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Original Research Paper

On Bimodal Polynomials with a Non-Hyperbolic Fixed Point

M. Rabii*
Alzahra University

M. Akbari
Shahid Rajaee Teacher Training University

Abstract. We consider the real polynomials of degree $d+1$ with a fixed point of multiplicity $d \geq 2$. Such polynomials are conjugate to $f_{a,d}(x) = ax^d(x-1)+x,\ a \in \mathbb{R} \setminus \{0\}$. In this family, the point 0 is always a non-hyperbolic fixed point. We prove that for given d, d', and a, where d and d' are positive even numbers and a belongs to a special subset of $\mathbb{R}^-$, there is $a' < 0$ such that $f_{a,d}$ is topologically conjugate to $f_{a',d'}$. Then we extend the properties that we have studied in case $d = 2$ to this family for every even $d > 2$.

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1 Introduction

Among the C^1 multi-modal maps, polynomials are typical. It has been shown that each C^1 l-modal map is semi-conjugate to a polynomial l-modal map (see [4, Chapter II, Theorem 6.4]). Therefore it is useful to

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*Corresponding Author

investigate the dynamical behavior of polynomials. In [8], the dynamics of the family of complex polynomials $f(z) = z^3 + az^2 + z = z^2(z+a) + z$, with $f'(0) = 1$ is studied. In [1], a family of real polynomials $f_a(x) = ax^2(x-1)+x$ is studied. Note that in this case we also have $f'_a(0) = 1$. In this paper we consider the family of polynomials $f_{a,d}(x) = ax^d(x-1)+x$ where $a < 0$ is a real number and $d \geq 2$ is an even integer. Each map of this family is a bimodal polynomials with a non-hyperbolic fixed point. The main features of $f_{a,d}$ are similar to $f_{a',2}$. The question is whether these similarities make $f_{a,d}$ and $f_{a',d'}$ conjugate. The main tool in this paper is Corollary 3.1 of Chapter II of [4], which states the conditions under which two l-modal maps are conjugate on a compact interval.

This paper is organized in four sections: In Section 2 we discuss some common properties of the family $f_{a,d}(x) = ax^d(x-1)+x$, for even $d \geq 2$ and $a < 0$. To determine the position of the orbit of each critical point, in Section 3, for given even $d \geq 2$, the parameter line $a < 0$ is partitioned into some subintervals such that the behavior of the critical points are different in these subintervals. Our information about the position of the critical orbits respect to each other enables us to define an order preserving bijection that is applied in Corollary 3.1 of Chapter II of [4]. In Section 4 we present more observations about the function $f_{a,d}$. In these observations, we discuss the topological entropy of this function by comparing it with $f_{a,2}$, for special negative a's.

Here we explain some terminology and preliminaries used in this paper. Let I be an interval and $f : I \to I$ be a continuous map. By f^n we mean $f \circ f^{n-1}$, where f^0 is the identity map. The orbit of $x \in I$ is the sequence $\big(f^n(x)\big)_{n\geq 0}$. A point x_0 is called a *periodic point* of f of period n if n is the least natural number that $f^n(x_0) = x_0$. If $n = 1$, x_0 is a *fixed point* of f. If x_0 is a periodic point of f of period n, then the set $O(x_0) = \big\{x_0, f(x_0), \cdots, f^{n-1}(x_0)\big\}$ is called a *cycle* of f of period n. In this case the *basin* of $O(x_0)$ is $B(x_0) = \cup_{i=0}^{n-1}\{x : \lim_{k\to\infty} f^{kn}(x) = f^i(x_0)\}$. The *immediate basin* $B_0(x_0)$ of $O(x_0)$ is the union of the components of $B(x_0)$ which contain points of $O(x_0)$. The cycle is a *periodic attractor* if $B_0(x_0)$ contains an open set. It is called a *two-sided periodic attractor* if each point of $O(x_0)$ is an interior point of $B_0(x_0)$, otherwise it is a *one-sided attractor*. We denote the union of the immediate basins of periodic attractors of f by $B_0(f)$.

The map f is called *l-modal* if it has exactly l turning point in the interior of the compact interval I and $f(\partial I) \subset \partial I$, where ∂I is the boundary of I. The point in which f has a local extremum is called a *turning point.*

An interval $J \subseteq I$ is called *wandering* if all its iterates $J, f(J), \cdots$ are disjoint and $\big(f^n(J)\big)_{n\geq 0}$ does not tend to a cycle.

For a C^1 function f, the fixed point x_0 is called *non-hyperbolic* if $|f'(x_0)| = 1$.

For a C^2 map f, the critical point c is called *non-flat* if there exists a C^2 local diffeomorphism ϕ with $\phi(c) = 0$ such that $f(x) = \pm|\phi(x)|^\alpha + f(c)$ for some $\alpha \geq 2$. Note that if f is C^∞ and some derivative of f is non-zero at c, then c is a non-flat critical point.

We say f is an increasing (a decreasing) function if $x < y$, then $f(x) < f(y)$ $(f(x) > f(y))$.

