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Original Research Paper

Orthogonal b-Metric Spaces and Best Proximity Points

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Abstract. The aim of this research is to define $\perp$-proximally increasing mapping and obtain several best proximity point results concerning this mapping in the framework of new spaces, which is called orthogonal b-metric spaces. Also, several well-known fixed point results in such spaces are established. All main results and new definitions are supported by some illustrative and interesting examples.

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1 Introduction

Fixed point theory is a significant tool for some various sections of analysis such as nonlinear functional analysis, convex analysis, and mathematical analysis. Also, it has many applications in mathematics and other sciences. After the first results of Banach in 1922, many authors

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studied on this theory in various spaces (for more details, see Rhoades [14], Rahimi and Soleimani Rad [12] and Kirk and Shahzad [10]). One of these spaces is a b-metric space defined by Bakhtin [3] (also, see [2, 5]).

Definition 1.1. [3, 5] Let $X \neq \emptyset$ and $s \geq 1$ be a real number. Assume that a mapping $d : X \times X \to [0, \infty)$ for every $k, l, p \in X$ satisfies in the following relations:

(d_1) $d(k, l) = 0$ iff $k = l$;

(d_2) $d(k, l) = d(l, k)$;

(d_3) $d(l, p) \leq s[d(l, k) + d(k, p)]$.

Then d is named a b-metric on X and (X, d) is named a b-metric space.

Notice that a b-metric is a metric with $s = 1$. But, the inverse always is not valid. Hence, b-metric spaces is an expansion of metric spaces. Also, for notions in this space such as convergence, completeness, continuity, etc, we refer to Agarwal et al. [1].

In 2008, Eshaghi et al. [8] defined the idea of orthogonal sets and orthogonal metric spaces. Very recently, many authors extended orthogonal metric spaces and discussed on fixed points for several various contractive mappings on such spaces in [4, 7, 13] and references contained therein. In the sequel, we consider some definitions and notations about these concepts.

Definition 1.2. [8] Assume $X \neq \emptyset$ and consider a binary relation $\perp$ on $X \times X$ as follows:

$$\text{there exists a } a_0 \in X \text{ such that for all } \ b \in X: \ (b \perp a_0) \ \text{ or } \ (a_0 \perp b).$$

Then X is named an orthogonal set (or same O-set). Also, a_0 is named an orthogonal element.

Example 1.3. [6] Let $X = [2, \infty)$ and consider $e \perp f$ if $e \leq f$ for every $e, f \in X$. Then, by considering $a_0 = 2$, $(X, \perp)$ is an O-set.

Definition 1.4. [8] Consider an O-set $(X, \perp)$ with a sequence $\{a_n\}$ therein. Then $\{a_n\}$ is named orthogonal sequence (or same O-sequence) whenever

$$\text{for all } \ n \in \mathbb{N} : (a_n \perp a_{n+1}) \quad \text{or} \quad (a_{n+1} \perp a_n).$$

Analogously, a Cauchy sequence $\{a_n\}$ is named a Cauchy O-sequence whenever

$$\text{for all } \ n \in \mathbb{N} : (a_n \perp a_{n+1}) \quad \text{or} \quad (a_{n+1} \perp a_n).$$

Now, if we consider a b-metric instead a metric, then we can rewrite the following definitions that were previously introduced by other researchers. Consider $(X, \perp)$ and a b-metric d on X with a real number $s \geq 1$. The triple $(X, \perp, d)$ is named an orthogonal b-metric space.

Definition 1.5. [8] The triple $(X, \perp, d)$ is named a complete O-b-metric space (O-complete) whenever each Cauchy O-sequence converges in X.

Note that each complete b-metric space is O-complete. However, the converse is not value.

Definition 1.6. [8] Assume that $(X, \perp, d)$ is an orthogonal b-metric space and T is a self-mapping on X. T is named an orthogonal preserving ($\perp$-preserving) whenever $g \perp l$ implies $T(g) \perp T(l)$ for all $g, l \in X$.

