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On z-Ideals and z°-Ideals of Power Series Rings

A. Rezaei Aliabad*
Chamran University

R. Mohamadian
Chamran University

Abstract. Let R be a commutative ring with identity and $R[[x]]$ be the ring of formal power series with coefficients in R. In this article we consider sufficient conditions in order that $P[[x]]$ is a minimal prime ideal of $R[[x]]$ for every minimal prime ideal P of R and also every minimal prime ideal of $R[[x]]$ has the form $P[[x]]$ for some minimal prime ideal P of R. We show that a reduced ring R is a Noetherian ring if and only if every ideal of $R[[x]]$ is nicely-contractible (we call an ideal $\mathcal{I}$ of $R[[x]]$ a nicely-contractible ideal if $(\mathcal{I} \cap R)[[x]] \subseteq \mathcal{I}$). We will trivially see that an ideal $\mathcal{I}$ of $R[[x]]$ is a z-ideal if and only if we have $\mathcal{I} = (I, x)$ in which I is a z-ideal of R and also we show that whenever every minimal prime ideal of $R[[x]]$ is nicely-contractible, then $I[[x]]$ is a z°-ideal of $R[[x]]$ if and only if I is an $\aleph_0$-z°-ideal.

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1. Introduction

Throughout the paper R denotes a commutative ring with identity and $R[[x]]$ denotes the ring of formal power series over R. Whenever $f = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$, then we usually use f_n instead of a_n. Supposing $S \subseteq R[[x]]$, we denote by $C(S)$ the set of all coefficients of elements of S

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*Corresponding author

and use $C(f)$ instead of $C(\{f\})$. By $\mathrm{Min}(R)$ (resp. $\mathrm{Max}(R)$) we mean the set of all minimal prime (resp. maximal) ideals of R. If I is an ideal of R, then $\mathrm{Min}(I)$ denotes the set of all minimal prime ideals of I. For each $S \subseteq R$ let $\mathrm{Ann}(S) = \{r \in R : \ rs = 0, \ for \ all \ s \in S\}$, $< S >_R$ be the ideal generated by S in R and P_S (resp. M_S) be the intersection of all minimal prime (resp. maximal) ideals containing S. Clearly, if there is no minimal prime (resp. maximal) ideal containing S, then $P_S = R$ (resp. $M_S = R$). We use P_a (resp. M_a) instead of $P_{\{a\}}$ (resp. $M_{\{a\}}$) and $\mathrm{rad}(R)$ (resp. $\mathrm{Jac}(R)$) instead of P_0 (M_0); when we deal with rings of formal power series, we use $\mathcal{P}_S$ (resp. $\mathcal{M}_S$) instead of P_S (resp. M_S) where $S \subseteq R[[x]]$. An ideal I of a ring R is called a z°-ideal (resp. z-ideal) if for each $a \in I$ we have $P_a \subseteq I$ (resp. $M_a \subseteq I$). By I_z (resp. I^z) we mean the smallest z-ideal (resp. the largest z-ideal, if there exists) containing (resp. contained in) I.

By $\mathcal{C}(X)$ we mean the ring of continuous functions on a Tychonoff topological space X, $Z(f) = f^{-1}\{0\}$, $\mathrm{Coz}(f) = X \backslash Z(f)$ where $f \in \mathcal{C}(X)$, and $O^p(X)$ denotes the set $\{f \in \mathcal{C}(X) : \ p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$, where βX is the Ston-Čech compactification of X and $p \in \beta X$. If $p \in X$, then we usually use $O_p(X)$ instead of $O^p(X)$. The reader is referred to [3] and [11] for more information about the rings of formal power series, to [6] and [7] for more information about topological spaces and rings of continuous functions and to [9], [1], [4], [10], [5] and [2] for more information about z-ideals and z°-ideals.

In Section 1, we give preliminary statements about rings of formal power series. In Section 2, we characterize z-ideals of rings of formal power series. We show that an ideal $\mathcal{I}$ is a z-ideal if and only if $\mathcal{I} = (I, x)$, where I is a z-ideal of R. Section 3 is devoted to z°-ideals of $R[[x]]$. To consider z°-ideals, we need some facts about minimal prime ideals and so we investigate some conditions under which $P[[x]]$ is a minimal prime ideal in $R[[x]]$, where $P \in \mathrm{Min}(R)$. Also, we seek sufficient conditions so that every minimal prime ideal of $R[[x]]$ has the form $P[[x]]$, where $P \in \mathrm{Min}(R)$. Considering this, we introduce some new concepts such as "λ-annihilator exclusion property", "λ-z°-ideal", where λ is a cardinal number and "nicely-contractible". We conclude that R is a Noetherian ring if and only if every ideal of R is nicely-contractible. Also, we show

that if every minimal prime ideal of $R[[x]]$ is nicely-contractible then $I[[x]]$ is a z°-ideal of $R[[x]]$ if and only if I is a $\aleph_0$-z°-ideal.
The proof of the following proposition is straightforward and is omitted.

Proposition 1.1. *Let R be a ring, then the following statements hold.*

(a) $f = \sum_{n=0}^{\infty} f_n x^n$ is unit in $R[[x]]$ if and only if f_0 is a unit in R.