In this paper, we always assume that $a < 0$ is a real number and $d \geq 2$ is an even integer.

2 The Common Properties of $f_{a,d}$

The common properties of the family $f_{a,d}(x) = ax^d(x-1) + x$, which are stated in the following propositions, are proved by employing mathematical techniques of elementary calculus.

Proposition 2.1. *1. $f_{a,d}(x) = 0$ has only two non-zero solutions $x_0(a,d)$ and $x_1(a,d)$. Moreover, $x_0(a,d) < 0 < 1 < x_1(a,d)$.*

2. $f'_{a,d}(x) = 0$ has only two solutions $c_0(a,d)$ and $c_1(a,d)$ where $c_0(a,d)$ is a local minimum point and $c_1(a,d)$ is a local maximum point of $f_{a,d}$. Moreover, $x_0(a,d) < c_0(a,d) < 0 < \frac{d}{d+1} < c_1(a,d) < x_1(a,d)$.

The following proposition is about the dynamics of $f_{a,d}$ on some intervals. The proof of this proposition is identical to Lemma 1.2 in [1] and we do not present it here.

We use the notation $I \sqsubseteq J$ for two intervals I and J when $x < y$ for all $x \in I$ and all $y \in J$.

Proposition 2.2. *Suppose that $x_0(a,d) < x_1(a,d)$ are the non zero roots of $f_{a,d}$. Then there are an increasing bounded sequence $\big(x_{2n+1}(a,d)\big)_{n\geq 0}$ in $\big[x_1(a,d),\infty\big)$ and a decreasing bounded sequence $\big(x_{2n}(a,d)\big)_{n\geq 0}$ in $\big(-\infty, x_0(a,d)\big]$ such that (see Figure 1)*

$$f_{a,d}\big(x_{2n}(a,d)\big) = x_{2n-1}(a,d), \qquad n \geq 1, \tag{1}$$

and

$$f_{a,d}\big(x_{2n+3}(a,d)\big) = x_{2n}(a,d), \qquad n \geq 0. \tag{2}$$

Let

$$J_0(a,d) = \big[x_0(a,d), 0\big],\ I_0(a,d) = \big(0, x_1(a,d)\big),$$

$$p_0(a,d) = \lim_{n\to\infty} x_{2n}(a,d),\ p_1(a,d) = \lim_{n\to\infty} x_{2n+1}(a,d),$$

and for $n \geq 1$, set

$$J_n(a,d) = \big[x_{2n-1}(a,d), x_{2n+1}(a,d)\big],\ I_n(a,d) = \big(x_{2n}(a,d), x_{2n-2}(a,d)\big)$$

if n is odd and

$$J_n(a,d) = \big[x_{2n}(a,d), x_{2n-2}(a,d)\big],\ I_n(a,d) = \big(x_{2n-1}(a,d), x_{2n+1}(a,d)\big)$$

if n is even. Then

$$f_{a,d}\big(p_0(a,d)\big) = p_1(a,d), \qquad f_{a,d}\big(p_1(a,d)\big) = p_0(a,d),$$

$$f_{a,d}\big(J_n(a,d)\big) = J_{n-1}(a,d), \qquad f_{a,d}\big(I_n(a,d)\big) = I_{n-1}(a,d),$$

$$c_0(a,d) \in J_0(a,d), \qquad c_1(a,d) \in I_0(a,d),$$

and

$$\begin{aligned}\cdots I_{2n+1}(a,d) \sqsubseteq \quad & J_{2n}(a,d) \sqsubseteq \cdots \sqsubseteq J_0(a,d) \sqsubseteq \\ & I_0(a,d) \sqsubseteq \cdots \sqsubseteq I_{2n}(a,d) \sqsubseteq \quad J_{2n+1}(a,d) \cdots .\end{aligned}$$

Moreover, $\big(p_0(a,d), p_1(a,d)\big) = \big(\cup_{n\geq 0} I_n\big) \cup \big(\cup_{n\geq 0} J_n\big)$, and for every n the orbit of any point of the interval $J_n(a,d)$ converges to 0, and $\lim_{n\to\infty} |f^n_{a,d}(x)| = \infty$ for each $x \notin \big[p_0(a,d), p_1(a,d)\big]$.

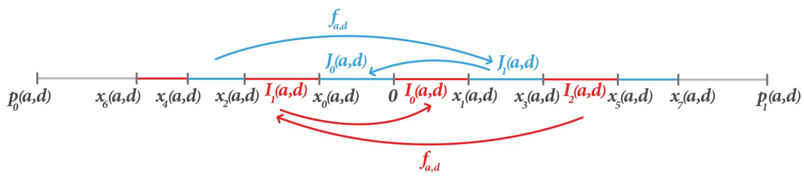

Figure 1: The images of $I_i(a,d)$'s and $J_i(a,d)$'s under $f_{a,d}$

By (1) and (2), it is straightforward to show that

$$f_{a,d}^{2n}\big(x_{4n}(a,d)\big) = f_{a,d}^{2n+1}\big(x_{4n+3}(a,d)\big) = x_0(a,d) \tag{3}$$

and

$$f_{a,d}^{2n}\big(x_{4n+1}(a,d)\big) = f_{a,d}^{2n+1}\big(x_{4n+2}(a,d)\big) = x_1(a,d). \tag{4}$$

