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Pascal-Like Triangle and Pascal-Like Functional Matrix

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Abstract. In this paper we shall first introduce the Pascal-like triangle, using a generalization of the recurrence relation for arrays of Pascal triangle. Then we define the Pascal-like functional and Fermat-like matrices and investigate their algebraic properties. Finally, we obtain some binomial identities, using these matrices.

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1. Introduction

The Pascal functional matrices for one, two and three variables have been introduced in [12]. Another generalization of these matrices as the Pascal k-eliminated functional matrices has been presented in [5]. Furthermore, considering the sequence $\{f_n(x)\}_{n\geqslant 1}$ which satisfy the following recurrence relation

$$f_n(x+y) = \sum_{i=0}^{n} f_i(x) f_{n-i}(y),$$

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and boundary condition

$$f_0(x) = 1,$$

another interesting generalization has been made in [15]. There are many interesting applications of the Pascal matrices in the literature [4]. Furthermore, several interesting triangular arrays are defined based on Pascal like recurrence relations.

Therefore introducing new kind of Pascal matrices using a generalization of the Pascal triangle is an impotent task.

In this paper we first proceed by generalizing the recurrence relation for the entries of the Pascal triangle. Then we define Pascal-like functional and the Fermat-like matrices and present their properties. Finally, using the linear algebra ideas, we obtain several interesting combinatorial identities.

2. Pascal-Like Triangle

Definition 2.1. *It is well-know that the Pascal triangle is obtained by the following two dimensional linear recurrence relation:*

$$p_{n+1,m+1} = p_{n,m} + p_{n,m+1} \quad (n \geqslant m \geqslant 0)$$

and boundary conditions,

$$p_{n,0} = 1 \quad (n \geqslant 0), \quad \text{and} \quad p_{n,n} = 1 \quad (n \geqslant 1).$$

The above recurrence relation forms the following triangular array of numbers:

Table 1. Pascal triangle

$\downarrow n \setminus m \rightarrow$	0	1	2	3	4	$\cdots$
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$

A natural generalization of the above triangular array of numbers is described in Figure 1.

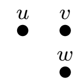

Figure 1. $w = u + \lambda v \qquad (\lambda \in \mathbb{Z})$

Clearly, in this case we have the following two dimensional linear recurrence relation: (It is important to note that we haven't lost the linearity yet):

$$\widetilde{p}_{n+1,m+1} = \widetilde{p}_{n,m} + \lambda \widetilde{p}_{n,m+1} \qquad (n \geqslant m \geqslant 0,\ \lambda \in \mathbb{Z}),$$
$$\widetilde{p}_{n,0} = 1\ (n \geqslant 0), \text{ and } \widetilde{p}_{n,n} = 1\ (n \geqslant 1) \tag{1}$$

For example, in the case $\lambda = 2$, we have the following array of numbers

Table 2. Pascal-like triangle for $\lambda = 2$

$\downarrow n \setminus m \rightarrow$	0	1	2	3	4	$\cdots$
0	1					
1	1	1				
2	1	3	1			
3	1	7	5	1		
4	1	15	17	7	1	
$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$

In general case λ, we call the above triangle the Pascal-like triangle associated with the parameter λ.
Also, on may, one may assume that $\lambda \in \mathbb{R} - \{0\}$. But the case $\lambda \in \mathbb{Z}$, for obtaining some combinatorial identities is very useful.

3. Main Properties

Lemma 3.1. *The ordinary horizontal generating function of* $\{\widetilde{p}_{n,k}\}$,

$$B_n(x) = \sum_{k \geqslant 0} \widetilde{p}_{n,k} x^k \qquad (n \geqslant 0) \tag{2}$$

satisfies the difference equation,

$$B_{n+1}(x) = (x+\lambda)B_n(x) + 1 - \lambda \tag{3}$$

where $n = 0, 1, 2, \ldots$; *and* $B_0(x) = 1$.

