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Original Research Paper

Bi-Singular Type of a Fractional-Order Multi-Points Boundary Value Condition Problem

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Abstract. Various fractional differential equations have been examined during the last decades. Among them, singular equations are more notable. In this article, by using control functions, the existence of a solution for a bi-singular fractional differential equation with multi-point initial value conditions is considered. In the following, some examples elucidate our main result. In this paper by using control functions method, we prove the existence of the solution.

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1 Introduction

Altough it is awhile that definitions for the fractional derivatives have been provided, differential equations with fractional order have played a prominent role in the researches of mathematicians (see, for example, [1]- [6]), among which singular ones are more significant.(see [7]- [11]). In fact, differential equations with fractional derivative order, can be considered as an extension of ordinary ones. One can see that in scientific and engineering problems, a exact mathematical modeling leads to

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a differential equation with fractional derivative order (see for examples [12]- [21]).

In 2013, the fractional problem $\mathcal{D}^r\nu(\xi) + y(t,\nu(\xi)) = 0$ with boundary conditions $\nu'(0) = \nu''(0) = \cdots = \nu^{(k_0-1)}(0) = 0$ and $\nu(1) = \int_0^1 \nu(s)d\gamma(s)$ was investigated, where $0 < \xi < 1$, $n \geq 2$, $r \in (k_0-1, k_0)$, $\gamma(s)$ is a function of bounded variation, y could be singular at $\xi = 1$ and $\int_0^1 d\gamma(s) < 1$ ([22]).

In 2015, the fractional problem $\mathcal{D}^\rho y(\zeta) = \psi(\zeta, y(\zeta), \mathcal{D}^\sigma y(\zeta))$ with boundary conditions $y(0)+y'(0) = g(x)$, $\int_0^1 y(\zeta)dt = m_0$ and $y''(0) = y^{(3)}(0) = \cdots = y^{(n_\rho-1)}(0) = 0$ was studied where, $0 < \zeta < 1$, m_0 is a real number, $n_\rho \geq 2$, $\rho \in (n_\rho - 1, n_\rho)$, $0 < \sigma < 1$, $\mathcal{D}^\rho$ and D^σ is the Caputo fractional derivatives, $g \in C([0,1],\mathbb{R}) \to \mathbb{R}$ and $\psi : (0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in which $\psi(\zeta, u, v)$ could has singularity at $\zeta = 0$ ([23]).

In 2018, the existence of a solution for the following three steps crisis problem was investigated:

$$\mathcal{D}^\eta z(\tau) + \psi(\tau, z(\tau), z'(\tau), \mathcal{D}^\sigma z(\tau), \int_0^\tau \Omega(\xi)z(\xi)d\xi, \omega(x(\tau))) = 0$$

with boundary conditions $z(1) = z(0) = z''(0) = z^{n_\eta}(0) = 0$, where $\eta \geq 2$, $\lambda, \mu, \sigma \in (0,1)$, $\Omega \in L^1[0,1]$, $\omega : C^1[0,1] \to C^1[0,1]$ is a mapping such that $\|\omega(x_1) - \omega(x_2)\| \leq \iota_0\|x_1 - x_2\| + \iota_1\|x_1' - x_2'\|$ for some ι_0, $\iota_1 \in [0,\infty)$ and all $x_1, x_2 \in C^1[0,1]$, $\mathcal{D}^\eta$ is the η-order Caputo fractional derivative, $\psi(\tau, z_1(\tau), ..., z_5(\tau)) = \psi_1(\tau, z_1(\tau), ..., z_5(\tau))$ for all $\tau \in [0,\lambda)$, $\psi(\tau, z_1(\tau), ..., z_5(\tau)) = \psi_2(\tau, z_1(\tau), ..., z_5(\tau))$ for all $\tau \in [\lambda, \mu]$ and $\psi(\tau, z_1(\tau), ..., z_5(\tau)) = \psi_3(\tau, z_1(\tau), ..., z_5(\tau))$ for all $\tau \in (\mu, 1]$, $\psi_1(\tau, ., ., ., ., .)$ and $\psi_3(\tau, ., ., ., ., .)$ are continuous on $[0,\lambda)$ and $(\mu, 1]$ and $\psi_2(\tau, ., ., ., ., .)$ is multi-singular ([24]).

In 2020, the existence of solutions for the strong singular fractional differential equation

$$\mathcal{D}^\alpha x(t) = f(t, x(t), \mathcal{I}^{p_1}x(t), ..., \mathcal{I}^{p_m}x(t)),$$

with boundary conditions $x^{(2)}(0) = \cdots = x^{(n-1)}(0) = 0$, $x(0) = \int_0^1 x(\xi)d\xi$ and $x(\mu) = \Sigma_{i=1}^k \lambda_i \mathcal{I}^{q_i} x(\gamma_i)$ was investigated, where $\alpha \geq 1$, $p_1, \ldots, p_m > 0$, $m \geq 1$, $\mathcal{D}^\alpha$ is the fractional Caputo derivative of order α, $\mathcal{I}^p$ is the Riemann-Liouville integral of order p and $f(t, ., ..., .)$ has strong singularity at some points $[0,1]$ ([25]).

Motivated by the mentioned articles, we investigate the non-controlled bi-singular fractional differential equation

$$\mathcal{D}^{\mathfrak{a}}(g(t)\mathcal{D}^{\mathfrak{r}}(\nu(t))) = \Theta(t, \nu(t), \nu'(t), \phi_\nu(t)) \qquad (1)$$

with boundary conditions $\mathcal{D}^{(\mathfrak{r}+j)}\nu(0) = \nu^{(j^*)}(0) = 0$ for all $1 \leq j^* \leq k-1, 0 \leq j \leq n-1$ and $\nu'(\eta) = \sum_{i=1}^{k_0} \lambda_i \nu(\gamma_i)$, for some $k_0 \in \mathbb{N}$, where $n = [\mathfrak{a}] + 1$, $k = [\mathfrak{r}] + 1$, $\mathfrak{a}, \mathfrak{r} \geq 1$, $\mathfrak{a} + \mathfrak{r} \geq 3$, $\lambda_i \in \mathbb{R}$, $\sum_{i=1}^{k_0} \lambda_i \neq 0$, $\eta, \gamma_i \in (0,1)$, $g : [0,1] \to \mathbb{R}$ is a function which can be zero at some points $t \in [0,1]$, $\phi : X \to \mathbb{R}$ is a function such that for all $u, v \in X$ and $t \in [0,1]$, satisfies the following inequality:

$$|\phi_u(t) - \phi_v(t)| \leq \omega_1|u(t) - v(t)| + \omega_2|u'(t) - v'(t)|,$$

$\omega_1, \omega_2 \in [0,\infty)$ and $X = C^1[0,1]$. $\mathcal{D}^{\mathfrak{a}}$ is the Caputo fractional derivative of order $\mathfrak{a}$ and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a function such that $\Theta(t, ., ., .)$ is singular at some points $t \in [0,1]$. This equation has the advantage that includes many similar ordinary differential equaltions and fractional order ones. The method which will be in the proposed article leads to control singular points. Actually, using inequalities and control functions, set fewer and weaker conditions to prove the existence of a solution. All types of singularity which occur in a differential equation are important. Bi-singularity ones have been less studied. In this article we introduce bi-singularity concept and consider a problem with this type of singularity. In fact, Θ is stated to be multi-sigular when it is singular at more than one point t. Note that the differential equation $\mathcal{D}^{\mathfrak{a}}(g(t)\mathcal{D}^{\mathfrak{r}}w(t)) = \mathcal{U}(t, w(t))$ is sigular when $\mathcal{U}$ is singular or $g(t) = 0$ at some points $t \in [0,1]$. When $\mathcal{U}$ is singular and $g(t) = 0$, we call the equation $\mathcal{D}^{\mathfrak{a}}(g(t)\mathcal{D}^{\mathfrak{r}}w(t)) = \mathcal{U}(t, w(t))$ to be bi-singular. Likewise, $\mathcal{D}^{\mathfrak{a}}w(t) + \mathcal{U}(t) = 0$ is pointwise defined equation on $[0,1]$ if there is the set $E \subset [0,1]$ such that its measure of complenment E^c is zero and equation on E is being hold. It is obvious that each equation is a pointwisly defined equation. In this paper, we use $\|.\|_1$ as the norm of $L^1[0,1]$, $\|.\|$ as the sup norm $Y = C[0,1]$ and $\|w\|_* = \max\{\|w\|, \|w'\|\}$ as the norm of $X = C^1[0,1]$.