(b) Let I be an ideal of R. Then the map $\varphi : R[[x]] \longrightarrow \frac{R}{I}[[x]]$ given by $\varphi(\sum_{n=0}^{\infty} f_n x^n) = \sum_{n=0}^{\infty}(f_n + I)x^n$ is an epimorphism with kernel $I[[x]]$ and $\frac{R[[x]]}{I[[x]]} \simeq \frac{R}{I}[[x]]$.

(c) $P[[x]]$ is a prime ideal of $R[[x]]$ if and only if P is a prime ideal of R.

(d) $\bigcap_{\alpha\in A}(I_\alpha[[x]]) = (\bigcap_{\alpha\in A} I_\alpha)[[x]]$ and so I is a semiprime ideal of R if and only if $I[[x]]$ is a semiprime ideal of $R[[x]]$.

(e) $\sqrt{I[[x]]} \subseteq \sqrt{I}[[x]]$ and so $rad(R[[x]]) \subseteq rad(R)[[x]]$.

(f) If R is a reduced ring, then so is $R[[x]]$; i.e., if $f = \sum_{n=0}^{\infty} f_n x^n$ is a nilpotent element in $R[[x]]$, then f_n is a nilpotent element in R for $n = 0, 1, \ldots$.

(g) $\mathcal{M} \in Max(R[[x]])$ if and only if there exists $M \in Max(R)$ such that $\mathcal{M} = (M, x)$.

(h) $Jac(R[[x]]) = (Jac(R), x)$. But since $x \in Jac(R[[x]]) \setminus rad(R[[x]])$, we always have $\mathrm{rad}(R[[x]]) \subsetneq \mathrm{Jac}(R[[x]])$.

It is well-known that the converse of the part (f) of Proposition 1.1 is not true. We show this fact, in another way, in Example 3.15.

Definition 1.2. *An ideal I of R is said to be a strongly z°-ideal, or briefly sz°-ideal, (resp. strongly z-ideal, or briefly sz-ideal) if $P_S \subseteq I$ (resp. $M_S \subseteq I$) for every finite subset S of I. Clearly, any intersection of sz°-ideals (resp. sz-ideals) is a sz°-ideal (resp. sz-ideal). Hence, the smallest sz°-ideal (resp. sz-ideal) containing I exists and we denote it by $I_{sz^{\circ}}$ (resp. I_{sz}), see [1] and [2].*

It is easy to see that every minimal prime ideal is sz°-ideal. Also, if I is a sz°-ideal (resp. z°-ideal) and $P \in \mathrm{Min}(I)$, then P is a sz°-ideal (resp. z°-ideal). This fact also holds for sz-ideal (resp. z-ideal), see [7], [2] and [5].

The following is a general form of Proposition 2.9 in [2].

Proposition 1.3. *The following statements are equivalent in any ring R.*

(a) $Jac(R) = rad(R)$.
(b) Every minimal prime ideal of R is a sz-ideal.
(c) P_S is a sz-ideal of R for every finite subset S of R.
(d) P_S is a z-ideal of R for every finite subset S of R.
(e) P_a is a z-ideal of R for every $a \in R$.
(f) Every minimal prime ideal of R is a z-ideal.
(g) Every z°-ideal of R is a z-ideal.
(h) Every sz°-ideal of R is a sz-ideal.
(i) P_a is a sz-ideal of R for every $a \in R$.

Proof. (a) $\Rightarrow$ (b). Suppose that $Q \in \mathrm{Min}(R)$ and S is a finite subset of Q. Thus, there exists $a \notin Q$ such that $aS \subseteq \mathrm{rad}(R)$, so $aS \subseteq \mathrm{Jac}(R)$. Therefore

$$M_a \cap M_S = M_{aS} \subseteq \mathrm{Jac}(R) = \mathrm{rad}(R) \subseteq Q.$$

It follows that $M_S \subseteq Q$.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) are trivial.

(g) $\Rightarrow$ (a). Clearly $\mathrm{rad}(R)$ is a z°-ideal and so is a z-ideal. Since $\mathrm{Jac}(R)$ is the smallest z-ideal in R, we have $\mathrm{Jac}(R) \subseteq \mathrm{rad}(R)$.

(b) $\Rightarrow$ (h). Let I be a sz°-ideal and S is a finite subset of I. Since every minimal prime ideal is a sz-ideal, it follows that $M_S \subseteq P_S \subseteq I$.

(h) $\Rightarrow$ (i). It is clear.

(i) $\Rightarrow$ (g). Suppose that I is a z°-ideal and $a \in I$. Since P_a is a sz-ideal and consequently a z-ideal, $M_a \subseteq P_a \subseteq I$. $\square$

2. z-Ideals of the Rings of Formal Power Series

We have studied z°-ideals of $R[x]$, see [2]. But it seems (at least to us) that the z-ideals of $R[x]$ are, in general, difficult object to be dealt with. In this section, we will characterize the class of z-ideals of $R[[x]]$ in terms of those of R.

Lemma 2.1. *Assuming that R and S are two rings and $\phi : R \longrightarrow S$ are an onto homomorphism, if J is a z-ideal (resp. sz-ideal) of S, then*

$\phi^{-1}(J)$ is a z-ideal (resp. sz-ideal) of R.