Corollary 2.3. *Suppose that $f_{a,d}\big(c_1(a,d)\big) \leq p_1(a,d)$. Then $f_{a,d} : \big[p_0(a,d), p_1(a,d)\big] \to \big[p_0(a,d), p_1(a,d)\big]$ is a bimodal polynomial.*

3 Conjugacy in the Family $f_{a,d}$

In this section we are going to prove that $f_{a,d}$'s are conjugate in some cases. To achieve this, in the first subsection we state the required theorems, in the second subsection we study some properties of the family $f_{a,d}$ on the parameter space, and in the third subsection we define the conjugacy.

3.1 Some known theorems

First, we apply the following theorem to determine when two functions are conjugate. Note that we can also apply the following theorem when the domains of f and g are two different intervals.

Theorem 3.1. *[4, Chapter II, Corollary 3.1.] Suppose that $f, g : I \to I$ are two l-modal maps with turning points $c_1 < \cdots < c_l$ respectively $\tilde{c}_1 < \cdots < \tilde{c}_l$ and assume that*

(1) the map

$$h : \cup_{i=1}^{l} \cup_{n\geq 0} f^n(c_i) \to \cup_{i=1}^{l} \cup_{n\geq 0} g^n(\tilde{c}_i) \tag{5}$$

defined by $h(f^n(c_i)) = g^n(\tilde{c}_i)$ is an order preserving bijection;

(2) *the basin of each periodic attractor of* f *and* g *contains a turning point and each periodic turning point is an attractor;*

(3) *the immediate basins of two periodic attractors have no boundary point in common.*

Then there exists a one-to-one correspondence between periodic attractors of f *and* g*. Moreover, if a periodic attractor of* f *is one-sided if and only if the same holds for the corresponding periodic attractor of* g*, then the monotone bijection* h *from (5) can be extended to a conjugacy between* $f|_{B_0(f)}$ *and* $g|_{B_0(g)}$*. In particular, if* f *and* g *have no wandering intervals and have no intervals consisting of periodic points of constant period, then* f *and* g *are conjugate.*

We also employ the following theorem to show that $f_{a,d}$ has no wandering interval.

Theorem 3.2. *[4, ChapterII, Theorem 6.2.] Let* $f : I \to I$ *be a* C^2 *map such that each critical point of* f *is non-flat. Then* f *has no wandering interval.*

We use the following theorems of complex dynamics to show that, besides the attractor 0, the function $f_{a,d}$ can have at most one real periodic attractor. Note that if a C^3 real function has negative Schwarzian derivative, then each bounded immediate basin of a periodic attractor contains a critical point (see [4, 5]), however, for $d > 2$, the Schwarzian derivative of $f_{a,d}$ is positive at some points. In the next two theorems we assume R is a complex function.

Theorem 3.3. *[2, Theorem 9.3.1.] Let* R *be a rational map of degree at least two. Then the immediate basin of each attracting cycle of* R *contains a critical point of* R*.*

Theorem 3.4. *[6, Theorem 10.15.] If* $\hat{z}$ *is a parabolic fixed point with multiplier* $\lambda = 1$ *for a rational map* R *(i.e.,* $R'(\hat{z}) = 1$*), then each immediate basin for* $\hat{z}$ *contains at least one critical point of* R*.*

Recall that a cycle $O(z_0) = \{z_0, R(z_0), \cdots, R^{n-1}(z_0)\}$ of a rational map R, where $R^n(z_0) = z_0$, is called

- an *attracting cycle* if $|(R^n)'(z_0)| < 1$.
- a *rationally indifferent cycle* or a *parabolic cycle* if $(R^n)'(z_0) = e^{\frac{2\pi ip}{q}}$, where $(p, q) = 1$.
- a *repelling cycle* if $|(R^n)'(z_0)| > 1$.

The *immediate basin* and *the basin* of an attracting cycle $O(z_0)$ of a rational map are defined the same as the ones that are introduced in Introduction.

Suppose that near the origin $R(z) = z + az^{n+1} + \cdots, \quad a \neq 0$. We call a complex number v a *repulsion vector* for R at the origin if $nav^n = 1$ and an *attraction vector* if $nav^n = -1$. Thus there are n attraction vectors, $U_0, U_1, \cdots, U_{n-1}$ at the origin, separated by n repulsion vectors $V_0, \cdots, V_{n-1}$, such that $U_{j+1} = U_j e^{\frac{2\pi i}{n}}$ and $V_{j+1} = V_j e^{\frac{2\pi i}{n}}$ for $j = 0, 1, \cdots, n-2$. The *basin of 0 associated with* U_j consists of all z such that $R^k(z) \to 0$ and $R^k(z)$ is asymptotic to $\frac{U_j}{\sqrt[n]{k}}$ as $k \to \infty$. The connected component of the basin which maps into itself under R is called the *immediate basin associated with* U_j (see [2, 3, 6] for more information).