The Riemann-Liouville integral of order r with the lower limit $\nu \geq 0$ for a function $\mathcal{Y} : (\nu, \infty) \to \mathbb{R}$ is defined by $\mathcal{I}_{\nu^+}^r \mathcal{Y}(x) = \frac{1}{\Gamma(r)} \int_\nu^x (x -$

$\zeta)^{r-1}\mathcal{Y}(\zeta)d\zeta$ provided that the right-hand side is pointwise defined on (ν,∞). we denote $\mathcal{I}^r\mathcal{Y}(x)$ for $\mathcal{I}^r_{0+}\mathcal{Y}(x)$. Also, The Caputo fractional derivative of order $r>0$ of a function $\mathcal{Y}:(0,\infty)\to\mathbb{R}$ is defined by ${}^c\mathcal{D}^r\mathcal{Y}(x)=\frac{1}{\Gamma(n_r-r)}\int_0^x\frac{\mathcal{Y}^{n_r}(\zeta)}{(x-\zeta)^{r+1-n_r}}d\zeta$, where $n_r=[r]+1$ ([26]).
Let Ψ be the family of nondecreasing functions $\psi:[0,\infty)\to[0,\infty)$ such that $\sum_{j=1}^{\infty}\psi^j(\zeta)<\infty$ for all $\zeta>0$ ([27]). It is easy to see that $\psi(\zeta)<\zeta$ is held for all $\zeta>0$ ([27]). Let $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ and $\mathcal{A}:\mathcal{E}\times\mathcal{E}\to[0,\infty)$ be two maps. Then $\mathcal{T}$ is called an $\mathcal{A}$-admissible map whenever $\mathcal{A}(x,y)\geq 1$ implies $\mathcal{A}(\mathcal{T}x,\mathcal{T}y)\geq 1$ ([28]). Let $(\mathcal{E},d)$ be a complete metric space, $\psi\in\Psi$ and $\mathcal{A}:\mathcal{E}\times\mathcal{E}\to[0,\infty)$ a map. A self-map $\mathcal{T}:\mathcal{E}\to\mathcal{E}$ is called an $\mathcal{A}$-ψ-contraction whenever $\mathcal{A}(x,y)d(\mathcal{T}x,\mathcal{T}y)\leq\psi(d(x,y))$ for all $x,y\in\mathcal{E}$ ([28]). We need the following results.

Lemma 1.1. *([29]) Assume that* $0<n-1\leq r<n$ *and* $v\in C[0,1]\cap L^1[0,1]$. *Then* $\mathcal{I}^r\mathcal{D}^rv(\xi)=v(\xi)+\sum_{i=0}^{n-1}\iota_i\xi^i$ *for some constants* $\iota_0,\ldots,\iota_{n-1}\in\mathbb{R}$.

Lemma 1.2. *([30]) Consider a complete metric space* $(\mathcal{E},d)$, *a map* $\mathcal{A}:\mathcal{E}\times\mathcal{E}\to[0,\infty)$, $\psi\in\Psi$, *and* $\mathcal{L}\to\mathcal{E}$ *an* $\mathcal{A}$*-admissible* $\mathcal{A}$-ψ*-contraction. If* $\mathcal{L}$ *is continuous and there exists* $u_0\in\mathcal{E}$ *such that* $\mathcal{A}(u_0,\mathcal{L}u_0)\geq 1$, *then* $\mathcal{L}$ *has a fixed point.*

Lemma 1.3. *([31]) For all* $\zeta>-1$ *and* $w>0$, *we have* $\int_0^t(t-s)^{w-1}s^{\zeta}ds=\mathcal{B}(\zeta+1,w)t^{w+\zeta}$, *where* $\mathcal{B}(\zeta,w)=\frac{\Gamma(w)\Gamma(\zeta)}{\Gamma(w+\zeta)}$.

2 Main Results

Lemma 2.1. *Let* $\mathfrak{a}\geq 1$, $\mathfrak{r}\geq 2$, $\lambda_i\in\mathbb{R}$, $\eta\in(0,1)$ *for* $1\leq i\leq k_0$, $k_0\in\mathbb{N}$, $n=[\mathfrak{a}]+1$, $k=[\mathfrak{r}]+1$ *and* $f\in L^1[0,1]$. *Then* $x_0\in X$ *is a solution for the problem*

$$\mathcal{D}^{\mathfrak{a}}(g(t)\mathcal{D}^{\mathfrak{r}}(\nu(t)))=f(t) \tag{2}$$

with boundary conditions $\mathcal{D}^{(\mathfrak{r}+j)}\nu(0)=\nu^{(j^*)}(0)=0$ *for all* $1\leq j^*\leq k-1, 0\leq j\leq n-1$ *and* $\nu'(\eta)=\sum_{i=1}^{k_0}\lambda_i\nu(\gamma_i)$, *if and only if* x_0 *is given* *as follow*

$$
\begin{aligned}
x_0(t) = & \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t f(\zeta)\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)d\zeta \\
+ & \frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta f(\zeta)\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)d\zeta \\
- & \frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}\lambda_i\int_0^{\gamma_i} f(\zeta)\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)d\zeta,
\end{aligned}
$$

where

$$
\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta) = \int_s^t \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)}d\xi
$$

and $\Delta = \sum_{i=1}^{k_0}\lambda_i$.

Proof. Let x_0 be a solution for the problem (2), then regarding Lemma (1.1), it is evinced that

$$
g(t)\mathcal{D}^{\mathfrak{r}}x_0(t) = \frac{1}{\Gamma(\mathfrak{a})}\int_0^t (t-\zeta)^{\mathfrak{a}-1}f(\zeta)d\zeta + m_0 + m_1 t + \dots + m_{n-1}t^{n-1}.
$$

Since $\mathcal{D}^{\mathfrak{r}}x_0(0) = 0$, we $m_0 = 0$. Also we have $(g(t)\mathcal{D}^{\mathfrak{r}}x_0(t))'\Big|_{t=0} = m_1$, hence

$$
g'(0)\mathcal{D}^{\mathfrak{r}}x_0(0) + g(0)\mathcal{D}^{\mathfrak{r}+1}x_0(0) = m_1.
$$

Since for $0 \le j \le n-1$, $\mathcal{D}^{\mathfrak{r}+j}x_0(0) = 0$, we conclude that $m_1 = 0$. Using the same argument, it is concluded $m_2 = \dots = m_{n-1} = 0$. So

$$
\mathcal{D}^{\mathfrak{a}}x_0(t) = \frac{1}{g(t)\Gamma(\mathfrak{a})}\int_0^t (t-\zeta)^{\mathfrak{a}-1}f(\zeta)d\zeta.
$$

Using Lemma (1.1) again, it is resulted

$$
\begin{aligned}
x_0(t) \;\; = \;\; & \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t \frac{(t-\xi)^{\mathfrak{r}-1}}{g(\xi)}(\int_0^\xi (\xi-\zeta)^{\mathfrak{a}-1}f(\zeta)d\zeta)d\xi \\
& + \iota_0 + \iota_1 t + \dots + \iota_{k-1}t^{k-1}.
\end{aligned}
$$

As regarded $x^{(j^*)}(0) = 0$ for $1 \leq j^* \leq k-1$ then $\iota_1 = \iota_2 = ... = \iota_{k-1} = 0$. Replacing in the above equality, we have

$$x_0(t) = \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^t \frac{(t-\xi)^{\mathfrak{r}-1}}{g(\xi)} (\int_0^\xi (\xi-\zeta)^{\mathfrak{a}-1} f(\zeta) d\zeta) d\xi + \iota_0. \quad (3)$$

By differentiating from the last equality, it is deduced that

$$x_0'(t) = \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^t \frac{(t-\xi)^{\mathfrak{r}-2}}{g(\xi)} (\int_0^\xi (\xi-\zeta)^{\mathfrak{a}-1} f(\zeta) d\zeta) d\xi,$$

so

$$\begin{aligned} x_0'(\eta) &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta \frac{(\eta-\xi)^{\mathfrak{r}-2}}{g(\xi)} (\int_0^\xi (\xi-\zeta)^{\mathfrak{a}-1} f(\zeta) d\zeta) d\xi \\ &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta \int_0^\xi \frac{(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)} f(\zeta) d\zeta d\xi \\ &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta \int_\zeta^\eta \frac{(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)} f(\zeta) d\xi d\zeta \\ &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta f(\zeta) (\int_\zeta^\eta \frac{(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)} d\xi) d\zeta. \end{aligned}$$