Proof. The property $B_0(x) = 1$, is obvious from (1). Now making use of definition, we have

$$\sum_{k\geqslant 0} \widetilde{p}_{n+1,k+1} x^k = \sum_{k\geqslant 0} \widetilde{p}_{n,k} x^k + \lambda \sum_{k\geqslant 0} \widetilde{p}_{n,k+1} x^k.$$

This may be rewritten in the form,

$$\sum_{k\geqslant 1} \widetilde{p}_{n+1,k} x^{k-1} = \sum_{k\geqslant 0} \widetilde{p}_{n,k} x^k + \lambda \sum_{k\geqslant 0} \widetilde{p}_{n,k} x^{k-1},$$

which is identical to the following:

$$\begin{aligned} \sum_{k\geqslant 1} \widetilde{p}_{n+1,k} x^k &= xB_n(x) + \lambda \sum_{k\geqslant 1} \widetilde{p}_{n,k} x^k \\ \sum_{k\geqslant 0} \widetilde{p}_{n+1,k} x^k - \widetilde{p}_{n+1,0} &= xB_n(x) + \lambda\Big(\sum_{k\geqslant 0} \widetilde{p}_{n,k} x^k - \widetilde{p}_{n,0}\Big) \end{aligned}$$

or equivalently,

$$B_{n+1}(x) = xB_n(x) + \lambda B_n(x) + 1 - \lambda,$$

and finally,

$$B_{n+1}(x) = (x+\lambda)B_n(x) + 1 - \lambda.$$

Hence the lemma is proved. □

Theorem 3.2. *The Pascal-like triangle entries* $\widetilde{p}_{n,k}$*'s have the ordinary horizontal generating function,*

$$\sum_{k\geqslant 0} \widetilde{p}_{n,k} x^k = \frac{x(x+\lambda)^n}{x+\lambda-1} + \frac{\lambda - 1}{x+\lambda-1} \qquad (n \geqslant 0). \tag{4}$$

Moreover, we have the following explicit formula for $\widetilde{p}_{n,k}$'s:

$$\widetilde{p}_{n,k} = \sum_{l=k}^{n} \binom{l-1}{k-1} \lambda^{l-k} \quad (n \geqslant k \geqslant 1). \tag{5}$$

Proof. Let the right hand side (RHS) of (2) be denoted by $\Phi_n(x)$. Notice that (2) has the unique solution $\Phi_n(x)$ under the condition $\Phi_0(x) = 1$. Thus it suffices to show that $\Phi_n(x)$ is the unique solution of (2), such that $\Phi_n(x) = B_n(x)$. Evidently, $\Phi_0(x) = 1$. Moreover, using elementary algebraic computations, we can verify that

$$\Phi_n(x) = (x+\lambda)\Phi_{n-1}(x) + 1 - \lambda.$$

To prove formula (5), since $B_n(x)$ is equal to

$$\frac{x(x+\lambda)^n + \lambda - 1}{x + \lambda - 1} = \frac{x((x+\lambda)^n - 1) + x + \lambda - 1}{x + \lambda - 1},$$

then by division algorithm, we have

$$B_n(x) = x(x+\lambda)^{n-1} + x(x+\lambda)^{n-2} + \cdots + x(x+\lambda)^0 + 1.$$

Finally, using the Newton's binomial expansion, we obtain

$$\widetilde{p}_{n,k} = \binom{n-1}{k-1}\lambda^{n-k} + \binom{n-2}{k-1}\lambda^{n-k-1} + \cdots + \binom{k-1}{k-1}\lambda^0 \quad (n \geqslant k \geqslant 1),$$

or equivalently,

$$\widetilde{p}_{n,k} = \sum_{l=k}^{n} \binom{l-1}{k-1} \lambda^{l-k} \quad (n \geqslant k \geqslant 1). \quad \Box$$

Now, we intend to define the Pascal-like functional matrix, for one variable, using the Pascal-like triangle as we have done it for the Pascal functional matrix [4].

4. Pascal-Like Functional Matrix

Definition 4.1. *Suppose n is a natural number let $\lambda > 0$ be a real positive number. Let x be an element of also a real number. We define a Pascal-like functional matrix of order $(n+1)\times(n+1)$ with one variable x, as follows:*

$$(PL_n[x;\lambda])_{i,j} = \begin{cases} \widetilde{p}_{i,j}x^{i-j} & \text{if } i \geqslant j \geqslant 0, \\ 0 & \text{if } j > i, \end{cases}$$

in which $\widetilde{p}_{i,j}$ is the (i,j)-entry of the Pascal-like triangle.