3.2 Partitioning the parameter space

Now, the point 0 is a rationally indifferent fixed point of $f_{a,d}$ when it is considered as a complex map. Thus, there are $d-1$ attraction vectors and $d-1$ repulsion vectors at the origin. Therefore there are $d-1$ immediate basins at 0 which by Theorem 3.4 contain at least $d-1$ critical points. On the other hand since $a < 0$ and $d-1$ is odd, the negative real axis contains an attraction vector at 0 and the positive real axis contains a repulsion vector at 0. Now, if $f_{a,d}\big(c_1(a,d)\big) \in J_{2n-1}(a,d)$ for some $n \geq 1$, then $f^{2n}_{a,d}\big(c_1(a,d)\big)$ belongs to $J_0(a,d)$ that is the immediate basin at 0 and if $f_{a,d}\big(c_1(a,d)\big) = p_1(a,d)$, then $c_1(a,d)$ lands on the repelling periodic point $p_1(a,d)$, hence by Theorems 3.3 and 3.4, $f_{a,d}(x)$ has no other periodic attractor. Also, if $f_{a,d}\big(c_1(a,d)\big) \in I_{2n}(a,d)$ for some $n \geq 1$, and $f^{2n+1}_{a,d}\big(c_1(a,d)\big) = c_1(a,d)$, then $c_1(a,d)$ is a periodic attractor whose immediate basin has no common boundary with the immediate basin of the fixed point zero. Thus, we have the following lemma.

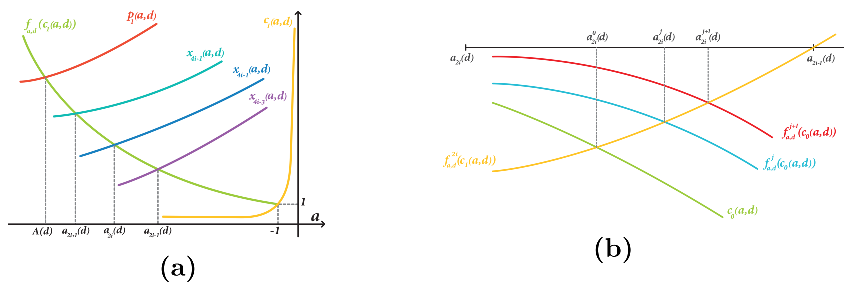

Figure 2: A schematic diagram of
(a) the locations of the sequence $(a_i(d))_{i\geq 1}$ and $A(d)$ and
(b) the locations of the sequence $\big(a^j_{2i}(d)\big)_{j\geq 0}$ in the interval $(a_{2i}(d), a_{2i-1}(d))$.

Lemma 3.5. *Let*

1. $f_{a,d}\big(c_1(a,d)\big) \in J_{2n-1}(a,d)$ *for some* $n \geq 1$, *or*
2. $f_{a,d}\big(c_1(a,d)\big) = p_1(a,d)$, *or*
3. $f_{a,d}\big(c_1(a,d)\big) \in I_{2n}(a,d)$ *for some* $n \geq 1$, *and* $f^{2n+1}_{a,d}\big(c_1(a,d)\big) = c_1(a,d)$.

Then conditions (2) and (3) of Theorem 3.1 are satisfied.

In the following we are going to state conditions needed for defining an order preserving bijection on the union of the orbits of critical points.

Proposition 3.6. *For given* d, *there is a unique* $a < -1$ *such that* $p_1(a,d) = f_{a,d}\big(c_1(a,d)\big)$. *We denote this unique negative number by* $A(d)$ *(see Figure 2(a)).*

Proof. We know $p_1(-1,d) > 1$ and $f_{-1,d}(c_1(-1,d)) = c_1(-1,d) = 1$. Hence, $p_1(-1,d) > f_{-1,d}\big(c_1(-1,d)\big)$. Also, as functions of the variable a, $p_1(a,d)$ is a continuous increasing function and for $a < -1$, $f_{a,d}\big(c_1(a,d)\big)$ is a continuous decreasing function. Moreover, $f_{a,d}\big(c_1(a,d)\big)$ tends to $+\infty$, when $a \to -\infty$. Therefore, by the Intermediate Value Theorem, the graph of $f_{a,d}\big(c_1(a,d)\big)$ intersects the graph of $p_1(a,d)$ at a unique point with the a-coordinate called $A(d)$. □

In the following we are going to divide the parameter interval $(A(d), 0)$ into an infinite number of subintervals. These subintervals determine the location of $f_{a,d}\big(c_1(a,d)\big)$.