Put

$$\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta) = \int_\zeta^t \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)} d\xi,$$

then, it is obtained that

$$x_0'(\eta) = \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta f(\zeta) \mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta) d\zeta.$$

Also by (3), we induce that

$$\begin{aligned} x_0(t) &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^\zeta \int_0^\xi \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)} f(\zeta) d\zeta d\xi + \iota_0 \\ &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^t \int_\zeta^t \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)} f(\zeta) d\xi d\zeta + \iota_0 \\ &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^t \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta) f(\zeta) d\zeta + \iota_0. \end{aligned}$$

Hence, for $1 \leq i \leq k_0$, we have

$$\lambda_i x_0(\gamma_i) = \frac{\lambda_i}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^{\gamma_i} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i, \zeta) f(\zeta) d\zeta + \lambda_i \iota_0.$$

Therefore

$$\sum_{i=1}^{k_0} \lambda_i x_0(\gamma_i) = \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{i=1}^{k_0} \lambda_i \int_0^{\gamma_i} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i, \zeta) f(\zeta) d\zeta + \iota_0 \sum_{i=1}^{k_0} \lambda_i.$$

By hypothesis $x_0'(\eta) = \sum_{i=1}^{k_0} \lambda_i x_0(\gamma_i)$, so we have

$$\begin{aligned} &\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^{\eta} f(\zeta) \mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta, \zeta) d\zeta \\ &= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{i=1}^{k_0} \lambda_i \int_0^{\gamma_i} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i, \zeta) f(\zeta) d\zeta + \iota_0 \sum_{i=1}^{k_0} \lambda_i, \end{aligned}$$

so ι_0 is obtained as follow

$$\begin{aligned} \iota_0 \quad = \quad & \frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^{\eta} f(\zeta) \mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta, \zeta) d\zeta \\ & - \frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{i=1}^{k_0} \lambda_i \int_0^{\gamma_i} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i, \zeta) f(\zeta) d\zeta, \end{aligned}$$

where $\Delta = \sum_{i=1}^{k_0} \lambda_i$. Hence $x_0(t)$ is given as

$$\begin{aligned} x_0(t) = & \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^{t} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t, \zeta) f(\zeta) d\zeta \\ & + \frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^{\eta} \mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t, \zeta) f(\zeta) d\zeta \\ & - \frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{i=1}^{k_0} \lambda_i \int_0^{\gamma_i} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i, \zeta) f(\zeta) d\zeta. \end{aligned}$$

□

Now, let $\mathcal{L}: X \to X$ be defined as

$$\mathcal{L}\mathfrak{u}(t) = \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))d\zeta$$
$$+\frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta \mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))d\zeta$$
$$-\frac{1}{\Delta\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}\lambda_i\int_0^{\gamma_i} \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))d\zeta.$$

where $\phi : X \to X$ is a mapping such that

$$|\phi_{\mathfrak{u}}(t) - \phi_{\mathfrak{v}}(t)| \leq \omega_1|\mathfrak{u}(t) - \mathfrak{v}(t)| + \omega_2|\mathfrak{u}'(t) - \mathfrak{v}'(t)|,$$

for all $\mathfrak{u}, \mathfrak{v} \in X$, $t \in [0,1]$ and some functions $\omega_1, \omega_2 \in [0,\infty)$. It easy to see that $\mathcal{L}'$ is given as follow

$$\mathcal{L}'\mathfrak{u}(t) = \frac{\partial \mathcal{L}\mathfrak{u}}{\partial t} = \frac{1}{\Gamma(\alpha)\Gamma(\mathfrak{r})}\int_0^t \frac{\partial \mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)}{\partial t}\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))d\zeta$$
$$= \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^t \mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))d\zeta.$$

Now, we investigate $\mathcal{L} : X \to X$, to prove the existence of a solution in X for the problem (1). Applying lemma (1.1), it is indicated that $\mathcal{L}$ has a fixed point in X. In the next results, by using some functions which are called control functions, we will control the singularity and then, inequalities help us to consider a sloution for the bi-singular fractional differential problem.

Theorem 2.2. *Let* $\mathfrak{a}, \mathfrak{r} \geq 1$, $\mathfrak{a} + \mathfrak{r} \geq 3$, $n = [\mathfrak{a}] + 1$, $k = [\mathfrak{r}] + 1$, $\lambda_i \in \mathbb{R}$, $\Delta := \sum_{i=1}^{k_0} \lambda_i \neq 0$, $\eta, \gamma_i \in (0,1)$, $\phi : X \to \mathbb{R}$ *is a function such that for all* $u, v \in X$ *and* $t \in [0,1]$,

$$|\phi_{\mathfrak{u}}(t) - \phi_{\mathfrak{v}}(t)| \leq \omega_1|\mathfrak{u}(t) - \mathfrak{v}(t)| + \omega_2|\mathfrak{u}'(t) - \mathfrak{v}'(t)|,$$

for some $\omega_1, \omega_2 \in [0,\infty)$, $g : [0,1] \to \mathbb{R}$ *may be zero at some points* $t_0 \in [0,1]$, $\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\| := \int_0^1 \frac{(1-\zeta)^{\mathfrak{r}-2}\zeta^{\mathfrak{a}}}{|g(\zeta)|}d\zeta < \infty$, $\Theta : [0,1] \times (C^1[0,1])^3 \to \mathbb{R}$ *be a singular function at some points* $t \in [0,1]$ *such that*

$$|\Theta(t,w_1,w_2,w_3) - \Theta(t,z_1,z_2,z_3)| \leq \sum_{j=1}^{k^*}\theta_j(t)\Lambda_j(|w_1 - z_1|, |w_2 - z_2|, |w_3 - z_3|),$$

for all $w_1, w_2, w_3, z_1, z_2, z_3 \in X$*, almost* $t \in [0,1]$ *and some* $k^* \in \mathbb{N}$*, where* $\Lambda_j : X^3 \to [0,\infty)$ *for each* $1 \le j \le k^*$*, is a nondecreasing function with respect to all their components,* $\theta_j : [0,1] \to [0,\infty)$*,* $\lim_{z\to 0^+} \frac{\Lambda(z,z,z)}{z} = q_j \in [0,\infty)$ *and* $\|\tilde{g}^{\mathfrak{a},\mathfrak{r}-1}_{\theta_j}\|_{[0,t]} := \int_0^t \frac{\hat{\theta}^{\mathfrak{a},\mathfrak{r}-1}_j(t,\xi)}{|g(\xi)|} d\xi \in L^1[0,1]$*, where* $\hat{\theta}^{\mathfrak{a},\mathfrak{r}}_j(t,\xi) = \int_0^\xi (t-\zeta)^{\mathfrak{a}+\mathfrak{r}-2}\theta_j(\zeta)d\zeta$*. Also let* $|\Theta(t,x_1,x_2,x_3)| \le \sum_{i=1}^3 \mathcal{N}_i(t,x_i)$*, where* $\mathcal{N}_i : [0,1]\times X \to [0,\infty)$ *for each* $1 \le i \le 3$ *is nondecreasing with respect to its second component and* $\lim_{z\to 0^+} \frac{\mathcal{N}_i(t,z)}{z} = \mathcal{V}_i(t)$ *a.e.* $[0,1]$*, such that* $\|\hat{\mathcal{V}}^{\mathfrak{a},\mathfrak{r}-1}_i\|_{[0,t]} \in L^1[0,1]$ *and*

$$\sum_{j=1}^{3}\Bigg(|\Delta|(\mathfrak{r}-1)\|\hat{g}^{\mathfrak{a},\mathfrak{r}-1}_{\mathcal{V}_j}\|_{[0,1]} + (\mathfrak{r}-1)\|\hat{g}^{\mathfrak{a},\mathfrak{r}}_{\mathcal{V}_j}\|_{[0,\eta]} + \sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}^{\mathfrak{a},\mathfrak{r}}_{\mathcal{V}_j}\|_{[0,\gamma_i]}\Bigg) \in [0, \frac{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}{\Xi}),$$

where $\Xi = max\{1, \omega_1+\omega_2\}$*. If*

$$\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*} q_j\|\tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*} q_j\|\tilde{g}^{\mathfrak{a},\mathfrak{r}-1}_{\theta_j}\|_{[0,1]} + \sum_{j=1}^{k^*}\sum_{i=1}^{k_0} q_j|\lambda_i|\|\tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}\|_{[0,1]}\Bigg) < 1,$$

then the singular fractional differential equation

$$\mathcal{D}^{\mathfrak{a}}(g(t)\mathcal{D}^{\mathfrak{r}}(\nu(t))) = \Theta(t,\nu(t),\nu'(t),\phi_\nu(t))$$

with boundary conditions $\mathcal{D}^{(\mathfrak{r}+j)}\nu(0) = \nu^{(j^*)}(0) = 0$ *for all* $1 \le j^* \le k-1, 0 \le j \le n-1$ *and* $\nu'(\eta) = \sum_{i=1}^{k_0}\lambda_i\nu(\gamma_i)$*, for some* $k_0 \in \mathbb{N}$*.*