Definition 1.7. [13] Assume that $(X, \perp, d)$ is an orthogonal b-metric space and T is a self-mapping on X. Then T is named orthogonal continuous ($\perp$-continuous) in $a \in X$ if for all O-sequences $\{a_n\}$ in X so that $a_n \longrightarrow a$ then $T(a_n) \longrightarrow T(a)$. Also, T is $\perp$-continuous on X if T is $\perp$-continuous for all $a \in X$.

Notice that a $\perp$-continuous mapping is not necessary continuous. Thus, continuity is stronger than O-continuity in b-metric spaces.

Now, we review some basic definitions regarding best proximity point theory. Assume that U, V are two non-empty subsets of a metric space (X, d) and define $d(U, V) = \inf\{d(u, v); u \in U, \ v \in V\}$. The point $u \in U$ is the best proximity point (briefly, bpp) of a non-self mapping $F : U \to V$ if $d(u, Fu) = d(U, V)$. Some bpp results in various metric spaces can be found in [9, 11, 15] and references therein.

For two arbitrary subset $U, V \neq \emptyset$, consider

$$
\begin{aligned}
U_0 &= \{u \in U : d(u,v) = d(U,V) \text{ for some } v \in V\},\\
V_0 &= \{v \in V : d(u,v) = d(U,V) \text{ for some } u \in U\}.
\end{aligned}
$$

Definition 1.8. [16] Assume that U, V are two non-empty subsets of (X, d) and $U_0 \neq \emptyset$. Then (U, V) has P-property iff

$$
\begin{cases} d(u_1, v_1) = d(U, V) \\ d(u_2, v_2) = d(U, V) \end{cases} \quad \text{implies} \quad d(u_1, u_2) = d(v_1, v_2),
$$

where $u_1, u_2 \in U_0$ and $v_1, v_2 \in V_0$.

Obviously, the pair (U, U) has P-property.

Definition 1.9. [15] Assume that $(X, \leq)$ is a partially ordered set and (U, V) is a pair of non-empty subsets of X. A mapping $F : U \to V$ is named proximally increasing if

$$
\begin{cases} v_1 \leq v_2 \\ d(u_1, Fv_1) = d(U, V) \\ d(u_2, Fv_2) = d(U, V) \end{cases} \quad \text{implies} \quad u_1 \leq u_2,
$$

where $u_1, u_2, v_1, v_2 \in U$.

In this section, we explained a record of both fixed point theory and bpp theory. Also, we considered some preliminaries which is needed in the sequel. In the next section, we define a new concept of proximity, which is named $\perp$-proximally increasing, and obtain some bpp theorems. Ultimately, we establish several fixed point results in orthogonal b-metric spaces in section 3.

2 Bpp Results

First we define the following concept.

Definition 2.1. Assume that $(X, \perp, d)$ is an orthogonal b-metric space. A mapping $F : U \longrightarrow V$ is named $\perp$-proximally increasing if

$$\begin{cases} v_1 \perp v_2 \\ d(u_1, Fv_1) = d(U, V) \\ d(u_2, Fv_2) = d(U, V) \end{cases} \quad \text{implies} \quad u_1 \perp u_2,$$

where $u_1, u_2, v_1, v_2 \in U$.

Here, we express and prove the main theorem of this section.

Theorem 2.2. *Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space. Also, suppose that (U, V) is a pair of non-empty closed subsets of X and $U_0 \neq \emptyset$. Moreover, let $T : U \longrightarrow V$ be a mapping so that*

i) T is a $\perp$-preserving and $\perp$-proximally increasing mapping provided that $T(U_0) \subseteq V_0$, and (U, V) has P-property;

ii) There exist orthogonal elements $a_0, a_1 \in U_0$ provided that

$$d(a_1, Ta_0) = d(U, V);$$

iii) T is an O-continuous mapping on U provided that

$$d(Ta, Tb) \leq \alpha Q(a, b) \tag{1}$$

with

$$Q(a, b) = \max\{d(a, b), d(a, Ta) - sd(U, V), d(b, Tb) - sd(U, V)\}$$

for all point $a, b \in U$ so that $a \perp b$ and $\alpha \in [0, \frac{1}{s^2})$.

Then T has a bpp $a' \in U$ so that $d(a', Ta') = d(U, V)$. Moreover, if for two bpp(s) $x, y \in U$ we have $x \perp y$, then T has a unique bpp in U.