Proof. Since $\phi^{-1}(M) \in \mathrm{Max}(R)$ for every $M \in \mathrm{Max}(S)$, it follows that $M_a \subseteq \phi^{-1}(M_{\phi(a)}) \subseteq \phi^{-1}(J)$ (resp. $M_S \subseteq \phi^{-1}(M_{\phi(S)}) \subseteq \phi^{-1}(J)$) for every $a \in \phi^{-1}(J)$ (resp. finite subset S of $\phi^{-1}(J)$). $\square$

Henceforth, by ϕ we mean the homomorphism from $R[[x]]$ onto R with $\phi(f) = f(0) = f_0$.

Lemma 2.2. *The following statements hold in any ring R.*

(a) $\phi^{-1}(I) = (I, x)$ for any ideal I of R.

(b) If $S \subseteq R[[x]]$ and $S_0 = \phi(S) = \{f_0 : \ f \in S\}$, then $\mathcal{M}_S = (M_{S_0}, x)$. In particular, $\mathcal{M}_f = (M_{f_0}, x)$ for every $f \in R[[x]]$.

Proof. (a) It is obvious.
(b) By part (a) and definition of ϕ, clearly, $\mathcal{M}_S \subseteq \phi^{-1}(M_{S_0}) = (M_{S_0}, x)$. Conversely, suppose that $\mathcal{M} = (M, x)$ is a maximal ideal of $R[[x]]$ containing S. It is easily seen that $S_0 \subseteq M$. Thus, $(M_{S_0}, x) \subseteq \mathcal{M}_S$ and so the equality holds. $\square$

Theorem 2.3. *An ideal $\mathcal{I}$ in $R[[x]]$ is a z-ideal (resp. sz-ideal) of $R[[x]]$ if and only if $\mathcal{I} = (I, x)$ where I is a z-ideal (resp. sz-ideal) of R.*

Proof. Suppose that $\mathcal{I}$ is a z-ideal in $R[[x]]$. By Lemma 2.2, $\mathcal{M}_0 = (\mathrm{Jac}(R), x) \subseteq \mathcal{I}$ and so $x \in \mathcal{I}$. It follows that there exists an ideal I in R such that $\mathcal{I} = (I, x)$. Now we show that I is a z-ideal of R. To see this, let $a \in I$. If we put $f = a + x$, then $(M_a, x) = \mathcal{M}_f \subseteq \mathcal{I} = (I, x)$ and consequently $M_a \subseteq I$. The converse is obvious, by Lemma 2.1. The case of sz-ideal is similar. $\square$

Note that if F is a field, then $< x >$ is the only maximal ideal of $F[[x]]$. Therefore, $< x >$ is the only z-ideal (resp. sz-ideal) of $F[[x]]$.
In view of Theorem 2.3, we infer that whenever I is a proper ideal of R, then $I[[x]]$ is never a z-ideal. Finally, we conclude this section, by considering the concept of the smallest (resp. greatest) z-ideal containing (resp. contained in) an ideal $\mathcal{I}$ of $R[[x]]$ in terms of the same properties in R.

Proposition 2.4. *The following statements hold for any ring R.*

(a) For every ideal $\mathcal{I}$ of $R[[x]]$ we have $\mathcal{I}_z = (J, x)$ where $J = (\phi(\mathcal{I}))_z$.

(b) $(I[[x]])_z = (I_z, x)$ for any ideal I of R.

(c) If $\mathcal{I}$ is an ideal of $R[[x]]$, then $\mathcal{I}^z$ exists if and only if $x \in \mathcal{I}$ and $J = (\mathcal{I} \cap R)^z$ exists. In this case we have $\mathcal{I}^z = (J, x)$.

Proof. (a) Clearly, if we take $J = (\phi(\mathcal{I}))_z$, then J is a z-ideal of R. Moreover, $\mathcal{I} \subseteq (J, x)$; since whenever we take $f = f_0 + xg \in \mathcal{I}$, then $f_0 \in J$ and consequently $f \in (J, x)$. Therefore, (J, x) is a z-ideal containing $\mathcal{I}$. Now, let $\mathcal{K} = (K, x)$ be a z-ideal containing $\mathcal{I}$. Obviously, $J = (\phi(\mathcal{I}))_z \subseteq K$ and so $(J, x) \subseteq (K, x) = \mathcal{K}$.

(b) By part (a), it is evident.

(c) By Theorem 2.3, there exists a z-ideal contained in $\mathcal{I}$ if and only if there exists a z-ideal K of R such that $(K, x) \subseteq \mathcal{I}$. Now, supposing that there exists a z-ideal contained in $\mathcal{I}$, we show that $\mathcal{I}^z = (J, x)$. It is clear that (J, x) is a z-ideal contained in $\mathcal{I}$. Let (K, x) be a z-ideal contained in $\mathcal{I}$, then $K \subseteq \mathcal{I} \cap R$ and so $K \subseteq (\mathcal{I} \cap R)^z = J$. Therefore, $(K, x) \subseteq (J, x)$. $\square$

3. z°-Ideals of Rings of Formal Power Series

It is easy to see that to investigate z°-ideals of a ring R, we need some information about minimal prime ideals of R. So we must first consider the set of minimal prime ideals of $R[[x]]$. In particular, if we want to find a close relation between the set of z°-ideals of $R[[x]]$ and the set of z°-ideals of R, it is natural to investigate the conditions under which any minimal prime of $R[[x]]$ is of the form $P[[x]]$ where P is a minimal prime ideal of R.