Proposition 3.7. *For given d, there is a decreasing sequence $\big(a_i(d)\big)_{i\geq 1}$ such that $A(d) < a_i(d) < -1$, $f_{a,d}\big(c_1(a,d)\big) \in J_{2i-1}(a,d)$ if $a_{2i}(d) \leq a \leq a_{2i-1}(d)$ and $f_{a,d}\big(c_1(a,d)\big) \in I_{2i}(a,d)$ if $a_{2i+1}(d) < a < a_{2i}(d)$ (see Figure 2(a)).*

Proof. By Proposition 2.2, for $i \geq 1$, we have $1 < x_{2i-1}(a,d) < p_1(a,d)$. On the other hand, we have $f_{-1,d}\big(c_1(-1,d)\big) = 1$ and by Proposition 3.6 we have $f_{A(d),d}\big(c_1(A(d),d)\big) = p_1(A(d),d)$. Now, since $x_{2i-1}(a,d)$ is a continuous increasing function and for $a < -1$, $f_{a,d}\big(c_1(a,d)\big)$ is a continuous decreasing function (as functions of the variable a), by the Intermediate Value Theorem, the graph of $f_{a,d}\big(c_1(a,d)\big)$ intersects the graph of $x_{2i-1}(a,d)$ at a unique point with the a-coordinate denoted by $a_i(d)$ such that $A(d) < a_i(d) < -1$. Also, $a_{i+1}(d) < a_i(d)$ since $x_{2i-1}(a,d) < x_{2i+1}(a,d)$. Therefore, if $a_{2i}(d) \leq a \leq a_{2i-1}(d)$, then $x_{4i-3}(a,d) \leq f_{a,d}\big(c_1(a,d)\big) \leq x_{4i-1}(a,d)$ and if $a_{2i+1}(d) < a < a_{2i}(d)$, then $x_{4i-1}(a,d) < f_{a,d}\big(c_1(a,d)\big) < x_{4i+1}(a,d)$. Thus, for $i \geq 1$, in the first case $f_{a,d}\big(c_1(a,d)\big) \in J_{2i-1}(a,d)$ and in the second case $f_{a,d}\big(c_1(a,d)\big) \in I_{2i}(a,d)$ (see the definitions of $I_n(a,d)$'s and $J_n(a,d)$'s in the Proposition 2.2). □

Lemma 3.8. *Let d be given and $a_i(d)$'s be the ones that are introduced in Proposition 3.7. Then for each $i \geq 1$ there is an increasing sequence $\big(a^j_{2i}(d)\big)_{j\geq 0}$ in the interval $(a_{2i}(d), a_{2i-1}(d))$ such that if $a \in \Big(a^j_{2i}(d), a^{j+1}_{2i}(d)\Big)$, then $f^j_{a,d}\big(c_0(a,d)\big) < f^{2i}_{a,d}\big(c_1(a,d)\big) < f^{j+1}_{a,d}\big(c_0(a,d)\big)$ (see Figure 2(b)).*

Proof. By (3) and (4), we have $f^{2i}_{a,d}\big(c_1(a,d)\big) = 0$ for $a = a_{2i-1}(d)$ and $f^{2i}_{a,d}\big(c_1(a,d)\big) = x_0(a,d)$ for $a = a_{2i}(d)$. Also, by employing induction on i, we can show that as a function of the variable a, for $i \geq 0$, $f^{2i}_{a,d}\big(c_1(a,d)\big)$ is increasing if $a < a_{2i-1}(d)$ and $f^{2i+1}_{a,d}\big(c_1(a,d)\big)$ is decreasing if $a < a_{2i}(d)$. On the other hand, as a function of the variable a, $f^n_{a,d}\big(c_0(a,d)\big)$ is decreasing and, moreover $\lim_{a\to-\infty} f^n_{a,d}\big(c_0(a,d)\big) = 0$, where $n \geq 0$.

Now, since $x_0(a,d) < f^j_{a,d}\big(c_0(a,d)\big) < 0$ for $j \geq 0$, by employing the Intermediate Value Theorem, the graph of $f^{2i}_{a,d}\big(c_1(a,d)\big)$ intersects the graph of $f^j_{a,d}\big(c_0(a,d)\big)$ at a unique point with the a-coordinate denoted by $a^j_{2i}(d)$ such that $a^j_{2i}(d) \in (a_{2i}(d), a_{2i-1}(d))$. Also, we know that $f^{j}_{a,d}\big(c_0(a,d) < f^{j+1}_{a,d}\big(c_0(a,d)$, therefore we conclude that $a^{j}_{2i}(d) < a^{j+1}_{2i}(d)$. $\square$

Definition 3.9. *For $j \geq 0$, we set $\widehat{A^{j}_{2i}}(d) = \Big(a^{j}_{2i}(d), a^{j+1}_{2i}(d)\Big)$, $\widetilde{A^{j}_{2i}}(d) = \{a^{j}_{2i}(d)\}$, $A^{j}_{2i}(d) = \widehat{A^{j}_{2i}}(d) \cup \widetilde{A^{j}_{2i}}(d)$ and for for $j \geq 1$, we set $\widehat{A^{-j}_{2i}}(d) = \{a \in \big(a_{2i}(d), a^{0}_{2i}(d)\big) : f^{j}_{a,d}\big(c_0(a,d)\big) < f^{2i+1}_{a,d}\big(c_1(a,d)\big) < f^{j+1}_{a,d}\big(c_0(a,d)\big)\}$, $\widetilde{A^{-j}_{2i}}(d) = \{a \in \big(a_{2i}(d), a^{0}_{2i}(d)\big) : f^{j}_{a,d}\big(c_0(a,d)\big) = f^{2i+1}_{a,d}\big(c_1(a,d)\big)\}$, and $A^{-j}_{2i}(d) = \widehat{A^{-j}_{2i}}(d) \cup \widetilde{A^{-j}_{2i}}(d)$.*