Proof. By (ii), there are $a_0, a_1 \in U_0$ provided that $d(a_1, Ta_0) = d(U, V)$ and $a_0 \perp a_1$. Since $Ta_1 \in T(U_0) \subseteq V_0$, there is element a_2 in U_0 provided that $d(a_2, Ta_1) = d(U, V)$. Since T is a $\perp$-proximally increasing mapping, we obtain $a_1 \perp a_2$. Continuing this process, we attain a sequence $\{a_n\}$ in U_0 so that

$$\begin{aligned} &d(a_{n+1}, Ta_n) = d(U, V) \qquad for\ all\ n \in \mathbb{N}, \\ &with \quad a_0 \perp a_1,\ a_1 \perp a_2,\ a_2 \perp a_3,\ \dots,\ a_n \perp a_{n+1},\ \dots. \end{aligned} \tag{2}$$

Thus, $\{a_n\}$ is an O-sequence. Since (U,V) has P-property, we gain

$$\begin{cases} d(a_{n+1},Ta_n)=d(U,V) \\ d(a_n,Ta_{n-1})=d(U,V) \end{cases} \quad \text{implies} \quad d(a_{n+1},a_n)=d(Ta_n,Ta_{n-1})$$

for each $n\in\mathbb{N}$. Now, we demonstrate $\{a_n\}$ is a Cauchy O-sequence. Using (1), we get

$$d(a_{n+1},a_n)=d(Ta_n,Ta_{n-1})\leq \alpha Q(a_n,a_{n-1}),$$

where

$$\begin{aligned} Q(a_n,a_{n-1})=\max\{&d(a_n,a_{n-1}),d(a_n,Ta_n)-sd(U,V),\\ &d(a_{n-1},Ta_{n-1})-sd(U,V)\}. \end{aligned}$$

Now,
- if $Q(a_n,a_{n-1})=d(a_n,a_{n-1})$, then $Q(a_n,a_{n-1})\leq sd(a_n,a_{n-1})$;
- if $Q(a_n,a_{n-1})=d(a_n,Ta_n)-sd(U,V)$, then

$$\begin{aligned} Q(a_n,a_{n-1})&=d(a_n,Ta_n)-sd(U,V)\\ &=d(a_n,Ta_n)-sd(a_{n+1},Ta_n)\\ &\leq s[d(a_n,a_{n+1})+d(a_{n+1},Ta_n)]-sd(a_{n+1},Ta_n)\\ &=sd(a_n,a_{n+1}), \end{aligned}$$

which is a contradiction;
- if $Q(a_n,a_{n-1})=d(a_{n-1},Ta_{n-1})-sd(U,V)$, then

$$\begin{aligned} Q(a_n,a_{n-1})&=d(a_{n-1},Ta_{n-1})-sd(U,V)\\ &=d(a_{n-1},Ta_{n-1})-sd(a_n,Ta_{n-1})\\ &\leq s[d(a_{n-1},a_n)+d(a_n,Ta_{n-1})]-sd(a_n,Ta_{n-1})\\ &=sd(a_{n-1},a_n). \end{aligned}$$

Therefore, $Q(a_n,a_{n-1})\leq sd(a_n,a_{n-1})$ for all $n\in\mathbb{N}$. Hence, we have

$$\begin{aligned} d(a_{n+1},a_n)=d(Ta_n,Ta_{n-1})&\leq \alpha Q(a_n,a_{n-1})\leq \alpha sd(a_n,a_{n-1})\\ &\leq\ldots\leq(\alpha s)^n d(a_1,a_0). \end{aligned}$$

Now, set $\beta = \alpha s$. Since $\alpha < \dfrac{1}{s^2}$ and $s \geq 1$, then $\beta s < 1$. Now, for every $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
d(a_n, a_m) &\leq s[d(a_n, a_{n+1}) + d(a_{n+1}, a_m)] \\
&\leq sd(a_n, a_{n+1}) + s[sd(a_{n+1}, a_{n+2}) + sd(a_{n+2}, a_m)] \\
&\ \ \vdots \\
&\leq sd(a_n, a_{n+1}) + s^2 d(a_{n+1}, a_{n+2}) + \ldots + s^{m-n-1} d(a_{m-1}, a_m) \\
&\leq [s\beta^n + s^2\beta^{n+1} + \ldots + s^{m-n-1}\beta^{m-1}] d(a_1, a_0) \\
&\leq (\frac{s\beta^n}{1-\beta s}) d(a_1, a_0) \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty. \qquad (3)
\end{aligned}$$

