ and $\varphi'_{u_n}(1) = 0$ for all n. Hence we must have $t_1 > 1$, but $t_1 u_1 \in N_\lambda^+$ and so $\tilde{I}(t_1 u_1) < \tilde{I}(u_1) < \lim_{n\to\infty} \tilde{I}(u_n) = \inf_{u\in N^+} \tilde{I}(u_n)$, which is a contradiction. Therefore $u_n \to u_1$ in X_0 and so $\tilde{I}(u_1) = \lim_{n\to\infty} \tilde{I}(u_n) = \inf_{u\in N^+} \tilde{I}(u)$. Thus u_1 is a minimizer for $\tilde{I}$ on N^+ and by using lemmas 4.2 and 4.3, u_1 is a nontrivial weak solution of (5). □

Proof of Theorem 2.4(ii). Let $\lambda^{**} := \min\{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, then by lemma 4.5 for all $u \in N^-$ we have $\tilde{I}(u) \geq \tilde{I}(t_{\max} u) > \epsilon > 0$ i.e., $\inf_{u\in N^-} \tilde{I}(u) \geq 0$. Hence there exists a minimizing sequence $\{u_n\} \subseteq N^-$ such that $\lim_{n\to\infty} \tilde{I}(u_n) = \inf_{u\in N^-} \tilde{I}(u) \geq 0$. Now similarly as in the proof of the pervious theorem we find that, $\{u_n\}$ is bounded in X_0, $u_n \rightharpoonup u_2$ in X_0 and $u_n \to u_2$ in $L^r(\Omega)$, $1 < r < p^*$. Since $u_n \in N^-$ so by lemma 4.7, $B(u_n) > 0$ and $B(u_2) \geq 0$. We claim that $B(u_2) > 0$. Suppose this is false, thus $(p-1)\|u_2\|_{\tilde{A}}^p < \lambda \|\bar{g_2}\|_\infty \tilde{S}_p^p \|u_2\|_{\tilde{A}}^p$, which gives a contradiction for $\lambda < \lambda_3$. So by corollary 4.6 there exists $t_2 > 0$ such that $t_2 u_2 \in N^-$. We claim that $u_n \to u_2$ in X_0; if it is supposed that this is false, so $\|u_2\|_{\tilde{A}}^p < \liminf_{n\to\infty} \|u_n\|_{\tilde{A}}^p$. But $u_n \in N^-$ and so $\tilde{I}(u_n) \geq \tilde{I}(tu_n)$ for all $t \geq 0$. Therefore, using remark 4.1 we get

$$\begin{aligned}
\tilde{I}(t_2 u_2) &= \frac{1}{p} t_2^p \|u_2\|_{\tilde{A}}^p - \frac{1}{q} t_2^q B(u_2) - \lambda \int_\Omega G(x, t_2|u_2|)dx + \frac{t_2^r}{r} H(u_2) \\
&< \lim_{n\to\infty} \big(\frac{1}{p} t_2^p \|u_n\|_{\tilde{A}}^p - \frac{1}{q} t_2^q B(u_n) - \lambda \int_\Omega G(x, t_2|u_n|)dx + \frac{t_2^r}{r} H(u_n) \big) \\
&= \lim_{n\to\infty} \tilde{I}(t_2 u_n) \leq \lim_{n\to\infty} \tilde{I}(u_n) = \inf_{u\in N^-} \tilde{I}(u),
\end{aligned}$$

which is a contradiction, hence by lemmas 4.2 and 4.3, u_2 is a nontrivial weak solution of (5) which belongs to N^-. □

Conclusion. This paper has two impotant Theorems; in Section 3, we establish the existence of a solution for problem(1) by using Lagrange multiplier theorem. Also in Section 4, by using the Nehari manifold and the fibering maps, we prove the existence of two distinct weak solutions for problem (5).

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