Proof. Firstly, we prove that $\mathcal{L}$ is continuous on X. Let $\mathfrak{u}, \mathfrak{v} \in X$, then for all $t \in [0,1]$ we have

$$
\begin{aligned}
|\mathcal{L}\mathfrak{u}(t) - \mathcal{L}\mathfrak{v}(t)| &\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)| \Big| \Theta(\zeta, \mathfrak{u}(\zeta), \mathfrak{u}'(\zeta), \phi_{\mathfrak{u}}(\zeta)) \\
&-\Theta(\zeta, \mathfrak{v}(\zeta), \mathfrak{v}'(\zeta), \phi_{\mathfrak{v}}(\zeta)) \Big| d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)| \Big| \Theta(\zeta, \mathfrak{u}(\zeta), \mathfrak{u}'(\zeta), \phi_{\mathfrak{u}}(\zeta)) \\
&-\Theta(\zeta, \mathfrak{v}(\zeta), \mathfrak{v}'(\zeta), \phi_{\mathfrak{v}}(\zeta)) \Big| d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{i=1}^{k_0} |\lambda_i| \int_0^{\gamma_i} |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)| \Big| \Theta(\zeta, \mathfrak{u}(\zeta), \mathfrak{u}'(\zeta), \phi_{\mathfrak{u}}(\zeta)) \\
&-\Theta(\zeta, \mathfrak{v}(\zeta), \mathfrak{v}'(\zeta), \phi_{\mathfrak{v}}(\zeta)) \Big| d\zeta \\
&\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^t \bigg(|\mathcal{H}_{\alpha,\mathfrak{r}}(t,\zeta)| \\
&\times \sum_{j=1}^{k^*} \theta_j(\zeta) \Lambda_j(|\mathfrak{u}(\zeta) - \mathfrak{v}(\zeta)|, |\mathfrak{u}'(\zeta) - \mathfrak{v}'(\zeta)|, |\phi_{\mathfrak{u}}(\zeta) - \phi_{\mathfrak{v}}(\zeta)|) \bigg) d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^\eta \bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)| \\
&\times \sum_{j=1}^{k^*} \theta_j(\zeta) \Lambda_j(|\mathfrak{u}(\zeta) - \mathfrak{v}(\zeta)|, |\mathfrak{u}'(\zeta) - \mathfrak{v}'(\zeta)|, |\phi_{\mathfrak{u}}(\zeta) - \phi_{\mathfrak{v}}(\zeta)|) \bigg) d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{i=1}^{k_0} |\lambda_i| \int_0^{\gamma_i} \bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)| \\
&\times \sum_{j=1}^{k^*} \theta_j(\zeta) \Lambda_j(|\mathfrak{u}(\zeta) - \mathfrak{v}(\zeta)|, |\mathfrak{u}'(\zeta) - \mathfrak{v}'(\zeta)|, |\phi_{\mathfrak{u}}(\zeta) - \phi_{\mathfrak{v}}(\zeta)|) \bigg) d\zeta \\
&\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \int_0^t \bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)| \\
&\times \sum_{j=1}^{k^*} \theta_j(\zeta) \Lambda_j(\|\mathfrak{u} - \mathfrak{v}\|, \|\mathfrak{u}' - \mathfrak{v}'\|, \omega_1 \|\mathfrak{u} - \mathfrak{v}\| + \omega_2 \|\mathfrak{u}' - \mathfrak{v}'\|) \bigg) d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^{\eta}\Bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\\
&\times\sum_{j=1}^{k^*}\theta_j(\zeta)\Lambda_j(\|\mathfrak{u}-\mathfrak{v}\|,\|\mathfrak{u}'-\mathfrak{v}'\|,\omega_1\|\mathfrak{u}-\mathfrak{v}\|+\omega_2\|\mathfrak{u}'-\mathfrak{v}'\|)\Bigg)d\zeta\\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}\Bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\\
&\times\sum_{j=1}^{k^*}\theta_j(\zeta)\Lambda_j(\|\mathfrak{u}-\mathfrak{v}\|,\|\mathfrak{u}'-\mathfrak{v}'\|,\omega_1\|\mathfrak{u}-\mathfrak{v}\|+\omega_2\|\mathfrak{u}'-\mathfrak{v}'\|)\Bigg)d\zeta\\
&\leq\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^{t}\Bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\\
&\times\sum_{j=1}^{k^*}\theta_j(\zeta)\Lambda_j(\|\mathfrak{u}-\mathfrak{v}\|_*,\|\mathfrak{u}-\mathfrak{v}\|_*,(\omega_1+\omega_2)\|\mathfrak{u}-\mathfrak{v}\|_*)\Bigg)d\zeta\\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^{\eta}\Bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\\
&\times\sum_{j=1}^{k^*}\theta_j(\zeta)\Lambda_j(\|\mathfrak{u}-\mathfrak{v}\|_*,\|\mathfrak{u}-\mathfrak{v}\|_*,(\omega_1+\omega_2)\|\mathfrak{u}-\mathfrak{v}\|_*)\Bigg)d\zeta\\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}\Bigg(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\\
&\times\sum_{j=1}^{k^*}\theta_j(\zeta)\Lambda_j(\|\mathfrak{u}-\mathfrak{v}\|_*,\|\mathfrak{u}-\mathfrak{v}\|_*,(\omega_1+\omega_2)\|\mathfrak{u}-\mathfrak{v}\|_*)\Bigg)d\zeta.
\end{aligned}
$$

Let $\Xi := max\{1,\omega_1+\omega_2\}$, then by the last equality, for all $t\in[0,1]$ it is concluded that

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t)-\mathcal{L}\mathfrak{v}(t)|\\
&\leq \sum_{j=1}^{k^*}\bigg(\frac{\Lambda_j(\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*)}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\theta_j(\zeta)d\zeta\bigg)\\
&+\sum_{j=1}^{k^*}\bigg(\frac{\Lambda_j(\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*)}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\\
&\times\int_0^\eta|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\theta_j(\zeta)d\zeta\bigg)\\
&+\sum_{j=1}^{k^*}\bigg(\frac{\Lambda_j(\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*)}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\\
&\times(\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\theta_j(\zeta)d\zeta)\bigg).
\end{aligned}
$$

Regarding the properties $\lim_{z\to 0^+}\frac{\Lambda(\Xi z,\Xi z,\Xi z)}{\Xi z}=q_j$ for all $1\leq j\leq k^*$, for $\epsilon>0$ there exists $0<\delta(\epsilon)>0$ such that $z\in(0,\delta(\epsilon)]$ implies $0<|\frac{\Lambda(\Xi z,\Xi z,\Xi z)}{\Xi z}|\leq q_j+\epsilon$, for all $1\leq j\leq k^*$, so $0<\Lambda(\Xi z,\Xi z,\Xi z)\leq(q_j+\epsilon)\Xi z$, for all $z\in(0,\delta(\epsilon)]$ and $1\leq j\leq k^*$. Let $\delta_m(\epsilon)=\min\{\epsilon,\delta(\epsilon)\}$, then $\|\mathfrak{u}-\mathfrak{v}\|_*<\delta_m(\epsilon)$ implies

$$
\Lambda(\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*,\Xi\|\mathfrak{u}-\mathfrak{v}\|_*)\leq\Xi(q_j+\epsilon)\|\mathfrak{u}-\mathfrak{v}\|_*, \tag{4}
$$

so when $\|\mathfrak{u}-\mathfrak{v}\|_*<\delta_m(\epsilon)$, then for all $t\in[0,1]$

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t)-\mathcal{L}\mathfrak{v}(t)|\\
&\leq\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}\bigg[(q_j+\epsilon)\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\theta_j(\zeta)d\zeta\bigg]\\
&+\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}\bigg[(q_j+\epsilon)\int_0^\eta|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\theta_j(\zeta)d\zeta\bigg]\\
&+\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}\bigg[(q_j+\epsilon)\bigg(\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\theta_j(\zeta)d\zeta\bigg)\bigg].
\end{aligned}
$$