So, (3) implies that $\{a_n\}$ is a Cauchy O-sequence in U. But, $(X, \perp, d)$ is O-complete and U is a closed subset of X. Thus, the Cauchy O-sequence $\{a_n\}$ converges to some $a' \in U$. Since T is an O-continuous mapping on U, we get $Ta_n \longrightarrow Ta'$. Since the mapping d is continuous, we get $d(a_{n+1}, Ta_n) \longrightarrow d(a', Ta')$. On the other hand, by (2), $d(a_{n+1}, Ta_n)$ is a constant sequence converges to $d(U, V)$. Consequently, $d(a', Ta') = d(U, V)$, i.e., $a' \in U$ is a bpp for T.

To demonstrate the uniqueness, assume that a'' is another bpp of the mapping T so that $a' \perp a''$. Since (U, V) has P-property, we get

$$\begin{cases} d(a', Ta') = d(U, V) \\ d(a'', Ta'') = d(U, V) \end{cases} \qquad \text{implies } d(a', a'') = d(Ta', Ta'')$$

for $a', a'' \in U_0$ and $Ta', Ta'' \in V_0$. Hence, by (1) and $0 \leq \alpha < \frac{1}{s^2}$ with $s \geq 1$, we obtain

$$\begin{aligned}
d(a', a'') &= d(Ta', Ta'') \\
&\leq \alpha \max\{d(a', a''), d(a', Ta') - sd(U, V), \\
&\qquad d(a'', Ta'') - sd(U, V)\} = \alpha d(a', a''),
\end{aligned}$$

which implies that $d(a', a'') = 0$; that is $a' = a''$, and the proof ends. □

Theorem 2.3. *Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space. Also, suppose that (U, V) is a pair of non-empty closed subsets of X and $U_0 \neq \emptyset$. Moreover, let $T : U \longrightarrow V$ be a mapping so that* *i) T is a $\perp$-preserving and $\perp$-proximally increasing mapping provided that $T(U_0) \subseteq V_0$, and (U, V) has P-property;*

ii) There exist orthogonal elements $a_0, a_1 \in U_0$ provided that

$$d(a_1, Ta_0) = d(U, V);$$

iii) T is an O-continuous mapping on U provided that

$$d(Ta, Tb) \leq \alpha d(a, b) + \beta d(a, Ta) + \gamma d(b, Tb) - (\beta + \gamma) s d(U, V) \tag{4}$$

for all $a, b \in U$ so that $a \perp b$ and $\alpha, \beta, \gamma \geq 0$ with $\alpha s + \beta s + \gamma s^2 < 1$.

Then T has a bpp $a' \in U$ so that $d(a', Ta') = d(U, V)$.

Proof. As in the proof of previous theorem, we have an O-sequence $\{a_n\}$ in U_0 so that $d(a_{n+1}, Ta_n) = d(U, V)$ for all $n \in \mathbb{N}$, where $a_0 \perp a_1$, $a_1 \perp a_2$, $a_2 \perp a_3$, …, $a_n \perp a_{n+1}$, …. Since (U, V) has P-property, we gain

$$\begin{cases} d(a_{n+1}, Ta_n) = d(U, V) \\ d(a_n, Ta_{n-1}) = d(U, V) \end{cases} \quad \text{implies} \quad d(a_{n+1}, a_n) = d(Ta_n, Ta_{n-1})$$

for each $n \in \mathbb{N}$. First we demonstrate that $\{a_n\}$ is a Cauchy O-sequence. Now, by (4), we get

$$\begin{aligned} d(a_2, a_1) = d(Ta_1, Ta_0) &\leq \alpha d(a_1, a_0) + \beta d(a_1, Ta_1) + \gamma d(a_0, Ta_0) \\ &\quad - (\beta + \gamma) s d(U, V) \\ &\leq \alpha d(a_1, a_0) + \beta s d(a_1, Ta_0) + \beta s d(Ta_0, Ta_1) \\ &\quad + \gamma s d(a_0, a_1) + \gamma s d(a_1, Ta_0) \\ &\quad - (\beta + \gamma) s \underbrace{d(a_1, Ta_0)}_{d(U,V)}. \end{aligned} \tag{5}$$