Example 4.2. The Pascal-like functional matrix of order 4×4 and $\lambda = 2$ is,

$$PL_3[x;2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 3x & 1 & 0 \\ x^3 & 7x^2 & 5x & 1 \end{bmatrix}.$$

Remark 4.3. *Consider the properties of the Pascal-like triangle, we are able to present the Pascal-like functional matrix by the following explicit formula*

$$(PL_n[x;\lambda])_{i,j} = \begin{cases} x^i & \text{if } i \geqslant 0, j = 0, \\ x^{i-j}\left(\sum_{l=j}^{i}\binom{l-1}{j-1}\lambda^{l-j}\right) & \text{if } i \geqslant j \geqslant 1, \\ 0 & \text{if } j > i. \end{cases}$$

Remark 4.4. *Using the identity $\sum_{t=0}^{n}\binom{t+a}{a} = \binom{n+a+1}{a+1}$ (see [8]), in the special case $\lambda = 1$, we have the following simple formula for $PL_n[x;1]$*

$$(P_n[x;1])_{i,j} = \begin{cases} \binom{i}{j}x^{i-j} & \text{if } i \geqslant j \geqslant 0, \\ 0 & \text{if } j > i, \end{cases}$$

the above matrix is also called the Pascal functional matrix and is denoted by $P_n[x]$(see [4]).

5. Main Results

In [7] the authors have shown that the matrix $P_n[x]$ has an exponential property, i.e.

$$P_n[x]P_n[y] = P_n[x+y].$$

Unfortunately in general case ($\lambda \neq 1$), we have never an exponential property, but we have another interesting property. Indeed, these matrices are factored into the Pascal functional matrices.
Before starting to present our main theorem, we need to state the following lemma.

Lemma 5.1. *Suppose $\alpha, \alpha', \beta, \beta'$ are four real numbers. Also let $A = [a_{ij}]$, $B = [b_{ij}]$ be two lower triangular matrices, where defined by the following recurrence relations respectively,*

$$\begin{cases} a_{n,k} = \alpha a_{n-1,k-1} + \beta a_{n-1,k}, & (n \geqslant k \geqslant 1), \\ a_{n,0} = 1 & n \geqslant 0, \\ a_{n,k} = 0 & k > n, \end{cases} \tag{6}$$

$$\begin{cases} b_{n,k} = \alpha' b_{n-1,k-1} + \beta' b_{n-1,k}, & (n \geqslant k \geqslant 1), \\ b_{n,0} = 1 & n \geqslant 0, \\ b_{n,k} = 0 & k > n. \end{cases} \tag{7}$$

If $AB = [c_{ij}]$ then, there exist real numbers $\alpha'' = \alpha\alpha'$ and $\beta'' = \beta + \alpha\beta'$, such that

$$\begin{cases} c_{n,k} = \alpha'' c_{n-1,k-1} + \beta'' c_{n-1,k}, & (n \geqslant k \geqslant 1), \\ c_{n,0} = \sum_{i=0}^{n} a_{n,i} & n \geqslant 0, \\ c_{n,k} = 0 & k > n. \end{cases} \tag{8}$$

Proof. Considering the uniqueness of the solution of (1), it suffices to show that

$$\alpha\alpha' c_{n-1,k-1} + (\beta + \alpha\beta')c_{n-1,k} = c_{n,k}$$

but using the definition of the matrix product and relations (6)-(8), we have

$$\begin{aligned}
\alpha\alpha' c_{n-1,k-1} &+ (\beta+\alpha\beta')c_{n-1,k} \\
&= \alpha\alpha'\sum_{l=0}^{n-1} a_{n-1,l}b_{l,k-1} + (\beta+\alpha\beta')\sum_{l=0}^{n-1} a_{n-1,l}b_{l,k} \\
&= \sum_{l=0}^{n-1}[\alpha\alpha' a_{n-1,l}b_{l,k-1} + \alpha\beta' a_{n-1,l}b_{l,k} + \beta a_{n-1,l}b_{l,k}] \\
&= \sum_{l=0}^{n-1}[\alpha a_{n-1,l}(\alpha' b_{l,k-1} + \beta' b_{l,k}) + \beta a_{n-1,l}b_{l,k}] \\
&= \sum_{l=0}^{n-1}[\alpha a_{n-1,l}b_{l+1,k} + \beta a_{n-1,l}b_{l,k}] \\
&= \sum_{l=0}^{n-1}\alpha a_{n-1,l}b_{l+1,k} + \sum_{l=0}^{n-1}\beta a_{n-1,l}b_{l,k} \\
&= \sum_{l=1}^{n}\alpha a_{n-1,l-1}b_{l,k} + \sum_{l=1}^{n}\beta a_{n-1,l}b_{l,k} \\
&= \sum_{l=1}^{n}(\alpha a_{n-1,l-1} + \beta a_{n-1,l})b_{l,k} = \sum_{l=1}^{n} a_{n,l}b_{l,k} \\
&= \sum_{l=0}^{n} a_{n,l}b_{l,k} = c_{n,k}. \quad \Box
\end{aligned}$$