The graph of $f^{2i+1}_{a,d}\big(c_1(a,d)\big)$ intersects the graph of $f^{j+1}_{a,d}\big(c_0(a,d)\big)$, hence $\widetilde{A^{-j}_{2i}}(d) \neq \emptyset$ and $\widehat{A^{-j}_{2i}}(d)$ contains an interval. For each $i \geq 1$, we have the following corollary that determines the location of $f^{2i+1}_{a,d}\left(c_1(a,d)\right)$ among the intervals $\Big(f^{n}_{a,d}\big(c_0(a,d)\big), f^{n+1}_{a,d}\big(c_0(a,d)\big)\Big)$.

Corollary 3.10. *Let $a_i(d)$'s be the ones that are introduced in Proposition 3.7. Then for each $i \geq 1$ there is a sequence $\Big(A^{j}_{2i}(d)\Big)_{j\in\mathbb{Z}}$ of sets that partitions the interval $(a_{2i}(d), a_{2i-1}(d))$ such that if $a \in A^{j}_{2i}(d)$ for some $j \geq 0$, then $f^{j}_{a,d}\big(c_0(a,d)\big) \leq f^{2i}_{a,d}\big(c_1(a,d)\big) < f^{j+1}_{a,d}\big(c_0(a,d)\big)$ and if $a \in A^{-j}_{2i}(d)$ for some $j \geq 1$, then $f^{j}_{a,d}\big(c_0(a,d)\big) \leq f^{2i+1}_{a,d}\big(c_1(a,d)\big) < f^{j+1}_{a,d}\big(c_0(a,d)\big)$.*

Proposition 3.11. *Suppose that $d \geq 2$ is given. For each $i \geq 0$ there is a unique $a \in (a_{2i+1}(d), a_{2i}(d))$ such that $f^{2i+1}_{a,d}\big(c_1(a,d)\big) = c_1(a,d)$.*

Proof. From (3) and (4) we conclude that $f^{2i+1}_{a,d}\big(c_1(a,d)\big) = 0$ for $a = a_{2i}(d)$ and $f^{2i+1}_{a,d}\big(c_1(a,d)\big) = x_1(a,d)$ for $a = a_{2i+1}(d)$, when $i \geq 1$. Now, since $c_1(a,d)$ is increasing and for $a < -1$ and $f_{a,d}\big(c_1(a,d)\big)$ is decreasing, by Intermediate Value Theorem, the graph of $f^{2i+1}_{a,d}\big(c_1(a,d)\big)$ intersects the graph of $c_1(a,d)$ at a unique point that belongs to the interval $(a_{2i+1}(d), a_{2i}(d))$. Note that for $i = 0$ we have $f_{-1,d}(c_1(-1,d)) = c_1(-1,d)$. $\square$

3.3 Conjugacy

In the following we state three theorems that help us to prove the Main Theorem. To prove these theorems, first we define a topological conjugacy from $[p_0(a,d), p_1(a,d)]$ to $[p_0(a',d'), p_1(a',d')]$ by employing Theorem 3.1. Next, to define a topological conjugacy from $\mathbb{R} \setminus [p_0(a,d), p_1(a,d)]$ to $\mathbb{R} \setminus [p_0(a',d'), p_1(a',d')]$ we consider fundamental domains in $(p_1(a,d), +\infty)$ and $(p_1(a',d'), +\infty)$ and define a topological conjugacy between them, then we extend this conjugacy from $\mathbb{R} \setminus [p_0(a,d), p_1(a,d)]$ to $\mathbb{R} \setminus [p_0(a',d'), p_1(a',d')]$ (see [5] for more information).

Theorem 3.12. *Suppose that the even integers $d, d' \geq 2$ are given. Let $a, a' \in \mathbb{R}^-$ satisfy one of the following conditions.*

(1) $a = a_i(d)$ and $a' = a_i(d')$ for some $i \geq 1$.

(2) $a \in \widehat{A^j_{2i}}(d)$ and $a' \in \widehat{A^j_{2i}}(d')$ for some $i \geq 1$ and some $j \in \mathbb{Z}$.

(3) $a \in \widetilde{A^j_{2i}}(d)$ and $a' \in \widetilde{A^j_{2i}}(d')$ for some $i \geq 1$ and some $j \in \mathbb{Z}$.

Then $f_{a,d}$ and $f_{a',d'}$ are conjugate.