Thus, (5) implies that $d(a_2, a_1) \leq (\frac{\alpha + \gamma s}{1 - \beta s}) d(a_1, a_0)$. By continuing this procedure, we get $d(a_{n+1}, a_n) \leq \lambda^n d(a_1, a_0)$ for each $n \in \mathbb{N}$, where $\lambda = \frac{\alpha + \gamma s}{1 - \beta s} < \dfrac{1}{s}$ (since $\alpha s + \beta s + \gamma s^2 < 1$). Now, for every $m, n \in \mathbb{N}$ with $m > n$, we gain

$$d(a_n, a_m) \leq (\frac{s\lambda^n}{1 - s\lambda}) d(a_1, a_0) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

The continue of the proof is like the Theorem 2.2. $\square$

Corollary 2.4. *Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space, and (U, V) is a pair of non-empty closed subsets of X with $U_0 \neq \emptyset$. Moreover, suppose that $T : U \longrightarrow V$ is a $\perp$-preserving and $\perp$-proximally increasing mapping so that $T(U_0) \subseteq V_0$, and (U, V) has P-property. Also, assume that there exist orthogonal elements $a_0, a_1 \in U_0$ so that $d(a_1, Ta_0) = d(U, V)$ and T is an O-continuous function on U satisfying in one of the following cases:*

(I) for each $a, b \in U$ so that $a \perp b$ and $\beta \in [0, \frac{1}{s^2+s})$, we have

$$d(Ta, Tb) \leq \beta[d(a, Ta) + d(b, Tb) - 2sd(U, V)];$$

(II) for every $a, b \in U$ so that $a \perp b$ and $\alpha, \beta \geq 0$ with $\alpha s + (s^2+s)\beta < 1$, we have

$$d(Ta, Tb) \leq \alpha d(a, b) + \beta[d(a, Ta) + d(b, Tb) - 2sd(U, V)]$$

Then T has a bpp $a' \in U$ so that $d(a', Ta') = d(U, V)$.

Proof. We can prove this corollary by considering $\alpha = 0$ and $\beta = \gamma$ (for the relation (I)), and $\beta = \gamma$ (for the relation (II)) in Theorem 2.3. $\square$

Example 2.5. Consider $X = \mathbb{R}^2$ and define $d : X \times X \to [0, +\infty)$ by

$$d\big((a_1, b_1), (a_2, b_2)\big) = (a_1 - a_2)^2 + (b_1 - b_2)^2$$

for all $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$. Then (X, d) is a b-metric space, where $s = 2$. Also, Let $U = \big\{(a, 1) : \ a \in [0, 1]\big\}$ and $V = \big\{(b, 0) : b \in [0, 1]\big\}$. Clearly, $d(U, V) = 1$, $U = U_0$ and $V = V_0$. In particular, U_0 is non-empty.

Consider a binary relation $\perp$ on X by $(a_1, b_1) \perp (a_2, b_2)$ if $|a_1 - a_2| \geq \frac{5}{6}$ or $b_1 b_2 = 0$. Then $(a_1, 0)$ is same element a_0 in Definition 1.2. Also, we define the mapping $T : U \to V$ by

$$T(a, 1) = \begin{cases} (0, 0) & 0 \leq a < 1, \\ (\frac{1}{3}, 0) & a = 1 \end{cases} \qquad \big(a \in [0, 1]\big).$$

Then T is a $\perp$-preserving. Also, assume $(a_1,1),(a_2,1)\in U$, $(a_1,1)\perp(a_2,1)$ and $\alpha=\dfrac{4}{25}$. Then we have