Now, we are at the position to state our main theorem.

Theorem 5.2. *For any positive integer $\lambda \neq 1$, we have*

$$P_n[-x]PL_n[x;\lambda] = \bar{P}_1[(\lambda-1)x],$$

where $P_n[x]$ is the Pascal functional matrix and $\bar{P}_k(x)$ is defined by

$$\bar{P}_k(x) = \begin{bmatrix} I_{n-k} & O \\ O & P_k(x) \end{bmatrix}.$$

Proof. Put $C_{ij}(x) = (P_n[-x]PL_n[x;\lambda])_{i,j}$. Considering the well-known identity $\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l} = \delta_{i,0}$, we obtain that $C_{i,0}(x) = \delta_{i,0}$. Thus it is necessary to show that $P_{n,1}[-x]PL_{n,1}[x;\lambda] = P_n[(\lambda-1)x]$ in which $P_{n,1}[x]$ and $PL_{n,1}[x;\lambda]$ are the Pascal 1-eliminated and the Pascal-like 1-eliminated functional matrices which are obtained from $P_n[x]$ and $PL_n[x;\lambda]$ by omitting their first row and column respectively. Now, applying Lemma 5, since the entries of $P_{n,1}[x;-1]$ and $PL_{n,1}[x;\lambda]$ satisfy the following recurrence relations respectively

$$\begin{aligned} a_{n,k} &= a_{n-1,k-1} - 1a_{n-1,k} \qquad (n \geqslant k \geqslant 1) \\ b_{n,k} &= b_{n-1,k-1} + \lambda b_{n-1,k}. \end{aligned}$$

Thus, $\alpha'' = 1$ and $\beta'' = 1-\lambda$, we obtain

$$c_{n,k} = c_{n-1,k-1} + (1-\lambda)c_{n-1,k}.$$

Finally, using the definition of the Pascal-like triangle and the uniqueness of the solution of the above difference equation under the mentioned boundary conditions, the proof is complete. $\square$

Considering the Pascal functional matrix property [4], we have immediately the following results:

Corollary 5.3.

$$PL_n[x;\lambda] = P_n[x]\bar{P}_{n,1}[(\lambda-1)x].$$

Corollary 5.4.

$$PL_n^{-1}[x;\lambda] = \bar{P}_{n,1}[-(\lambda-1)x]P_n[-x].$$

Example 5.5.

$$PL_3[x;2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 3x & 1 & 0 \\ x^3 & 7x^2 & 5x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x^2 & 2x & 1 \end{bmatrix}.$$

6. Generalization of the Pascal-Like Functional Matrix for Three Variables

Using the same idea which is used in the definition of the Pascal function matrix for three variables, we generalize the Pascal-like functional matrix as follows:

Definition 6.1. *Suppose n is a natural number and $x, y, z,$ are real numbers. Then we define the matrix $PL_n[x,y,z;\lambda]$ by*

$$(PL_n[x,y,z;\lambda])_{i,j} = \begin{cases} z^{n-i}x^i & \text{if } i \geqslant 0, j=0, \\ \sum_{l=j}^{i} \binom{l-1}{j-1} \lambda^{l-i} x^{i-j} y^j z^{n-i} & \text{if } i \geqslant j \geqslant 1, \\ 0 & \text{if } j > i. \end{cases}$$

Example 6.2.

$$PL_3[x,y,z;2] = \begin{bmatrix} z^3 & 0 & 0 & 0 \\ xz^2 & yz^2 & 0 & 0 \\ x^2z & 3xyz & y^2z & 0 \\ x^3 & 7x^2y & 5xy^2 & y^3 \end{bmatrix}.$$