On the other hand,

$$\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\theta_j(\zeta)d\zeta = \int_0^t |\int_s^t \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{g(\xi)}d\xi|\theta_j(\zeta)d\zeta$$
$$\leq \int_0^t \int_\zeta^t \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\theta_j(\zeta)d\xi d\zeta$$
$$= \int_0^t \int_0^\xi \frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\theta_j(\zeta)d\zeta d\xi.$$

When $\mathfrak{a},\mathfrak{r} \geq 1$ and $\xi \in [\zeta,t]$, we have $(t-\zeta)^{\mathfrak{r}-1} \geq (t-\xi)^{\mathfrak{r}-1}$ and $(t-\zeta)^{\mathfrak{a}-1} \geq (\xi-\zeta)^{\mathfrak{a}-1}$, so by the above inequality, we conclude that

$$\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\theta_j(\zeta)d\zeta \leq \int_0^t \frac{1}{|g(\xi)|}\left(\int_0^\xi (t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}\theta_j(\zeta)d\zeta\right)d\xi$$
$$\leq \int_0^t \frac{1}{|g(\xi)|}\left(\int_0^\xi (t-\zeta)^{\mathfrak{a}+\mathfrak{r}-2}\theta_j(\zeta)d\zeta\right)d\xi = \int_0^t \frac{\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(t,\xi)}{|g(\xi)|}d\xi,$$

where $\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(t,\xi) = \int_0^\xi (t-\zeta)^{\mathfrak{a}+\mathfrak{r}-2}\theta_j(\zeta)d\zeta$. It is evident that $\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(t,\xi)$ is nondecreasing with respect to their components, also $\hat{\theta}_{\mathfrak{a},\mathfrak{r}} \leq \hat{\theta}_{\mathfrak{a},\mathfrak{r}^*}$ when $\mathfrak{r} \geq \mathfrak{r}^*$. By the same manner, it is resulted that

$$\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\theta_j(\zeta)d\zeta \leq \int_0^\eta \frac{\hat{\theta}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\xi)}{|g(\xi)|}d\xi$$

and for all $1 \leq i \leq k_0$, we have

$$\int_0^{\gamma_i} |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\theta_j(\zeta)d\zeta \leq \int_0^{\gamma_i} \frac{\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\xi)}{|g(\xi)|}d\xi.$$

Hence for all $t \in [0,1]$ and $\mathfrak{u},\mathfrak{v} \in X$ in which $\|\mathfrak{u}-\mathfrak{v}\|_* \leq \delta_m(\epsilon)$, the following inequality can be concluded

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t) - \mathcal{L}\mathfrak{v}(t)| \\
&\leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{j=1}^{k^*} (q_j+\epsilon) \int_0^t \frac{\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(t,\xi)}{|g(\xi)|} d\xi \\
&+ \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \sum_{j=1}^{k^*} (q_j+\epsilon) \int_0^\eta \frac{\hat{\theta}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\xi)}{|g(\xi)|} d\xi \\
&+ \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{j=1}^{k^*} \left[(q_j+\epsilon) \left(\sum_{i=1}^{k_0} |\lambda_i| \int_0^{\gamma_i} \frac{\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\xi)}{|g(\xi)|} d\xi \right) \right].
\end{aligned}
$$

Let $\tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}(t,\xi) := \frac{\hat{\theta}^{\mathfrak{a},\mathfrak{r}}_j(t,\xi)}{|g(\xi)|}$ and $\|\tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}\|_{[0,1]} := \int_0^1 \tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}(1,\xi)d\xi$. Since $\hat{\theta}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\xi)$ is nondecreasing with respect to t, $\tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}(t,\xi)$ also is nondecreasing with respect to t. Also since $\hat{\theta}_{\mathfrak{a},\mathfrak{r}-1}(\gamma_i,\xi) \geq \hat{\theta}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\xi)$ for all $t,\xi \in [0,1]$, we conclude that $\tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}(t,\xi) \leq \tilde{g}^{\mathfrak{a},\mathfrak{r}-1}_{\theta_j}(t,\xi)$ for all $t,\xi \in [0,1]$, hence $\tilde{g}^{\alpha,\mathfrak{r}}_{\theta_j}(1,\xi) \in L^1[0,1]$ implies that $\tilde{g}^{\mathfrak{a},\mathfrak{r}-1}_{\theta_j}(1,\xi) \in L^1[0,1]$. So for all $t \in [0,1]$ and $\mathfrak{u}, \in X$ in which $\|\mathfrak{u}-\mathfrak{v}\|_* \leq \delta_m(\epsilon)$, we have

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t) - \mathcal{L}\mathfrak{v}(t)| \leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{j=1}^{k^*} (q_j+\epsilon) \int_0^1 \tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}(1,\xi) d\xi \\
&+ \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \sum_{j=1}^{k^*} (q_j+\epsilon) \int_0^1 \tilde{g}^{\mathfrak{a},\mathfrak{r}-1}_{\theta_j}(1,\xi) d\xi \\
&+ \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})} \sum_{j=1}^{k^*} \left[(q_j+\epsilon) \left(\sum_{i=1}^{k_0} |\lambda_i| \int_0^1 \tilde{g}^{\mathfrak{a},\mathfrak{r}}_{\theta_j}(1,\xi) d\xi \right) \right]. \qquad (5)
\end{aligned}
$$

Therefore

$$|\mathcal{L}\mathfrak{u}(t)-\mathcal{L}\mathfrak{v}(t)| \leq \frac{\Xi\delta_m(\epsilon)}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}$$

$$+\frac{\Xi\delta_m(\epsilon)}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}$$

$$+\frac{\Xi\delta_m(\epsilon)}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}\sum_{i=1}^{k_0}|\lambda_i|(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}$$

$$\leq \frac{\Xi\epsilon}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}$$

$$+(\mathfrak{r}-1)\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}+\sum_{j=1}^{k^*}(\sum_{i=1}^{k_0}|\lambda_i|)(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}\Bigg).$$

So

$$\|\mathcal{L}\mathfrak{u}-\mathcal{L}\mathfrak{v}\| \leq \frac{\Xi\epsilon}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}$$

$$+(\mathfrak{r}-1)\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}+\sum_{j=1}^{k^*}(\sum_{i=1}^{k_0}|\lambda_i|)(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}\Bigg).$$

Also for all $t\in[0,1]$ and $u,v\in X$, we have

$$|\mathcal{L}'\mathfrak{u}(t)-\mathcal{L}'\mathfrak{v}(t)| \leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t|\frac{\partial\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)}{\partial t}|\Big|\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))$$

$$-\Theta(\zeta,\mathfrak{v}(\zeta),\mathfrak{v}'(\zeta),\phi_{\mathfrak{v}}(\zeta))\Big|d\zeta.$$

Note that for $\mathfrak{z}\in X$, we have

$$\mathcal{L}'\mathfrak{z}(t)=\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t\frac{\partial\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)}{\partial t}\Theta(\zeta,\mathfrak{z}(\zeta),\mathfrak{z}'(\zeta),\phi_{\mathfrak{z}}(\zeta))d\zeta$$

$$=\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^t\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)\Theta(\zeta,\mathfrak{z}(\zeta),\mathfrak{z}'(\zeta),\phi_{\mathfrak{z}}(\zeta))d\zeta.$$

For all $t \in [0,1]$ and $\mathfrak{u}, \mathfrak{v} \in X$, it is concluded that

$$
\begin{aligned}
&|\mathcal{L}'\mathfrak{u}(t) - \mathcal{L}'\mathfrak{v}(t)| \leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^t \Big(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)| \\
&\times |\Theta(\zeta, \mathfrak{u}(\zeta), \mathfrak{u}'(\zeta), \phi_{\mathfrak{u}}(\zeta)) - \Theta(\zeta, \mathfrak{v}(\zeta), \mathfrak{v}'(\zeta), \phi_{\mathfrak{v}}(\zeta))| \Big) d\zeta \\
&\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^t \Big(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)| \\
&\times \sum_{j=1}^{k^*} \theta_j(\zeta) \Lambda_j(|\mathfrak{u}(\zeta) - \mathfrak{v}(\zeta)|, |\mathfrak{u}'(\zeta) - \mathfrak{v}'(\zeta)|, |\phi_{\mathfrak{u}}(\zeta) - \phi_{\mathfrak{v}}(\zeta)|) \Big) d\zeta \\
&\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^t \Big(|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)| \\
&\times \sum_{j=1}^{k^*} \theta_j(\zeta) \Lambda_j(\|\mathfrak{u} - \mathfrak{v}\|, \|\mathfrak{u}' - \mathfrak{v}'\|, \omega_1 \|\mathfrak{u} - \mathfrak{v}\| + \omega_2 \|\mathfrak{u}' - \mathfrak{v}'\|) \Big) d\zeta \\
&\leq \sum_{j=1}^{k^*} \Bigg[\frac{\Lambda_j(\Xi\|\mathfrak{u} - \mathfrak{v}\|_*, \Xi\|\mathfrak{u} - \mathfrak{v}\|_*, \Xi\|\mathfrak{u} - \mathfrak{v}\|_*)}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \\
&\times \int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)| \theta_j(\zeta) d\zeta \Bigg].
\end{aligned}
$$