• if $a_1,a_2\in[0,1)$, then

$$\begin{aligned}
d\big(T(a_1,1),T(a_2,1)\big) &= d\big((0,0),(0,0)\big)=0\\
&\le \frac{4}{25}(a_1-a_2)^2\\
&= \frac{4}{25}\max\Big\{(a_1-a_2)^2,\underbrace{a_1^2+1-2}_{\le 0},\underbrace{a_2^2+1-2}_{\le 0}\Big\}\\
&= \frac{4}{25}\max\Big\{d\big((a_1,1),(a_2,1)\big),\\
&\quad d\big((a_1,1),T(a_1,1)\big)-2d(U,V),\\
&\quad d\big((a_2,1),T(a_2,1)\big)-2d(U,V)\Big\},\\
&= \frac{4}{25}Q\big((a_1,1),(a_2,1)\big),
\end{aligned}$$

• if $a_1\vee a_2=1$ (let $a_2=1$), then $|a_1-a_2|\ge\frac{5}{6}$ and

$$\begin{aligned}
d\big(T(a_1,1),T(1,1)\big) &= d\big((0,0),(\frac{1}{3},0)\big)=\frac{1}{9}\\
&\le \frac{4}{25}(a_1-1)^2\\
&= \frac{4}{25}\max\Big\{(a_1-1)^2,\underbrace{a_1^2+1-2}_{\le 0},\underbrace{\frac{4}{9}+1-2}_{\le 0}\Big\}\\
&= \frac{4}{25}\max\Big\{d\big((a_1,1),(1,1)\big),d\big((a_1,1),(0,0)\big)-2d(U,V),\\
&\quad d\big((1,1),(\frac{1}{3},0)\big)-2d(U,V)\Big\}\\
&= \frac{4}{25}\max\Big\{d\big((a_1,1),(1,1)\big),\\
&\quad d\big((a_1,1),T(a_1,1)\big)-2d(U,V),\\
&\quad d\big((1,1),T(1,1)\big)-2d(U,V)\Big\}\\
&= \frac{4}{25}Q\big((a_1,1),(1,1)\big).
\end{aligned}$$

Thus, T satisfies the relation (1). Moreover, all hypotheses of Theorem 2.2 with $\alpha = \frac{4}{25}$ is satisfied. Hence, T has a bpp $a' = (0,1)$.

Now, suppose $a'' = (a,1) \in U$ with $a \in (0,1]$ is another bpp of T. If $a \in (0,1)$, then

$$d\big((a,1), T(a,1)\big) = d\big((a,1),(0,0)\big) = a^2 + 1 > d(U,V).$$

Otherwise, let $a = 1$. Then

$$d\big((1,1), T(1,1)\big) = d\big((1,1),(\frac{1}{3},0)\big) = \frac{13}{9} > d(U,V),$$

which is a contradiction. Thus, $(0,1)$ is the unique bpp of T.

3 Fixed Point Results

Here, we express several fixed point results in an orthogonal complete b-metric space.

Theorem 3.1. *(Rus-type contraction) Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ is a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha s + \beta(1+s) + \gamma(s^2+s) < 1$ provided that*

$$\begin{aligned} d(Ta,Tb) \leq{}& \alpha d(a,b) + \beta[d(a,Ta) + d(b,Tb)] \\ &+ \gamma[d(a,Tb) + d(b,Ta)] \end{aligned} \tag{6}$$

for every $a, b \in X$, where $a \perp b$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for each $a \in X$.

Proof. Suppose a_0 is an orthogonal element in X so that

$$\text{for all } \ b \in X : \ (a_0 \perp b) \quad \text{or} \quad (b \perp a_0).$$

Consider a sequence $\{a_n\}$ by $a_n = T(a_{n-1}) = T^n a_0$. By using the property $\perp$-preserving of T, $\{a_n\}$ is an O-sequence, i.e.,

$$\text{for all } \ n \in \mathbb{N} : \ (a_n \perp a_{n+1}) \quad \text{or} \quad (a_{n+1} \perp a_n).$$