Remark 6.3. *In special case $\lambda = 1$, $PL_n[x,y,z;1]$ is called the Pascal functional matrix for three variables and we denote it by $P_n[x,y,z]$. As an immediate consequence of the above definition, we have the following lemma.*

Lemma 6.4. *The matrix $PL_n[x,y,z;\lambda]$ can be factored as:*

$$PL_n[x,y,z;\lambda] = \mathrm{diag}(z^n, \cdots, z, 1) PL_n[x;\lambda] \mathrm{diag}(1, y, \cdots, y^n).$$

Lemma 6.5.

$$P_n[x,y,z] = \mathrm{diag}(z^n, \cdots, z, 1) P_n[x] \mathrm{diag}(1, y, \cdots, y^n).$$

Considering the property of the Pascal functional matrix for three variables [12], we get the following result:

Theorem 6.6. *For any positive integer $\lambda \neq 0$, we have*

$$PL_n[x, y, z; \lambda] = PL_n[x, y, z]\bar{P}_{n,1}\Big[(\lambda - 1)\frac{x}{y}\Big].$$

Proof. By Lemma 11 and Theorem 6 , we have

$$\begin{aligned} PL_n[x, y, z; \lambda] &= \operatorname{diag}(z^n, \cdots, z, 1)PL_n[x; \lambda]\operatorname{diag}(1, y, \cdots, y^n) \\ &= \operatorname{diag}(z^n, \cdots, z, 1)P_n[x]\bar{P}_{n,1}[(\lambda - 1)x]\operatorname{diag}(1, y, \cdots, y^n), \end{aligned}$$

but it can be easily seen that

$$\bar{P}_{n,1}[x, y; \lambda]\operatorname{diag}(1, y, \cdots, y^n) = \operatorname{diag}(1, y, \cdots, y^n)\bar{P}_{n,1}\Big[(\lambda - 1)\frac{x}{y}\Big].$$

Thus, using the Lemma 11, we get

$$\begin{aligned} PL_n[x, y, z; \lambda] &= \operatorname{diag}(z^n, \cdots, z, 1)P_n[x]\operatorname{diag}(1, y, \cdots, y^n)\bar{P}_{n,1}\Big[(\lambda - 1)\frac{x}{y}\Big] \\ &= P_n[x, y, z]\bar{P}_{n,1}\Big[(\lambda - 1)\frac{x}{y}\Big]. \quad \square \end{aligned}$$

New, considering properties of the Pascal functional matrix of three variables [12], we obtain the following result:

Corollary 6.7.

$$PL_n^{-1}[x, y, z; \lambda] = \bar{P}_{n,1}\Big[-(\lambda - 1)\frac{x}{y}\Big]PL_n\Big[-xy^{-1}z^{-1}, y^{-1}, z^{-1}\Big].$$

Unfortunately, since the Pascal functional matrix and the Pascal Block functional matrix are not commuted, we are not able to compute the m-th power of the Pascal-like functional matrix by means of Corollary 7 of our main theorem.

In the next section, we move to define another interesting matrix which is closely related to the Pascal-like functional matrix.

7. The Fermat-Like Matrix

The Fermat matrix F_n is defined by [5],

$$(F_n)_{i,j} = \binom{i+j}{j} \quad (i, j = 0, 1, \cdots, n).$$

We have the following well-known Cholesky factorization of F_n (see [5]),

$$F_n = P_n[1]P_n^T[1],$$

in which $P_n[1]$ is a Pascal functional matrix.

Example 7.1.

$$F_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We observe that the diagonals of the Fermat matrix are the rows of the Pascal triangle. Using, the same idea, we define the Fermat-like matrix by

$$(FL_n)_{i,j}[\lambda] = \begin{cases} 1 & \text{if } i \geqslant 0, j = 0, \\ \sum_{l=j}^{i+j} \binom{l-1}{j-1} \lambda^{l-j} & \text{if } i \geqslant 0, j \geqslant 1, \end{cases}$$

Example 7.2. *For* $\lambda = 2$

$$FL_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 \\ 1 & 7 & 17 & 31 \\ 1 & 15 & 49 & 111 \end{bmatrix}.$$

Now we obtain a multiplicative decomposition for Fermat-like matrices.