First we need a new definition.

Definition 3.1. *Supposing that λ is a cardinal number and P a minimal prime ideal of a ring R, we say that P has λ-annihilator exclusion property if for every $S \subseteq P$ with $|S| \leqslant \lambda$, there exist $n \in \mathbb{N}$ and $c \notin P$ such that $(cS)^n = \{0\}$.*

It is easily seen that if P is minimal prime ideal of a reduced ring R, then P has λ-annihilator exclusion property if and only if $\mathrm{Ann}(S) \nsubseteq P$

for any $S \subseteq P$ with $|S| \leqslant \lambda$. Also, it is obvious that every minimal prime ideal P has n-annihilator exclusion property for every $n \in \mathbb{N}$. Moreover, if a minimal prime ideal P has λ-annihilator exclusion property, then it has α-annihilator exclusion property for every $\alpha \leqslant \lambda$.

Lemma 3.2. *Let R be a reduced ring. Then the following statements hold.*

(a) Assuming $f, g \in R[[x]]$, we have $fg = 0$ if and only if $f_n g_m = 0$ for $m, n = 0, 1, \ldots$.

(b) $f \in R[[x]]$ is zero divisor if and only if there exists $0 \neq c \in R$ such that $cf = 0$.

(c) For any $f \in R[[x]]$ and any ideal I of R, we have $Ann(f) \nsubseteq I[[x]]$ if and only if there exists $c \notin I$ such that $cf = 0$.

Proof. The proof is straightforward. $\square$

Corollary 3.3. *Suppose that R is a reduced ring. If $\mathcal{P} \in Min(R[[x]])$, then $\mathcal{P} \subseteq (\mathcal{P} \cap R)[[x]]$.*

Proof. Assume that $f = \sum_{n=0}^{\infty} f_n x^n \in \mathcal{P}$. By hypothesis, there exists $g \notin \mathcal{P}$ such that $fg = 0$ and so by Lemma 3.2, $f_n g = 0$ for $n = 0, 1\ldots$. Therefore, $f_n \in \mathcal{P} \cap R$ for $n = 0, 1, \ldots$ and so $\mathcal{P} \subseteq (\mathcal{P} \cap R)[[x]]$. $\square$

Theorem 3.4. *Suppose that R is a reduced ring and $P \in Min(R)$ such that P is an infinite set. Then the following statements are equivalent.*

(a) P has $\aleph_0$-annihilator exclusion property.

(b) $P[[x]]$ is a minimal prime ideal of $R[[x]]$ and has $\aleph_0$-annihilator exclusion property.

(c) $P[[x]]$ is a minimal prime ideal of $R[[x]]$.

Proof. (a) $\Rightarrow$ (b). Let $S \subseteq P[[x]]$ be countable and $T = C(S)$. Since T is countable, there exists $c \notin P$ such that $cT = \{0\}$ and therefore, $cS = \{0\}$.

(b) $\Rightarrow$ (c). It is clear.

(c) $\Rightarrow$ (a). Let $S \subseteq P$ be countable. Taking $f \in R[[x]]$ such that $S = C(f)$, clearly, $f \in P[[x]]$ and thus, there exists $g \notin P[[x]]$ such that $fg = 0$. Obviously, by Lemma 3.2, $f_n g_m = 0$ for every $n \in \mathbb{N}$, and

$g_m \notin P$ for some $m \in \mathbb{N}$. Clearly, if we put $c = g_m$, then $c \notin P$ and $cS = \{0\}$. $\square$

Corollary 3.5. *Suppose that R is a reduced ring and $\mathcal{P}$ is a prime ideal $R[[x]]$. Then the following statements are equivalent.*

(a) $\mathcal{P} \in Min(R[[x]])$ and moreover, there exist $P \in Min(R)$ such that $\mathcal{P} = P[[x]]$.

(b) $\mathcal{P} \cap R$ is a minimal prime of R and has $\aleph_0$-annihilator exclusion property.

Proof. (a) $\Rightarrow$ (b). If $P \in \mathrm{Min}(R)$ and $\mathcal{P} = P[[x]] \in \mathrm{Min}(R[[x]])$, then by Theorem 3.4, $\mathcal{P} \cap R$ is a minimal prime ideal of R and has $\aleph_0$-annihilator exclusion property.

(b) $\Rightarrow$ (a). By Theorem 3.4 we have $(\mathcal{P} \cap R)[[x]] \in \mathrm{Min}(R[[x]])$. On the other hand, by Corollary 3.3, $\mathcal{P} \subseteq (\mathcal{P} \cap R)[[x]]$ and so $\mathcal{P} = (\mathcal{P} \cap R)[[x]]$. $\square$

In the following two examples, we first show that there exists a ring R such that the set of the minimal prime ideals of $R[[x]]$ that are of the form $P[[x]]$ is uncountable. Next we prove that there is a ring R such that $P[[x]] \notin \mathrm{Min}(R[[x]])$ for every $P \in \mathrm{Min}(R)$.

