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Original Research Paper

A Modified Picard Iteration Method to Solve Fractional Optimal Control Problems

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Abstract. An effective modification of the Picard iteration method (PIM) is presented for solving linear and nonlinear fractional optimal control problems (FOCP) in the Caputo sense. Here, the control function is first approximated by a finite series with unknown coefficients. Then the modified PIM is utilized to simulate the resulting fractional equations. Finally, the unknown coefficients could be computed by applying an optimization procedure. Some test examples are given to show the accuracy and validity of the method.

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Keywords and Phrases: Picard iteration method, Fractional optimal control problem.

1 Introduction

The classical calculus provides a powerful tool for modeling many important dynamical processes of sciences. However, there are many complex systems in real world with anomalous dynamics, which their dynamics could not be characterized by classical derivative models [8, 7]. Also, it has been shown that the fractional order derivatives can provide more accurate models for many applied systems than integer order ones [14, 9, 11].

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The numerical simulations for such problems have been investigated in, for example, [3, 5, 13, 4]. In the present article, we consider an FOCP with the following general performance index [6]:

$$J(u) = \int_a^b f\left(y(t), u(t), t\right) dt, \tag{1}$$

subject to the dynamical system with the Caputo fractional derivative [2]

$${}_a^C D_t^\alpha y(t) = g\left(y(t), u(t), t\right), \tag{2}$$

and the condition

$$y(a) = y_a. \tag{3}$$

Here we assume that f and g are two continuously differentiable functions w.r.t the time t, the state variable $y(t)$ and the control variable $u(t)$. We intend to directly solve (1)-(3) without using Hamiltonian formulas. Our tools for this purpose are the Taylor expansion for the control variable and the PIM.

Here we present some basic definitions. The left fractional R–L integral operator of order α of a function $z(t)$ can be defined as below [10]:

$${}_a I_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} z(\tau) d\tau, \quad \alpha > 0, \tag{4}$$

and also the definition of the left fractional derivative of $z(t)$ in the Caputo sense is as [10]:

$${}_a^C D_t^\alpha z(t) = \frac{1}{\Gamma(k-\alpha)} \int_a^t (t-\tau)^{k-\alpha-1} z^{(k)}(\tau) d\tau, \quad k-1 < \alpha < k.$$

Some properties of them could be listed here as [10]:

$$\begin{aligned} {}_a I_t^\alpha \left[{}_a^C D_t^\alpha z(t)\right] &= z(t) - \sum_{i=0}^{k-1} z^{(i)}(a) \frac{t^i}{i!}, \quad k-1 < \alpha \leq k, \\ {}_a^C D_t^\alpha \left[{}_a I_t^\alpha z(t)\right] &= z(t). \end{aligned} \tag{5}$$

2 Main Results

The method we would like to introduce here is based upon expanding $u(t)$ by the Taylor finite series with some unknown coefficients, i.e.,

$$u_N(t) = \sum_{k=0}^{N} d_k t^k + \mathrm{O}(t^{N+1}), \tag{6}$$

where d_k are the unknown coefficients to be determined. Then a fractional version of the PIM for (2) can be utilized to approximate the state variable $y(t)$. Finally, by substituting the above approximations of $y(t)$ and $u(t)$ in (2), we will have an optimization problem. It could be then solved by means of any classical optimization algorithm.

The relations of (2) and (3), in view of (5), can be expressed as below:

$$y(t) = y(a) + {}_aI_t^{\alpha} \left[g\left(y(t), u(t), t \right) \right],$$

and, therefore, the fractional version of the PIM for solving the equation (2) and (3) can be resulted as:

$$y_{n+1}(t) = y_a + {}_aI_t^{\alpha} \left[g\left(y_n(t), u(t), t \right) \right],$$

or, according to (6),

$$y_{n+1}(t) = y_a + {}_aI_t^{\alpha} \left[g\left(y_n(t), \sum_{k=0}^{N} d_k t^k, t \right) \right],$$

where $y_0(t) = y_a$ is the initial guess. Accordingly, having determined the initial approximation, the approximations $y_{n+1}(t), n \geq 0$, of the solution $y(t)$ can be readily gained. So we will have:

$$y(t) = \lim_{n \to \infty} y_n(t).$$

Now, according to (4), we will have the following iterative procedure for solving (2) and (3):

$$y_{n+1}(t) = y_a + \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} g\left(y_n(\tau), \sum_{k=0}^{N} d_k \tau^k, \tau \right) d\tau. \tag{7}$$

Now, a betterment of the PIM (7) can be given by using the Taylor series around a of the integrand as:

$$y_{n+1}(t) = y_a + \frac{1}{\Gamma(\alpha)} \int_a^t G_n\left(t, \tau\right) d\tau,$$

where

$$(t-\tau)^{\alpha-1} g\left(y_n(\tau), \sum_{k=0}^{N} d_k \tau^k, \tau\right) = G_n\left(t, \tau\right) + \mathrm{O}\left((\tau - a)^{n+1}\right).$$

By using the above iterative relation, we gain $y_M(t)$, which depends on $d_0, d_1, ..., d_N$ i.e., $y_M(t) := \phi(t; d_0, d_1, ..., d_N)$. Therefore, by substituting $y_M(t)$ and $u_N(t)$ into the cost functional (1), we will have:

$$J\left(d_0, d_1, ..., d_N\right) = \int_a^b f\left(\phi(t; d_0, d_1, ..., d_N), \sum_{k=0}^{N} d_k t^k, t\right) dt,$$

which J can be minimized in a satisfactory manner. Thus, we can get the approximations of $y_M(t) \simeq y(t)$ and $u_N(t) \simeq u(t)$ by substituting the determined coefficients $d_0, d_1, ..., d_N$.