Theorem 7.3. *The Fermat-like matrix has the following decomposition*

$$FL_n[\lambda] = PL_n[1;\lambda]\mathrm{diag}(1, \lambda, \cdots, \lambda^n)P_n^T[1].$$

Proof. Clearly, by the definition, $(FL_n)_{i,0} = 1 \ \ (i \geqslant 0)$. Thus it suffices to show that,

$$\sum_{l=j}^{i+j} \binom{l-1}{j-1} \lambda^{i-j} = 1 + \sum_{r=1}^{i} \left(\sum_{l=r}^{j} \binom{l-1}{r-1} 2^{l-r} \right) \binom{i}{r} \lambda^r$$

or equivalently,

$$\sum_{l=j+1}^{i+j} \binom{l-1}{j-1} \lambda^{l-j} = \sum_{r=1}^{i} \sum_{l=r}^{i} \binom{l-1}{r-1} \binom{j}{r} \lambda^{l}.$$

To do this, we have

$$\begin{aligned} \text{RHS} &= \sum_{l=1}^{i} \sum_{r=1}^{l} \binom{l-1}{r-1} \binom{j}{r} \lambda^{l} \\ &= \sum_{l=j+1}^{i+j} \sum_{r=1}^{l-j} \binom{l-j-1}{r-1} \binom{j}{r} \lambda^{l-j}. \end{aligned}$$

Now, it is necessary to show that

$$\sum_{r=1}^{l-j} \binom{l-j-1}{r-1} \binom{j}{r} = \sum_{r=0}^{l-j-1} \binom{l-j-1}{r} \binom{j}{j-r-1},$$

but it can be easily proved using the following identity [8]

$$\sum_{k} \binom{n}{k} \binom{p}{m-k} = \binom{n+p}{m}. \quad \square$$

Now, considering properties of the Pascal-like functional matrix and the Pascal functional matrix, we immediately get the following results:

Corollary 7.4.

$$FL_n[\lambda] = P_n[1]\bar{P}_{n,1}[\lambda - 1]\text{diag}(1, \lambda, \cdots, \lambda^n)P_n^T[1].$$

Corollary 7.5.

$$FL_n^{-1}[\lambda] = P_n^T[-1]\text{diag}\Big(1, \frac{1}{\lambda}, \cdots, \frac{1}{\lambda^n}\Big)PB_{n,1}[1-\lambda]P_n[-1].$$

Corollary 7.6.

$$\det(FL_n[\lambda]) = \lambda^{\binom{n+1}{2}}.$$

8. Some Combinatorial Identities

Considering the previous discussions and linear algebra ideas, we obtain some beautiful binomial coefficients identities.

Theorem 8.1. *For any positive real number λ (with $\lambda \neq 1$) and $n \geqslant k \geqslant 1$, we have*

$$\begin{aligned}\sum_{l=k}^{n}\binom{l-1}{k-1}\lambda^{l} &= \sum_{r=k}^{n}\binom{r-1}{k-1}\binom{n}{r}\lambda^{k}(\lambda-1)^{r-k} \\ &= (\frac{\lambda}{\lambda-1})^{k}\sum_{r=k}^{n}\binom{r-1}{k-1}\binom{n}{r}(\lambda-1)^{r}.\end{aligned}$$

Proof. The proof is straightforward by considering the Corollary 16 and the definition of the matrix product. $\square$

Theorem 8.2. *For any nonnegative integer n and positive real number $\lambda \neq 1$, we have*

$$\sum_{l=0}^{n}\binom{l+n-1}{n-1}\lambda^{l} = \sum_{r=0}^{n}\lambda^{r}\binom{n}{r}^{2}.$$

Proof. Considering the matrix equality of Corollary 16, we have

$$(PL_n)_{i,j} = \Big(P_n[1]\bar{P}_{n,1}[\lambda-1]\mathrm{diag}(1,\lambda,\cdots,\lambda^{n})P_n^{T}[1]\Big)_{i,j}$$

and the definition of the matrix product, after a simplification, we obtain

$$\sum_{l=i}^{2i}\binom{l-1}{i-1}\lambda^{l-i} = \sum_{r=0}^{i}\lambda^{r}\binom{i}{r}^{2} \qquad (i=0,1,\cdots,n),$$

or equivalently

$$\sum_{l=0}^{i}\binom{l+i-1}{i-1}\lambda^{l} = \sum_{r=0}^{i}\lambda^{r}\binom{i}{r}^{2}.$$

This completes the proof. $\square$

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