By (4), when $\|\mathfrak{u} - \mathfrak{v}\|_* \leq \delta_m(\epsilon)$, for all $t \in [0,1]$ we infer that

$$
\begin{aligned}
&|\mathcal{L}'\mathfrak{u}(t) - \mathcal{L}'\mathfrak{v}(t)| \\
&\leq \sum_{j=1}^{k^*} \frac{\Xi(q_j + \epsilon)\|\mathfrak{u} - \mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \int_0^t |\mathcal{H}_{\alpha,\mathfrak{r}-1}(t,\zeta)| \theta_j(\zeta) d\zeta \\
&\leq \frac{\Xi\|\mathfrak{u} - \mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \sum_{j=1}^{k^*} (q_j + \epsilon) \int_0^t \int_\zeta^t \frac{(t-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|} d\xi \theta_j(\zeta) d\zeta \\
&\leq \frac{\Xi\|\mathfrak{u} - \mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)} \sum_{j=1}^{k^*} (q_j + \epsilon) \int_0^t \int_0^\xi \frac{(t-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|} \theta_j(\zeta) d\zeta d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon)\int_0^t \frac{1}{|g(\xi)|}(\int_0^\xi (t-\zeta)^{\mathfrak{a}+\mathfrak{r}-3}\theta_j(\zeta)d\zeta)d\xi \\
&= \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon)\int_0^t \frac{\hat{\theta}_j^{\mathfrak{a},\mathfrak{r}-1}(t,\xi)}{|g(\xi)|}d\xi \\
&\leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} \\
&\leq \frac{\Xi\delta_m(\epsilon)}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} \\
&\leq \frac{\Xi\epsilon}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} \\
&= \frac{\Xi\epsilon(\mathfrak{r}-1)}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}.
\end{aligned}
$$

Which leads to

$$
||\mathcal{L}'\mathfrak{u}-\mathcal{L}'\mathfrak{v}\| \leq \frac{\Xi\epsilon(\mathfrak{r}-1)}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}.
$$

Therefore

$$
\begin{aligned}
&\|\mathcal{L}\mathfrak{u}-\mathcal{L}\mathfrak{v}\|_* = \max\{\|\mathcal{L}\mathfrak{u}-\mathcal{L}\mathfrak{v}\|, \|\mathcal{L}'\mathfrak{u}-\mathcal{L}'\mathfrak{v}\|\} \\
&\leq \frac{\Xi\epsilon}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]} \\
&+(\mathfrak{r}-1)\max\{1,|\Delta|\}\sum_{j=1}^{k^*}(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} \\
&+\sum_{j=1}^{k^*}(\sum_{i=1}^{k_0}|\lambda_i|)(q_j+\epsilon)\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}\Bigg).
\end{aligned}
$$

Since $\epsilon > 0$ is arbitary, $\mathfrak{u} \to \mathfrak{v}$ in X implies $\mathcal{L}\mathfrak{u} \to \mathcal{L}\mathfrak{v}$ in X, therefore $\mathcal{L}$ is continuous on X. Now since $\lim_{\|z\|\to 0^+} \frac{\mathcal{N}_i(t,z)}{\|z\|} = \mathcal{V}_i(t)$, for all $1 \le i \le 3$ and almost all $t \in [0,1]$, $\lim_{z\to 0^+} \frac{\mathcal{N}_i(t,\Xi z)}{\Xi z} = \mathcal{V}_i(t)$. Therefore for $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $0 < z \le \delta(\epsilon)$ implies $\frac{\mathcal{N}_i(t,\Xi z)}{\Xi z} < \mathcal{V}_i(t) + \epsilon$ and thus $\mathcal{N}_i(t, \Xi z) < (\mathcal{V}_i(t) + \epsilon)\Xi z$, for all $1 \le i \le 3$ and almost all $t \in [0,1]$. Since

$$\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg[\sum_{j=1}^{3}\Bigg(|\Delta|(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} + (\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\eta]}$$
$$+\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\Bigg)\Bigg] < 1$$

then there exists $\epsilon_0 > 0$ such that

$$\Bigg(\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg[\sum_{j=1}^{3}\Bigg(|\Delta|(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} + (\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\eta]}$$
$$+\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\Bigg)\Bigg] + \frac{3\Xi\epsilon_0}{|\Delta|\Gamma(\mathfrak{a}+1)\Gamma(\mathfrak{r})}\Bigg[|\Delta|(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|$$
$$+(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\| + (\sum_{i=1}^{k_0}|\lambda_i|)\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\Bigg]\Bigg) < 1,$$

similarly since

$$\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}q_j\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*}q_j\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}$$
$$+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0}q_j|\lambda_i|\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}\Bigg) < 1$$

there exists $\epsilon_1 > 0$, such that

$$\Bigg[\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}$$
$$+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0} q_j|\lambda_i|\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}\Bigg)$$
$$+\frac{\Xi\epsilon_1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}$$
$$+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0}|\lambda_i|\|\tilde{g}_{\theta_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}\Bigg)\Bigg] < 1.$$

Let $R_0 = \min\{\epsilon_0, \frac{\delta_m(\epsilon_1)}{2}\}$, so $\mathcal{N}_i(t,\rho z) < (\mathcal{V}_i(t)+\epsilon_0)\rho z$, for all $0 < z \leq R_0$. Put $\Omega = \{\mathfrak{u} \in X : \|\mathfrak{u}\|_* \leq R_0\}$. Define the map $\mathcal{A} : X^2 \to [0,\infty)$ by $\mathcal{A}(\mathfrak{u},\mathfrak{v}) = 1$ when $\mathfrak{u},\mathfrak{v} \in \Omega$, otherwise let $\mathcal{A}(\mathfrak{u},\mathfrak{v}) = 0$. Suppose that $\mathfrak{u},\mathfrak{v} \in X$ be such that $\mathcal{A}(\mathfrak{u},\mathfrak{v}) \geq 1$, so $\mathfrak{u},\mathfrak{v} \in \Omega$, $\|\mathfrak{u}\|_* \leq R_0$ and $\|\mathfrak{v}\|_* \leq R_0$. Then for all $t \in [0,1]$, we have

$$|\mathcal{L}\mathfrak{u}(t)| \leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)||\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))|d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)||\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))|d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}\lambda_i\int_0^{\gamma_i} |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)||\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))|d\zeta$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\mathfrak{u}(\zeta)) + \mathcal{N}_2(\zeta,\mathfrak{u}'(\zeta)) + \mathcal{N}_3(\zeta,\phi_{\mathfrak{u}}(\zeta))\Big)d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\mathfrak{u}(\zeta)) + \mathcal{N}_2(\zeta,\mathfrak{u}'(\zeta)) + \mathcal{N}_3(\zeta,\phi_{\mathfrak{u}}(\zeta))\Big)d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i} |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\mathfrak{u}(\zeta)) + \mathcal{N}_2(\zeta,\mathfrak{u}'(\zeta)) + \mathcal{N}_3(\zeta,\phi_{\mathfrak{u}}(\zeta))\Big)d\zeta.
\end{aligned}
$$

Consequently, for $u \in \Omega$, hence

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t)| \leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\|\mathfrak{u}\|) + \mathcal{N}_2(\zeta,\|\mathfrak{u}'\|) + \mathcal{N}_3(\zeta,\omega_1\|\mathfrak{u}\| + \omega_2\|\mathfrak{u}'\|)\Big)d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\|\mathfrak{u}\|) + \mathcal{N}_2(\zeta,\|\mathfrak{u}'\|) + \mathcal{N}_3(\zeta,\omega_1\|\mathfrak{u}\| + \omega_2\|\mathfrak{u}'\|)\Big)d\zeta \\
&+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i} |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\|\mathfrak{u}\|) + \mathcal{N}_2(\zeta,\|\mathfrak{u}'\|) + \mathcal{N}_3(\zeta,\omega_1\|\mathfrak{u}\| + \omega_2\|\mathfrak{u}'\|)\Big)d\zeta \\
&\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)| \\
&\times\Big(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*) + \mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*) + \mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\Big)d\zeta
\end{aligned}
$$