Proof. Define $h : \cup_{i=0}^{1} \cup_{n \geq 0} f^n_{a,d}(c_i(a,d)) \to \cup_{i=0}^{1} \cup_{n \geq 0} f^n_{a',d'}(c_i(a',d'))$ such that $h(f^n_{a,d}(c_i(a,d))) = f^n_{a',d'}(c_i(a',d'))$. By Proposition 3.7 and Corollary 3.10, h is an order preserving bijection and in all cases, by Lemma 3.5, the conditions (2) and (3) of Theorem 3.1 are satisfied for the functions $f_{a,d}$ and $f_{a',d'}$.

Moreover, 0 is a one-sided attractor for both $f_{a,d}$ and $f_{a',d'}$. Also, $f_{a,d}$ and $f_{a',d'}$ have no intervals consisting of periodic points of constant period and, by Theorem 3.2, have no wandering intervals. Thus, by Theorem 3.1, $f_{a,d}$ on $[p_0(a,d), p_1(a,d)]$ and $f_{a',d'}$ on $[p_0(a',d'), p_1(a',d')]$ are conjugate, in all cases.

Now we extend the conjugacy to $\mathbb{R}$ by the following method. We choose $t_0 > p_1(a,d)$ and $t'_0 > p_1(a',d')$ arbitrarily. Let $\mathcal{F} = \big(t_0, f^2_{a,d}(t_0)\big]$ and $\mathcal{F}' = \big(t'_0, f^2_{a',d'}(t'_0)\big]$. Define $\mathcal{K} : \overline{\mathcal{F}} \to \overline{\mathcal{F}'}$ linearly such that $\mathcal{K}(t_0) = t'_0$ and $\mathcal{K}\big(f^2_{a,d}(t_0)\big) = f^2_{a',d'}(t'_0)$. Since $f^2_{a,d}$ and $f^2_{a',d'}$ are increasing on $\big(p_1(a,d), \infty\big)$ and $\big(p_1(a',d'), \infty\big)$, respectively, we can find the sequences

$(t_{2n})_{n\in\mathbb{Z}}$ and $(t'_{2n})_{n\in\mathbb{Z}}$ such that for $j \geq 0$, $t_{2j} = f_{a,d}^{2j}(t_0)$, $f_{a,d}^{2j}(t_{-2j}) = t_0$, $t'_{2j} = f_{a',d'}^{2j}(t'_0)$, and $f_{a',d'}^{2j}(t'_{-2j}) = t'_0$. Therefore, $\big(p_1(a,d),\infty\big) = \cup_{j\in\mathbb{Z}}(t_{2j}, t_{2(j+1)}]$ and $\big(p_1(a',d'),\infty\big) = \cup_{j\in\mathbb{Z}}(t'_{2j}, t'_{2(j+1)}]$. For every $x \in \big(p_1(a,d),\infty\big)$ there is a unique $j \in \mathbb{Z}$ such that $x \in (t_{2j}, t_{2(j+1)}]$. For simplicity, we show the function $f_{a,d}|_{\left(-\infty,p_0(a,d)\right)\cup\left(p_1(a,d),\infty\right)}$ by $f_{a,d}$ and the function $f_{a',d'}|_{\left(-\infty,p_0(a',d')\right)\cup\left(p_1(a',d'),\infty\right)}$ by $f_{a',d'}$. Then we define $\mathcal{K}(x) = f_{a',d'}^{2j}\Big(\mathcal{K}\big(f_{a,d}^{-2j}(x)\big)\Big)$. Finally for $x \in \big(-\infty, p_0(a,d)\big)$, we define $\mathcal{K}(x) = f_{a',d'}^{-1}\big(\mathcal{K}(f_{a,d}(x)\big)$, $\mathcal{K}\big(p_0(a,d)\big) = p_0(a',d')$, and $\mathcal{K}\big(p_1(a,d)\big) = p_1(a',d')$. □

Theorem 3.13. *Suppose that $d, d' \geq 2$ are even integers, $a_{2i+1}(d) < a < a_{2i}(d)$, and $a_{2i+1}(d') < a' < a_{2i}(d')$, for some $i \geq 0$, where $a_0(d) = a_0(d') = 0$. If a and a' are such that $f_{a,d}^{2i+1}\big(c_1(a,d)\big) = c_1(a,d)$, and $f_{a',d'}^{2i+1}\big(c_1(a',d')\big) = c_1(a',d')$, then $f_{a,d}$ and $f_{a',d'}$ are conjugate.*

Proof. For $i \geq 1$, one can define the order preserving bijection h as Theorem 3.12. By Lemma 3.5 all the conditions of Theorem 3.1 are satisfied. Hence, $h : [p_0(a,d), p_1(a,d)] \to [p_0(a',d'), p_1(a',d')]$ is a conjugacy that can be extended to $\mathbb{R}$.