Example 3.6. Assume that D is a uncountable discrete space and $X = D \cup \{a\}$ is the one point compactification of D. Putting $R = \mathcal{C}(X)$, we know that $O_d(X) \in \mathrm{Min}(R)$ for every $d \in D$. By Theorem 3.4, it is enough to show that if we take $P = O_d(X)$, then P has $\aleph_0$-annihilator exclusion property. To see this, suppose that $f_n \in P$ for $n = 0, 1, \dots$. If we define $g : X \longrightarrow \mathbb{R}$ with $g(d) = 1$ and $g(x) = 0$ for $x \neq d$, then, clearly, $g \in \mathcal{C}(X) \setminus P$ and $gf_n = 0$ for $n = 0, 1, \dots$.

In the above example, we can show that if we take $P \in \mathrm{Min}(R)$ such that $O_a(X) \subseteq P$, then $P[[x]] \notin \mathrm{Min}(R[[x]])$. In what follows, we present a problem for which we have not yet found any answer: Is there a space X such that every minimal prime ideal of $\mathcal{C}(X)$ has $\aleph_0$-annihilator exclusion property?

Example 3.7. Assume that $R = \mathcal{C}(\mathbb{R})$ and $P \in \mathrm{Min}(R)$. We show

that P has not $\aleph_0$-annihilator exclusion property. We have two cases for P. In the first case that $O_r(X) \subseteq P$ for some $r \in \mathbb{R}$, since $\mathbb{R}$ is first-countable space, for every $n \in \mathbb{N}$ there exists $f_n \in O_r(\mathbb{R})$ such that $Z = \bigcap_{n\in\mathbb{N}} Z(f_n) = \{r\}$. If, on the contrary, P has $\aleph_0$-annihilator exclusion property, then $g \notin P$ exists such that $gf_n = 0$ for $n = 0, 1, ...$. Therefore, $\mathrm{Coz}(g) \subseteq Z(f_n)$ for $n = 0, 1, ...$ and so $\varnothing \neq \mathrm{Coz}(g) \subseteq Z = \{r\}$, a contradiction. In the second case that $O^p(\mathbb{R}) \subseteq P$ for a point $p \in \beta\mathbb{R} \setminus \mathbb{R}$, we define $f_n \in \mathcal{C}(\mathbb{R})$ so that $Z(f_n) = (-\infty, -n] \cup [n, +\infty)$, for every $n \in \mathbb{N}$. Then, clearly, $f_n \in O^p(\mathbb{R}) \subseteq P$ for every $n \in \mathbb{N}$. Obviously, $Z = \bigcap_{n\in\mathbb{N}} Z(f_n) = \varnothing$ and similar to the first case, we can find that P has not $\aleph_0$-annihilator exclusion property.
Next, we consider a relation between $\mathrm{Min}(R)$ and $\mathrm{Min}(R[[x]])$.

Proposition 3.8. *Let R be a ring. Then the following statements hold.*

(a) $Min(R) \subseteq \{\mathcal{P} \cap R : \mathcal{P} \in Min(R[[x]])\}$ and so $|Min(R)| \leqslant |Min(R[[x]])|$.

(b) If the ring R is reduced, then $Min(R)$ is finite if and only if $Min(R[[x]])$ is too and in this case we have $Min(R[[x]]) = \{P[[x]] : P \in Min(R)\}$.

Proof. (a). Supposing $P \in \mathrm{Min}(R)$, clearly there exists $\mathcal{P} \in \mathrm{Min}(R[[x]])$ such that $\mathcal{P} \subseteq P[[x]]$. Therefore, $\mathcal{P} \cap R \subseteq P[[x]] \cap R = P$ and so $\mathcal{P} \cap R = P$, consequently, $\mathrm{Min}(R) \subseteq \{\mathcal{P} \cap R : \mathcal{P} \in \mathrm{Min}(R[[x]])\}$.
(b). Suppose that Min(R) is finite and $\mathcal{P} \in \mathrm{Min}(R[[x]])$. Then

$$\bigcap_{P\in\mathrm{Min}(R)} P[[x]] = (\bigcap_{P\in\mathrm{Min}(R)} P)[[x]] = (0) \subseteq \mathcal{P}.$$

Therefore, there exists $P \in \mathrm{Min}(R)$ such that $P[[x]] \subseteq \mathcal{P}$ and hence $P[[x]] = \mathcal{P}$. Thus, $\mathrm{Min}(R[[x]])$ is finite. To complete the proof, it is enough to show that $P[[x]] \in \mathrm{Min}(R[[x]])$ for every $P \in \mathrm{Min}(R)$. To see this, suppose that $P \in \mathrm{Min}(R)$ and $\mathcal{Q} \subseteq P[[x]]$ is a minimal prime ideal of $R[[x]]$. Hence, similarly, it follows that $\mathcal{Q} = Q[[x]]$ for a $Q \in \mathrm{Min}(R)$. Therefore, $Q[[x]] = \mathcal{Q} \subseteq P[[x]]$ and so $Q \subseteq P$, consequently $Q = P$ and hence $P[[x]] = \mathcal{Q} \in \mathrm{Min}(R[[x]])$. Therefore, $\mathrm{Min}(R[[x]]) = \{P[[x]] : P \in \mathrm{Min}(R)\}$. By part (a), the converse is trivial. $\square$

An immediate consequence of Theorem 3.4 and Proposition 3.8 is that if R is a reduced ring and $\mathrm{Min}(R)$ is finite, then every $P \in \mathrm{Min}(R)$ has $\aleph_0$-annihilator exclusion property.