$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|$$
$$\times\Big(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\Big)d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|$$
$$\times\Big(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\Big)d\zeta$$
$$\leq\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|$$
$$\times\Big(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\Big)d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|$$
$$\times\Big(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\Big)d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|$$
$$\times\Big(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\Big)d\zeta$$
$$\leq\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\Xi\|\mathfrak{u}\|_*(\sum_{j=1}^3\mathcal{V}_j(\zeta)+\epsilon_0)d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^\eta |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\Xi\|\mathfrak{u}\|_*(\sum_{j=1}^3\mathcal{V}_j(\zeta)+\epsilon_0)d\zeta$$
$$+\frac{1}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\Xi\|\mathfrak{u}\|_*(\sum_{j=1}^3\mathcal{V}_j(\zeta)+\epsilon_0)d\zeta$$
$$=\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Big(\sum_{j=1}^3(\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|\mathcal{V}_j(\zeta)d\zeta)+3\epsilon_0\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(t,\zeta)|d\zeta\Big)$$
$$+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\Big(\sum_{j=1}^3(\int_0^\eta|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|\mathcal{V}_j(\zeta)d\zeta)$$
$$+3\epsilon_0\int_0^\eta|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(\eta,\zeta)|d\zeta\Big)$$

$$
\begin{aligned}
&+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\bigg(\sum_{j=1}^{3}(\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|\mathcal{V}_j(\zeta)d\zeta)\\
&+3\epsilon_0\int_0^{\gamma_i}|\mathcal{H}_{\mathfrak{a},\mathfrak{r}}(\gamma_i,\zeta)|d\zeta\bigg)\\
&=\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}(\int_0^t\int_\zeta^t\frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi\mathcal{V}_j(\zeta)d\zeta)\\
&+3\epsilon_0\int_0^t\int_\zeta^t\frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi d\zeta\bigg)\\
&+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\bigg(\sum_{j=1}^{3}(\int_0^\eta\int_\zeta^t\frac{(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi\mathcal{V}_j(\zeta)d\zeta)\\
&+3\epsilon_0\int_0^\eta\int_\zeta^\eta\frac{(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi d\zeta\bigg)\\
&+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\bigg(\sum_{j=1}^{3}(\int_0^{\gamma_i}\int_\zeta^{\gamma_i}\frac{(\gamma_i-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi\mathcal{V}_j(\zeta)d\zeta)\\
&+3\epsilon_0\int_0^{\gamma_i}\int_\zeta^{\gamma_i}\frac{(\gamma_i-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi d\zeta\bigg),
\end{aligned}
$$

thus, it is concluded that for all $\mathfrak{u}\in\Omega$ and $t\in[0,1]$

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t)|\le\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}(\int_0^t\int_0^\xi\frac{(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\mathcal{V}_j(\zeta)d\zeta d\xi)\\
&+3\epsilon_0\int_0^t\frac{1}{|g(\xi)|}\int_0^\xi(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}d\zeta d\xi\bigg)\\
&+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\bigg(\sum_{j=1}^{3}(\int_0^\eta\int_0^\xi\frac{(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\mathcal{V}_j(\zeta)d\zeta d\xi)\\
&+3\epsilon_0\int_0^\eta\frac{1}{|g(\xi)|}\int_0^\xi(\eta-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}d\zeta d\xi\bigg)
\end{aligned}
$$

$$+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\bigg(\sum_{j=1}^{3}(\int_0^{\gamma_i}\int_0^{\xi}\frac{(\gamma_i-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\mathcal{V}_j(\zeta)d\zeta d\xi)$$
$$+3\epsilon_0\int_0^{\gamma_i}\frac{1}{|g(\xi)|}\int_0^{\xi}(\gamma_i-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}d\zeta d\xi\bigg).$$

Since $\xi\in[\zeta,t]$, then $(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}\leq(t-\zeta)^{\mathfrak{a}+\mathfrak{r}-2}$, hence for $\xi\in[\zeta,t]$ we have

$$\int_0^{\xi}\frac{(t-\xi)^{\mathfrak{r}-1}(\xi-s)^{\mathfrak{a}-1}}{|g(\xi)|}\mathcal{V}_j(\zeta)d\zeta\leq\frac{1}{|g(\xi)|}\int_0^{\xi}(t-\zeta)^{\mathfrak{a}+\mathfrak{r}-2}\mathcal{V}_j(\zeta)d\zeta$$
$$=\frac{1}{|g(\xi)|}\hat{\mathcal{V}}_j^{\mathfrak{a},\mathfrak{r}}(t,\xi),$$

also we have $\int_0^{\xi}(t-\xi)^{\mathfrak{r}-1}(\xi-\zeta)^{\mathfrak{a}-1}d\zeta=\frac{(t-\xi)^{\mathfrak{r}-1}\xi^{\mathfrak{a}}}{\mathfrak{a}}$. So for all $\mathfrak{u}\in\Omega$ and $t\in[0,1]$ it is infered that

$$|\mathcal{L}\mathfrak{u}(t)|\leq\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}\int_0^{t}\frac{\hat{\mathcal{V}}_j^{\mathfrak{a},\mathfrak{r}}(t,\xi)}{|g(\xi)|}d\xi+3\epsilon_0\int_0^{t}\frac{(t-\xi)^{\mathfrak{r}-1}\xi^{\mathfrak{a}}}{\mathfrak{a}|g(\xi)|}d\xi\bigg)$$
$$+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\bigg(\sum_{j=1}^{3}\int_0^{\eta}\frac{\hat{\mathcal{V}}_j^{\mathfrak{a},\mathfrak{r}}(\eta,\xi)}{|g(\xi)|}d\xi+3\epsilon_0\int_0^{\eta}\frac{(\eta-\xi)^{\mathfrak{r}-2}\xi^{\mathfrak{a}}}{\mathfrak{a}|g(\xi)|}d\xi\bigg)$$
$$+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\sum_{i=1}^{k_0}|\lambda_i|\bigg(\sum_{j=1}^{3}\int_0^{\gamma_i}\frac{\hat{\mathcal{V}}_j^{\mathfrak{a},\mathfrak{r}}(\gamma_i,\xi)}{|g(\xi)|}d\xi$$
$$+3\epsilon_0\int_0^{\gamma_i}\frac{(\gamma_i-\xi)^{\mathfrak{r}-1}\xi^{\mathfrak{a}}}{\mathfrak{a}|g(\xi)|}d\xi\bigg),$$

therefore

$$|\mathcal{L}\mathfrak{u}(t)|\leq\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,t]}+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\bigg)$$
$$+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\bigg(\sum_{j=1}^{3}\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\eta]}+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|\bigg)$$
$$+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}\bigg[\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\bigg]+\frac{3\epsilon_0}{\mathfrak{a}}(\sum_{i=1}^{k_0}|\lambda_i|)\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\bigg).$$

Taking the supremum norm over $[0,1]$, we conclude that

$$\begin{aligned}
&\|\mathcal{L}\mathfrak{u}| \leq \frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\bigg)\\
&+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\bigg(\sum_{j=1}^{3}\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\eta]}+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|\bigg)\\
&+\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg(\sum_{j=1}^{3}\bigg[\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\bigg]+\frac{3\epsilon_0}{\mathfrak{a}}(\sum_{i=1}^{k_0}|\lambda_i|)\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\bigg)\\
&=\frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg[\sum_{j=1}^{3}\bigg(|\Delta|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,1]}+(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\eta]}\\
&+\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\bigg)\bigg]\\
&+\frac{3\Xi\epsilon_0\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a}+1)\Gamma(\mathfrak{r})}\bigg[|\Delta\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|+(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|+(\sum_{i=1}^{k_0}|\lambda_i|)\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\bigg]\\
&\leq\bigg(\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\bigg[\sum_{j=1}^{3}\bigg(|\Delta|(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}+(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}}\|_{[0,\eta]}\\
&+\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\bigg)\bigg]\\
&+\frac{3\Xi\epsilon_0}{|\Delta|\Gamma(\mathfrak{a}+1)\Gamma(\mathfrak{r})}\bigg[|\Delta|(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|+(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|+(\sum_{i=1}^{k_0}|\lambda_i|)\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\bigg]\bigg)R_0\\
&\leq R_0.
\end{aligned}$$