For $i = 0$ the immediate basins of $c_1(a,d)$ and the non-hyperbolic fixed point 0 have common boundary, so, we consider the fundamental domains in $(0, c_1(a,d))$, $(c_0(a,d), 0)$, $(0, c_1(a',d'))$, and $(c_0(a',d'), 0)$ and define a topological conjugacy from $(0, c_1(a,d))$ to $(0, c_1(a',d'))$ and a topological conjugacy from $(c_0(a,d), 0)$ to $(c_0(a',d'), 0)$. Then we extend it to a topological conjugacy from $[p_0(a,d), p_1(a,d)]$ to $[p_0(a',d'), p_1(a',d')]$. To extend the topological conjugacy to $\mathbb{R}$, we consider fundamental domains in $(p_1(a,d), +\infty)$ and $(p_1(a',d'), +\infty)$. □

Theorem 3.14. *Suppose that $d, d' \geq 2$ are given. Then $f_{A(d),d}$ and $f_{A(d'),d'}$ are conjugate.*

Proof. In this case, also, by Lemma 3.5 all the conditions of Theorem 3.1 are satisfied. Similar to Theorem 3.12 we can construct the conjugacy. □

Now we are ready to state and prove the Main Theorem.

Theorem 3.15. *(**Main Theorem**) Suppose that even integers $d, d' \geq 2$ are given and for $a \in \mathbb{R}^-$ one of the following conditions is satisfied.*

1. $f_{a,d}(c_1(a,d)) \in J_{2i-1}(a,d)$ *for some* $i \geq 1$.

2. $f_{a,d}(c_1(a,d)) \in I_{2i}(a,d)$ *and* $f_{a,d}^{2i+1}\big(c_1(a,d)\big) = c_1(a,d)$ *for some* $i \geq 0$.

3. $f_{a,d}(c_1(a,d)) = p_1(a,d)$.

Then there is an $a' \in \mathbb{R}^-$ *such that* $f_{a,d}$ *and* $f_{a',d'}$ *are topologically conjugate.*

Proof. Let even integers $d, d' \geq 2$ are given and condition (3) of the Main Theorem is satisfied. By Proposition 3.6, $a = A(d)$. We set $a' = A(d')$. Now, by employing Theorem 3.14, the Main Theorem holds in this case.

Next, we partition $(A(d), 0)$ and $(A(d'), 0)$ into subsets defined in Proposition 3.7 and Corollary 3.10. Now if a satisfies condition (1) of the Main Theorem, then $a = a_i(d)$ for some $i \geq 1$, or there are $i \geq 1$ and $j \in \mathbb{Z}$ such that $a \in A_{2i}^j(d)$. In the first case we set $a' = a_i(d')$ and in the second case we choose $a' \in A_{2i}^j(d')$, then by Theorem 3.12, the Main Theorem holds in this case, as well.

Finally, let a satisfies condition (2) of the Main Theorem. We choose a' by Proposition 3.11. Then Theorem 3.13 guarantees that $f_{a,d}$ and $f_{a',d'}$ are conjugate. □

4 Topological Entropy in This Family

In [7] for $a < 0$, we presented some algorithms for computing the topological entropy of $f_{a,2}$ in the following cases

(1) $f_{a,2}\big(c_1(a,2)\big) \in J_{2n-1}(a,2)$ for some $n \geq 1$.

(2) $f_{a,2}\big(c_1(a,2)\big) \in I_{2n}(a,2)$ and $f_{a,2}^{2n+1}\big(c_1(a,2)\big) = c_1(a,2)$ for some $n \geq 1$.

(3) $f_{a,2}\big(c_1(a,2)\big) = p_1(a,2)$.

Table 1: The estimation of the topological entropy of $f_{a,d}$, for $1 \leq n \leq 5$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$f_{a,d}(c_1(a,d) \in J_{2n-1}(a,d)$	log 2	log 2.360	log 2.406	log 2.413	log 2.415
$f_{a,d}(c_1(a,d)) \in I_{2n}(a,d)$ $f_{a,d}^{2n+1}(c_1(a,d)) = c_1(a,d)$	log 2.207	log 2.384	log 2.410	log 2.414	log 2.415

We employ the Main Theorem of this paper and Corollaries 3.3, 3.4, and Proposition 1 of [7] and conclude the following corollary.

Corollary 4.1. *Let $d \geq 2$ be an even integer and a be a negative real number.*

1. *If $f_{a,d}(c_1(a,d)) \in J_1(a,d)$, then the entropy of $f_{a,d}$ is $\log 2$.*
2. *For each $n \geq 1$, the entropy of $f_{a,d}$ is constant when $f_{a,d}(c_1(a,d)) \in J_{2n-1}(a,d)$.*
3. *If $f_{a,d}(c_1(a,d)) = p_1(a,d)$, then the entropy of $f_{a,d}$ is $\log(1+\sqrt{2})$.*

Table 1 represented in [7] that shows an estimation of the topological entropy of $f_{a,2}$, in different cases for $1 \leq n \leq 5$, can be used for $f_{a,d}$ when $d \geq 2$ is an even number.

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Maryam Rabii
Assistant Professor of Mathematics
Department of Mathematics, Faculty of Mathematical Sciences
Alzahra University
Tehran, Iran
E-mail: mrabii@alzahra.ac.ir

Monireh Akbari
Assistant Professor of Mathematics
Department of Mathematics
Shahid Rajaee Teacher Training University
Tehran, Iran
E-mail: akbari@sru.ac.ir