3 Two Test Examples

Here, two test examples of the FOCPs are given to show the proficiency of the scheme. The Maple software was applied for the implementation of the two examples.

Example 3.1. For the first test problem, we give the following FOCP [12]:

$$J = \frac{1}{2} \int_0^1 \left[y^2(t) + u^2(t)\right] dt, \tag{8}$$

with

$${}_0D_t^{\alpha} y(t) = -y(t) + u(t), \quad y(0) = 1 \quad 0 < \alpha \leq 1.$$

The true solution (8), i.e., $J^* = 0.1929092981$ for $\alpha = 1$ could be observed in [12]. Now, if we put $u_N(t) = \sum\limits_{k=0}^{N} d_k t^k$, we will have the following fractional initial value problem:

$$ {}_0D_t^{\alpha} y(t) = -y(t) + \sum_{k=0}^{N} d_k t^k. $$

By using the formula (7), we will have the following iterative formula:

$$ y_{n+1}(t) = 1 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[y_n(\tau) - \sum_{k=0}^{N} d_k \tau^k \right] d\tau, $$

with the initial approximation $y_0(t) = 1$. Hence, by choosing $\alpha = 1$ and $N = 1, 2, 3$, we obtain the following approximations for $y(t)$:

$$ \begin{aligned} y_1(t) &= 1 + (d_0 - 1)t, \\ y_2(t) &= 1 + (d_0 - 1)t + \frac{1}{2}(-d_0 + d_1 + 1)t^2, \\ y_3(t) &= 1 + (d_0 - 1)t + \frac{1}{2}(-d_0 + d_1 + 1)t^2 + \frac{1}{3}(d_2 + \frac{1}{2}d_0 - \frac{1}{2}d_1 - \frac{1}{2})t^3. \end{aligned} $$

At this point, by using the Maple optimization toolbox, we can compute the unknown coefficients d_k. Table 1 indicates the optimal values J^* for the different approximations of the control function.

Table 1: The calculated optimal values of J^* for Example 3.1 for $\alpha = 1$ with different N

N	d_0	d_1	d_2	d_3	d_4	J^*
0	-0.17103818					0.1990804741
1	-0.35260048	0.37360123				0.1929833162
2	-0.37926750	0.53202950	-0.15856636			0.1929119841
3	-0.38539435	0.60519094	-0.34127799	0.12180922		0.1929093060
4	-0.38576842	0.61265211	-0.37482378	0.17397843	-0.02608461	0.1929092982

Besides, in Table 2, we listed J^* for the different values of α.

The numeric consequences of Tables 1 and 2 display obviously that the present modified PIM is accurate for investigating the FOCPs.

Example 3.2. For the second test problem, consider the following FOCP [12]:

Table 2: Comparisons of J^* with different choices of α for Example 3.1

α	Method of [1]	Method of [2]	Present method
1	0.192909	0.192909	0.1929092982
0.99	0.19153	0.19153	0.1915476611
0.9	0.17952	0.17953	0.1796176899
0.8	0.16729	0.16711	0.1674021655

$$\begin{aligned} J &= \int_0^1 \Big[y^2(t) - 2t^{\frac{3}{2}} y(t) + u^2(t) - \frac{3\sqrt{\pi}}{4} e^{-t} u(t) + e^{-t+t^{\frac{3}{2}}} u(t) \\ &+ t^3 + \frac{9\pi}{64} e^{-2t} - \frac{3\sqrt{\pi}}{8} e^{-2t+t^{\frac{3}{2}}} + \frac{1}{4} e^{-2t+2t^{\frac{3}{2}}} + e^{2t} \Big] dt, \end{aligned} \tag{9}$$

with

$$ {}_0D_t^{1.5} y(t) = e^{y(t)} + 2e^t u(t), \tag{10}$$

and the initial conditions:

$$y(0) = \dot{y}(0) = 0. \tag{11}$$

The true solution (9)-(11), i.e., $J^* = 3.194528049$ was reported in [12]. Proceeding as before, we can calculate the approximate solution for (10) and (11) using the following iterative relation:

$$\begin{aligned} y_{n+1}(t) &= \frac{1}{\Gamma(1.5)} \int_0^t (t-\tau)^{0.5} \left[e^{y_n(\tau)} + 2e^{\tau} \left(\sum_{k=0}^{N} d_k \tau^k \right) \right] d\tau, \\ y_0(t) &= 0. \end{aligned} \tag{12}$$

In Table 3, by implementing the present modified PIM of (12), we have listed the resulted optimal values of J^* for the different values N. From Table 3, one can notice that with increasing N, the obtained J^* approaches to the exact solution.

4 Conclusions

We established a beneficial PIM for a class of the FOCPs. By employing the polynomial basis for the control function and the modification of the

Table 3: Different values of J^* for Example 3.2

N	d_0	d_1	d_2	d_3	d_4	J^*
0	0.00352472					3.21240284
1	0.17628828	-0.42058649				3.19430483
2	0.15814969	-0.31103621	-0.11202805			3.19421192
3	0.15933296	-0.16383197	-0.68840731	0.47167339		3.19389048
4	0.15951084	-0.27885032	-0.32954128	0.34548036	-0.15425972	3.19453623

PIM, we diminished the primary optimal problem to the one of solving an initial value problem. Two test problems were given to express the superiority of the modified method to the PIM. The main advantage of the presented approach is the ability to reduce the computational work and to overcome the difficulty that arising in calculating integrals.

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