Definition 3.9. *An ideal $\mathcal{I}$ in $R[[x]]$ is said to be nicely-contractible if $(\mathcal{I} \cap R)[[x]] \subseteq \mathcal{I}$; i.e., $C(f) \subseteq \mathcal{I}$ implies that $f \in \mathcal{I}$. Also, an ideal I, in a ring R, in which every countable subset of I is contained in a finitely generated sub-ideal of I is called a c.f.g-ideal for brevity.*

We can easily see that $I[[x]]$ is a nicely-contractible for every ideal I of R. It is better to note that the inclusion that found in Definition 3.9 may be strict. For instance, if we take $\mathcal{I} =< x >$, then $(\mathcal{I} \cap R)[[x]] = (0) \subsetneq \mathcal{I}$.

Proposition 3.10. *Let I be an ideal in a ring R. Then the following statements are equivalent.*

(a) If $\mathcal{I}$ is an ideal of $R[[x]]$ such that $\mathcal{I} \cap R = I$, then $\mathcal{I}$ is nicely-contractible.

(b) $I[[x]] =< I >$.

(c) I is a c.f.g-ideal of R.

Proof. (a) $\Rightarrow$ (b). Clearly $< I > \subseteq I[[x]]$ and since $< I > \cap R = I$, it follows from (a) that $< I >$ is a nicely-contractible ideal, consequently $I[[x]] = (< I > \cap R)[[x]] \subseteq < I >$.

(b) $\Rightarrow$ (c). Let $S \subseteq I$ be countable, then we take $f = \sum_{n=0}^{\infty} f_n x^n \in R[[x]]$ so that $C(f) = S$. Therefore, by (b), we have $f \in I[[x]] =< I >$ and this implies that there exist $a_1, ..., a_k \in I$ such that $f \in < a_1, ..., a_k >_{R[[x]]}$. Hence, $S = C(f) \subseteq < a_1, ..., a_k >_R$.

(c) $\Rightarrow$ (a). Suppose that $\mathcal{I}$ is an ideal of $R[[x]]$ such that $\mathcal{I} \cap R = I$. Assume that $f \in I[[x]]$, then by part (c), there exist $a_1, ..., a_k \in I$ such that $C(f) \subseteq < a_1, ..., a_k >_R$ and so $f \in < a_1, ..., a_k >_R [[x]] \subseteq \mathcal{I}$. Hence, $(\mathcal{I} \cap R)[[x]] \subseteq \mathcal{I}$; i.e., $\mathcal{I}$ is a nicely-contractible ideal. $\square$

In what follows, we see that the concept of nicely-contractible is useful in the study of finding a relation between $\mathrm{Min}(R)$ and $\mathrm{Min}(R[[x]])$.

Proposition 3.11. *Suppose that $\mathcal{P} \in Min(R[[x]])$. Then the following statements are equivalent.*

(a) $\mathcal{P}$ is a nicely-contractible ideal.

(b) There exists a prime ideal P of R such that $\mathcal{P} = P[[x]]$.

(c) There exists $P \in Min(R)$ such that $\mathcal{P} = P[[x]]$.

Proof. (a) $\Rightarrow$ (b). Supposing $\mathcal{P} \in \mathrm{Min}(R[[x]])$, we put $P = \mathcal{P} \cap R$. Then, by (a), $P[[x]] \subseteq \mathcal{P}$ and so $\mathcal{P} = P[[x]]$.

(b) $\Rightarrow$ (c). Assuming $\mathcal{P} = P[[x]] \in \mathrm{Min}(R[[x]])$, we must show that $P \in \mathrm{Min}(R)$. Let $Q \subseteq P$ be a prime ideal of R. So, $Q[[x]] \subseteq P[[x]]$ and consequently, $Q[[x]] = P[[x]]$, this implies that $Q = P$.

(c) $\Rightarrow$ (a). It is clear, since $(I[[x]] \cap R)[[x]] = I[[x]]$ for every ideal I of R. $\square$

The following result is immediate.

Corollary 3.12. *Every minimal prime ideal of $R[[x]]$ is nicely-contractible if and only if $Min(R[[x]]) = \{P[[x]] :\ P \in Min(R)\}$.*

The following is an immediate consequence of part (b) of Proposition 1.1, Proposition 3.11 and Corollary 3.12.

Corollary 3.13. *Let I be an ideal in R. Then the following statements hold.*

(a) $\mathcal{P} \in Min(I[[x]])$ is nicely-contractible if and only if there exists a prime ideal P containing I such that $\mathcal{P} = P[[x]]$.