Now, set $a = a_{n-1}$ and $b = a_n$ in (6). Then, for any $n \in \mathbb{N}$, we get

$$\begin{aligned} d(a_n, a_{n+1}) = d(Ta_{n-1}, Ta_n) &\leq \alpha d(a_{n-1}, a_n) \\ &\quad + \beta[d(a_{n-1}, Ta_{n-1}) + d(a_n, Ta_n)] \\ &\quad + \gamma[d(a_{n-1}, Ta_n) + d(a_n, Ta_{n-1})] \\ &= \alpha d(a_{n-1}, a_n) \\ &\quad + \beta[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\ &\quad + \gamma[d(a_{n-1}, a_{n+1}) + d(a_n, a_n)] \\ &\leq \alpha d(a_{n-1}, a_n) \\ &\quad + \beta[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\ &\quad + \gamma s[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] \\ &\leq (\alpha + \beta + \gamma s) d(a_{n-1}, a_n) \\ &\quad + (\beta + \gamma s) d(a_n, a_{n+1})]. \end{aligned} \tag{7}$$

Now, (7) implies that $d(a_n, a_{n+1}) \leq \lambda d(a_{n-1}, a_n)$ for each $n \in \mathbb{N}$, where $\lambda = \frac{\alpha+\beta+\gamma s}{1-\beta-\gamma s} < \dfrac{1}{s}$. By continuing this process, we get $d(a_n, a_{n+1}) \leq \lambda^n d(a_0, a_1)$ for all $n \in \mathbb{N}$. Now, let $m, n \in \mathbb{N}$ with $m > n$. Then, by the same proof in Theorem 2.2, we get

$$d(a_n, a_m) \leq (\frac{s\lambda^n}{1 - \lambda s}) d(a_0, a_1) \longrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty,$$

which implies that $\{a_n\}$ is a Cauchy O-sequence in orthogonal O-complete b-metric space X. Thus, $\{a_n\}$ converges to element $a' \in X$. Now, since T is O-continuous and $Ta_n \longrightarrow Ta'$, we have

$$\begin{aligned} d(a', Ta') &= \lim_{n \longrightarrow \infty} d(a_{n+1}, Ta') \\ &= \lim_{n \longrightarrow \infty} d(Ta_n, Ta') = d(Ta', Ta') = 0. \end{aligned}$$

Thus, a' is a fixed point for T. Now, we demonstrate a' is unique. Assume b' is another fixed point of T. Then, we get

$$(a_0 \perp a' \;\text{ and }\; a_0 \perp b') \quad \text{or} \quad (a' \perp a_0 \;\text{ and }\; b' \perp a_0).$$

Since T is a $\perp$-preserving mapping, we obtain

$$(T^n a_0 \perp a' \;\text{ and }\; T^n a_0 \perp b') \quad \text{or} \quad (a' \perp T^n a_0 \;\text{ and }\; b' \perp T^n a_0)$$

for any $n \in \mathbb{N}$. Using (6), we obtain

$$d(a_n, a') = d(T^n a_0, T^n a') \leq \lambda^n d(a_0, a'),$$
$$d(a_n, b') = d(T^n a_0, T^n b') \leq \lambda^n d(a_0, b').$$

Now, $d(a', b') \leq sd(a', a_n) + sd(a_n, b')$ implies that $a' = b'$; i.e., T has a unique fixed point. □

Corollary 3.2. *(Chatterjea-type contraction) Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ is a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exists $\gamma \geq 0$ with $\gamma \in [0, \frac{1}{s^2+s})$ so that*

$$d(Ta, Tb) \leq \gamma[d(a, Tb) + d(b, Ta)]$$

for every $a, b \in X$, where $a \perp b$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for any point $a \in X$.

Proof. Set $\alpha = \beta = 0$ in (6) and apply Theorem 3.1. □

Corollary 3.3. *(Kannan-type contraction) Assume that $(X, \perp, d)$ is an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ is a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exists $\beta \geq 0$ with $\beta \in [0, \frac{1}{1+s})$ so that*

$$d(Ta, Tb) \leq \beta[d(a, Ta) + d(b, Tb)]$$

for each $a, b \in X$, where $a \perp b$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for each $a \in X$.

Proof. Set $\alpha = \gamma = 0$ in (6) and consider Theorem 3.1. □

Corollary 3.4. *(Reich-type contraction) Let $(X, \perp, d)$ be an orthogonal complete b-metric space. Also, suppose that $T : X \longrightarrow X$ be a $\perp$-preserving and O-continuous mapping. Moreover, assume that there exist $\alpha, \beta, \gamma \geq 0$ so that*

$$d(Ta, Tb) \leq \alpha d(a, b) + \beta d(a, Ta) + \gamma d(b, Tb)$$

for every $a, b \in X$ with $a \perp b$, where $\alpha s + \beta s + \gamma < 1$. Then T has a unique fixed point $a' \in X$ and $T^n a \longrightarrow a'$ for every $x \in X$.

Proof. Apply Theorem 3.1. $\square$

Example 3.5. Set $X = [0, 12]$ and define $d : X \times X \to [0, \infty)$ by

$$d(a, b) = |a - b|^2$$

for each $a, b \in X$. Consider the binary relation $\perp$ on X by $a \perp b$ if $ab \leq (a \vee b)$, where $a \vee b = a$ *or* b. Then $(X, d, \perp)$ is an O-complete b-metric space with $s = 2$. Consider the mapping $T : X \to X$ by

$$Ta = \begin{cases} \frac{a}{3} & 0 \leq a \leq 3, \\ 0 & 3 < a \leq 12 \end{cases} \qquad \big(a \in [0, 12]\big).$$

Let $a \perp b$ and $\alpha = \frac{1}{16}$, $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{24}$ in (6). Without loss of generality, we may consider $ab \leq b$. Now, we have

- if $a = 0$ and $0 \leq b \leq 3$, then $Ta = 0$ and $Tb = \frac{b}{3}$, and

$$\begin{aligned} d(Ta, Tb) &= \frac{b^2}{9} \\ &\leq \frac{1}{16} b^2 + \frac{1}{4} \cdot \frac{4b^2}{9} + \frac{1}{24} \cdot \frac{10b^2}{9} \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big), \end{aligned}$$

- if $a = 0$ and $3 \leq b \leq 12$, then $Ta = Tb = 0$, and

$$\begin{aligned} d(Ta, Tb) &= 0 \\ &\leq \frac{1}{16} b^2 + \frac{1}{4} b^2 + \frac{1}{24} b^2 \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big), \end{aligned}$$

- if $0 \leq b \leq 1$ and $0 \leq a \leq 3$, then $Ta = \frac{a}{3}$ and $Tb = \frac{b}{3}$, and

$$\begin{aligned} d(Ta, Tb) &= \frac{1}{9} |a - b|^2 \\ &\leq \frac{1}{9} (a^2 + b^2) \\ &= \frac{1}{4} \cdot \left(\frac{4a^2}{9} + \frac{4b^2}{9}\right) \\ &\leq \frac{1}{16} |a - b|^2 + \frac{1}{4} \cdot \left(\frac{4a^2}{9} + \frac{4b^2}{9}\right) + \frac{1}{24} \cdot \left(|a - \frac{b}{3}|^2 + |b - \frac{a}{3}|^2\right) \\ &= \alpha d(a, b) + \beta\big(d(a, Ta) + d(b, Tb)\big) + \gamma\big(d(a, Tb) + d(b, Ta)\big), \end{aligned}$$

• if $0 \leq b \leq 1$ and $3 < a \leq 12$, then $Ta = 0$ and $Tb = \frac{b}{3}$, and so

$$\begin{aligned} d(Ta, Tb) &= \frac{b^2}{9} \\ &\leq \frac{1}{16}|a-b|^2 + \frac{1}{4}\cdot(a^2 + \frac{4b^2}{9}) + \frac{1}{24}\cdot(|a - \frac{b}{3}|^2 + b^2) \\ &= \alpha d(a,b) + \beta\big(d(a,Ta) + d(b,Tb)\big) + \gamma\big(d(a,Tb) + d(b,Ta)\big). \end{aligned}$$

Thus, the relation (6) is valid. Clearly, T is $\perp$-continuous with $\alpha = \frac{1}{16}$, $\beta = \frac{1}{4}$ and $\gamma = \frac{1}{24}$. So, all hypotheses of Theorem 3.1 are held. Consequently, T has a unique fixed point $a = 0$ in X.

Remark 3.6. In Theorem 3.1 and its corollaries, set $s = 1$. Then we obtain same results in an orthogonal complete metric space.

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