(b) Every $\mathcal{P} \in Min(I[[x]])$ is nicely-contractible if and only if every $\mathcal{P} \in Min(I[[x]])$ is of the form $P[[x]]$ where $P \in Min(I)$.

In the following result, we show that the converse of the part (f) of Proposition 1.1, is also true, if assume that every minimal prime ideal of $R[[x]]$ is nicely-contractible.

Theorem 3.14. *Suppose that every minimal prime ideal of $R[[x]]$ is nicely-contractible. Then $f \in R[[x]]$ is nilpotent in $R[[x]]$ if and only if f_n is nilpotent in R for $n = 0, 1, ...$; i.e., $rad(R[[x]]) = (rad(R))[[x]]$.*

Proof. By Corollary 3.12, we can write

$$\mathrm{rad}(R[[x]]) = \bigcap_{P \in \mathrm{Min}(R)} P[[x]] = (\bigcap_{P \in \mathrm{Min}(R)} P)[[x]] = (\mathrm{rad}(R))[[x]]. \quad \square$$

However, in most of textbooks, we can find that the converse of the part

(f) of Proposition 1.1 is not true, but it seems that the following example is another simple way of showing this fact, at least for those who are familiar with $\mathcal{C}(X)$.

Example 3.15. Let $f \in \mathcal{C}(\mathbb{R})$ be the identity map, $I =< |f| >$ and $R = \frac{\mathcal{C}(\mathbb{R})}{I}$. For every $i \in \mathbb{N}$, we define $g_i = |f|^{\frac{1}{i}} + I$. Clearly, g_i is a nilpotent element of R for every $i \in \mathbb{N}$. It is enough to show that $g = \sum_{i=1}^{\infty} g_i x^i$ is not a nilpotent element of $R[[x]]$. On the contrary, suppose that $h = g^m = 0$ for an $m \in \mathbb{N}$. It is easy to see that there exists $n > m$ such that $h_n = |f|^r u + I = 0$ for some $0 < r < 1$ and some $u \in \mathcal{C}(\mathbb{R})$ with $u(0) = 1$. Therefore, $|f|^r u \in I$ and so $v \in \mathcal{C}(\mathbb{R})$ exists so that $|f|^r u = |f| v$. Hence, $\lim_{x \to 0} v(x) = \lim_{x \to 0} \frac{u(x)}{|f|^{1-r}(x)} = \infty$, a contradiction.

Corollary 3.16. *Let R be a reduced ring. Then R is a Noetherian ring if and only if every ideal of $R[[x]]$ is nicely-contractible.*

Proof. ($\Rightarrow$). Since R is a Noetherian ring, every ideal of R is finitely generated and by Proposition 3.10, we are done.

($\Leftarrow$). It is enough to show that every countable generated ideal of R is finitely generated, see [8]. To see this, suppose that I is a countably generated ideal of R. By hypothesis and Proposition 3.10, I is a c.f.g-ideal and hence it is finitely generated. $\square$

Definition 3.17. *Let λ be a cardinal number. An ideal I of a ring R is said to be a λ-z°-ideal whenever for every $S \subseteq I$ with $|S| \leqslant \lambda$, we have $P_S \subseteq I$. Evidently, the concept of sz°-ideal coincide with the "λ-z°-ideal" where λ is a finite cardinal.*

Clearly, the ideal (0) in any reduced ring R is a λ-z°-ideal for every cardinal number λ. Also, if I is a λ-z°-ideal and $\alpha \leqslant \lambda$, then I is an α-z°-ideal.

Theorem 3.18. *Suppose that every minimal prime ideal of $R[[x]]$ is nicely-contractible. Then the following statements hold.*

(a) If $S \subseteq R[[x]]$, then $\mathcal{P}_{C(S)} = \mathcal{P}_S = P_{C(S)}[[x]]$.

(b) If $S \subseteq R$, then $\mathcal{P}_S = P_S[[x]]$.

Proof. (a). Clearly, if $S \subseteq R[[x]]$, then we have

$$C(S) \subseteq P[[x]] \quad \Leftrightarrow \quad S \subseteq P[[x]] \quad \Leftrightarrow \quad C(S) \subseteq P.$$

Therefore, using Corollary 3.12, we can write

$$\mathcal{P}_{C(S)} = \{P[[x]] : \ P \in \operatorname{Min}(R), \ C(S) \subseteq P[[x]]\} = \mathcal{P}_S$$

$$= \bigcap_{C(S) \subseteq P \in \operatorname{Min}(R)} P[[x]] = (\bigcap_{C(S) \subseteq P \in \operatorname{Min}(R)} P)[[x]] = P_{C(S)}[[x]].$$

(b). It is obviously followed from (a). $\square$

Proposition 3.19. *Assume that every minimal prime ideal of* $R[[x]]$ *is nicely-contractible and* α *is infinite cardinal number. Then the following statements are equivalent.*

(a) I *is an* α-z°*-ideal of* R.

(b) $I[[x]]$ *is an* α-z°*-ideal of* $R[[x]]$.

Proof. (a) $\Rightarrow$ (b). Suppose that $T \subseteq I[[x]]$ and $|T| \leqslant \alpha$. Then, clearly, $|C(T)| \leqslant \alpha$ and so we can write

$$P_{C(T)} \subseteq I \quad \Rightarrow \quad \mathcal{P}_T = P_{C(T)}[[x]] \subseteq I[[x]].$$

(b) $\Rightarrow$ (a). Let $S \subseteq I$ and $|S| \leqslant \alpha$. Then, clearly, there exists $T \subseteq I[[x]]$ such that $|T| \leqslant \alpha$ and $C(T) = S$. Therefore, we can write

$$P_{C(T)}[[x]] = \mathcal{P}_T \subseteq I[[x]] \quad \Rightarrow \quad P_S = P_{C(T)} \subseteq I. \quad \square$$

Theorem 3.20. *Assume that every minimal prime ideal of* $R[[x]]$ *is nicely-contractible. Then* $I[[x]]$ *is a* z°*-ideal of* $R[[x]]$ *if and only if* I *is an* $\aleph_0$-z°*-ideal of* R.

Proof. ($\Rightarrow$). Let S be a countable subset of I. Clearly, there exists $f \in R[[x]]$ such that $C(f) = S$. Therefore, $P_S[[x]] = P_{C(f)}[[x]] = \mathcal{P}_f \subseteq I[[x]]$ and so $P_S \subseteq I$.

($\Leftarrow$). Suppose that $f \in I[[x]]$. Clearly, $C(f)$ is a countable subset of I. Thus, $P_{C(f)} \subseteq I$ and so $\mathcal{P}_f = P_{C(f)}[[x]] \subseteq I[[x]]$. $\square$

Theorem 3.21. *Suppose that every minimal prime ideal of $R[[x]]$ is nicely-contractible and $\mathcal{I}$ is an ideal of $R[[x]]$. Then the following statements hold.*

(a) If $\mathcal{I}$ is a z°-ideal, then $\mathcal{I} \subseteq (\mathcal{I} \cap R)[[x]]$.

(b) If $\mathcal{I}$ is an $\aleph_0$-z°-ideal, then $\mathcal{I} = (\mathcal{I} \cap R)[[x]]$.

Proof. (a). Suppose that $f \in \mathcal{I}$, then $P_{C(f)}[[x]] = \mathcal{P}_f \subseteq \mathcal{I}$ and so $f_n \in \mathcal{I} \cap R$ for $n = 0, 1, ...$, consequently $f \in (\mathcal{I} \cap R)[[x]]$.

(b). By part (a), it is enough to prove that $(\mathcal{I} \cap R)[[x]] \subseteq \mathcal{I}$; i.e., $\mathcal{I}$ is nicely-contractible. To show this, suppose that $f \in (\mathcal{I} \cap R)[[x]]$. Hence, $C(f) \subseteq \mathcal{I} \cap R \subseteq \mathcal{I}$ and since $C(f)$ is countable, we can conclude that $\mathcal{P}_f = \mathcal{P}_{C(f)} \subseteq \mathcal{I}$ and so $f \in \mathcal{I}$. $\square$

The following is an immediate consequence of Proposition 3.19 and Theorem 3.21.

Corollary 3.22. *Assume that every minimal prime ideal of $R[[x]]$ is nicely-contractible, α is an infinite cardinal number and $\mathcal{I}$ is an α-z°-ideal. Then $\mathcal{I} \cap R$ is an α-z°-ideal.*

Recall that R is semisimple if $\mathrm{Jac}(R) = (0)$. A ring R is called a z-radical ring if whenever $\sqrt{I}$ is a z-ideal, then it follows that $\sqrt{I} = I$, see [5].

Corollary 3.23. *Suppose that R is a z-radical semisimple ring, every minimal prime ideal in $R[[x]]$ is nicely-contractible, $\mathcal{I}$ is a nicely-contractible and $\sqrt{\mathcal{I}}$ is an $\aleph_0$-z°-ideal. Then $\sqrt{\mathcal{I}} = \mathcal{I}$.*

Proof. By part (b) of Theorem 3.21, it follows that $\sqrt{\mathcal{I}} = (\sqrt{\mathcal{I}} \cap R)[[x]]$ and by Corollary 3.22, $\sqrt{\mathcal{I}} \cap R = \sqrt{\mathcal{I} \cap R}$ is an $\aleph_0$-z°-ideal and so is a z-ideal. Therefore, $\sqrt{\mathcal{I} \cap R} = \mathcal{I} \cap R$. Hence, we can write

$$\sqrt{\mathcal{I}} = (\sqrt{\mathcal{I}} \cap R)[[x]] = \sqrt{\mathcal{I} \cap R}[[x]] = (\mathcal{I} \cap R)[[x]] \subseteq \mathcal{I}. \quad \square$$

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Ali Rezaei Aliabad
Department of Mathematics
Associate Professor of Mathematics
Chamran University
Ahvaz, Iran
E-mail: aliabady_r@scu.ac.ir

Rostam Mohamadian
Department of Mathematics
Assistant Professor of Mathematics
Chamran University
Ahvaz, Iran
E-mail: mohamadian_r@scu.ac.ir