Likewise, for all $t\in[0,1]$ and $\mathfrak{u}\in\Omega$ we have

$$\begin{aligned}
&|\mathcal{L}'\mathfrak{u}(t)|\leq\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)||\Theta(\zeta,\mathfrak{u}(\zeta),\mathfrak{u}'(\zeta),\phi_{\mathfrak{u}}(\zeta))|d\zeta\\
&\leq\frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)|\\
&\times\bigg(\mathcal{N}_1(\zeta,\|\mathfrak{u}\|)+\mathcal{N}_2(\zeta,\|\mathfrak{u}'\|)+\mathcal{N}_3(\zeta,\omega_1\|\mathfrak{u}\|+\omega_2\|\mathfrak{u}'\|)\bigg)d\zeta
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)|$$
$$\times\bigg(\mathcal{N}_1(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_2(\zeta,\Xi\|\mathfrak{u}\|_*)+\mathcal{N}_3(\zeta,\Xi\|\mathfrak{u}\|_*)\bigg)d\zeta$$
$$\leq \frac{1}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)|\Xi\|\mathfrak{u}\|_*(\sum_{j=1}^3\mathcal{V}_j(\zeta)+\epsilon_0)d\zeta$$
$$=\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3\int_0^t |\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)|\mathcal{V}_j(\zeta)d\zeta+3\epsilon_0\int_0^t|\mathcal{H}_{\mathfrak{a},\mathfrak{r}-1}(t,\zeta)|d\zeta\right)$$
$$=\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3\int_0^t\int_\zeta^t\frac{(t-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\mathcal{V}_j(\zeta)d\xi d\zeta\right.$$
$$\left.+3\epsilon_0\int_0^t\int_\zeta^t\frac{(t-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}d\xi d\zeta\right)$$
$$=\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3(\int_0^t\int_0^\xi\frac{(t-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}}{|g(\xi)|}\mathcal{V}_j(\zeta)d\zeta d\xi)\right.$$
$$\left.+3\epsilon_0\int_0^t\frac{1}{|g(\xi)|}\int_0^\xi(t-\xi)^{\mathfrak{r}-2}(\xi-\zeta)^{\mathfrak{a}-1}d\zeta d\xi\right)$$
$$\leq\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3\int_0^t\frac{1}{|g(\xi)|}(\int_0^\xi(t-\zeta)^{\mathfrak{a}+\mathfrak{r}-3}\mathcal{V}_j(\zeta)d\zeta)d\xi\right.$$
$$\left.+\frac{3\epsilon_0}{\mathfrak{a}}\int_0^t\frac{(t-\xi)^{\mathfrak{r}-2}\xi^{\mathfrak{a}}}{|g(\xi)|}d\xi\right)$$
$$\leq\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3\int_0^t\frac{\hat{\mathcal{V}}_j^{\mathfrak{a},\mathfrak{r}-1}(t,\xi)}{|g(\xi)|}d\xi+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|\right)$$
$$=\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,t]}+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|\right).$$

Therefore

$$\|\mathcal{L}'\mathfrak{u}\|\leq\frac{\Xi\|\mathfrak{u}\|_*}{\Gamma(\mathfrak{a})\Gamma(\mathfrak{r}-1)}\left(\sum_{j=1}^3\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]}+\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|\right)$$

$$= \frac{\Xi\|\mathfrak{u}\|_*}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Big(|\Delta|(\mathfrak{r}-1)\sum_{j=1}^{3}\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} + |\Delta|(\mathfrak{r}-1)\frac{3\epsilon_0}{\mathfrak{a}}\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|\Big)$$

$$\leq \Bigg(\frac{\Xi}{|\Delta|\Gamma(\mathfrak{a})\Gamma(\mathfrak{r})}\Bigg[\sum_{j=1}^{3}\Bigg(|\Delta|(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}-1}\|_{[0,1]} + (\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\eta]}$$

$$+\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\mathfrak{a},\mathfrak{r}}\|_{[0,\gamma_i]}\Bigg)\Bigg] + \frac{3\Xi\epsilon_0}{|\Delta|\Gamma(\mathfrak{r}+1)\Gamma(\mathfrak{r})}\Bigg[|\Delta|(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\|$$

$$+(\mathfrak{r}-1)\|\bar{g}_{\mathfrak{a},\mathfrak{r}-1}\| + (\sum_{i=1}^{k_0}|\lambda_i|)\|\bar{g}_{\mathfrak{a},\mathfrak{r}}\|\Bigg]\Bigg)R_0 \leq R_0.$$

Thus, we conclude that

$$\|\mathcal{L}\mathfrak{u}\|_* = \max\{\|\mathcal{L}\mathfrak{u}\|, \|\mathcal{L}'\mathfrak{u}\|\} \leq R_0,$$

hence $\mathcal{L}\mathfrak{u} \in \Omega$. By a similar way, it is resulted in $\mathcal{L}\mathfrak{v} \in \Omega$, this implies that $\mathcal{A}(\mathcal{L}\mathfrak{u}, \mathcal{L}\mathfrak{v}) \geq 1$, therefore $\mathcal{L}$ is $\mathcal{A}$- admissible. Evidently Ω is nonempty, so there exists $\mathfrak{u}_0 \in \Omega$, we further proved that $\mathcal{L}\mathfrak{u}_0 \in \Omega$, which leads to $\mathcal{A}(\mathfrak{u}_0, \mathcal{L}\mathfrak{u}_0) \geq 1$. Let $\mathfrak{u}, \mathfrak{v} \in X$, if $\mathcal{A}(\mathfrak{u}, \mathfrak{v}) \neq 0$, then $\mathfrak{u}, \mathfrak{v} \in \Omega$, therefore

$$d(\mathfrak{u}, \mathfrak{v}) \leq \|\mathfrak{u}\|_* + \|\mathfrak{v}\|_* \leq 2\frac{\delta_m(\epsilon_1)}{2} = \delta_m(\epsilon_1).$$

By (4), the following inequality is held

$$\Lambda(\Xi\|\mathfrak{u}-\mathfrak{v}\|_*, \Xi\|\mathfrak{u}-\mathfrak{v}\|_*, \Xi\|\mathfrak{u}-\mathfrak{v}\|_*) \leq \Xi(q_j+\epsilon_1)\|\mathfrak{u}-\mathfrak{v}\|_*,$$

so for all $t \in [0,1]$, (5) implies that

$$
\begin{aligned}
&|\mathcal{L}\mathfrak{u}(t)-\mathcal{L}\mathfrak{v}(t)| \leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\alpha)\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}(q_j+\epsilon_1)\int_0^1 \tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}(1,\xi)d\xi\\
&+\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon_1)\int_0^1 \tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}(1,\xi)d\xi\\
&+\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}\left[(q_j+\epsilon_1)\left(\sum_{i=1}^{k_0}|\lambda_i|\int_0^1 \tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}(1,\xi)d\xi\right)\right]\\
&\leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\alpha)\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}(q_j+\epsilon_1)\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\\
&+\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon_1)\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}\\
&+\frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\sum_{j=1}^{k^*}\sum_{i=1}^{k_0}(q_j+\epsilon_1)|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\\
&=\Bigg[\frac{\Xi}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}+(\mathfrak{r}-1)\sum_{j=1}^{k^*}q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}\\
&+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0}q_j|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\Bigg)\\
&+\frac{\Xi\epsilon_1}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}+(\mathfrak{r}-1)\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}\\
&+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0}|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\Bigg)\Bigg]\|\mathfrak{u}-\mathfrak{v}\|_*.
\end{aligned}
$$

Let

$$
\begin{aligned}
\lambda := \Bigg[&\frac{\Xi}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\alpha,\beta-1}\|_{[0,1]}\\
&+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0} q_j|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\Bigg) + \frac{\Xi\epsilon_1}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Bigg(|\Delta|\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\\
&+(\mathfrak{r}-1)\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]} + \sum_{j=1}^{k^*}\sum_{i=1}^{k_0}|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\Bigg)\Bigg] < 1.
\end{aligned}
$$

So $\|\mathcal{L}\mathfrak{u} - \mathcal{L}\mathfrak{v}\| \leq \lambda\|\mathfrak{u}-\mathfrak{v}\|_*$. By similar way, for $u, v \in X$ in which $\mathcal{A}(\mathfrak{u},\mathfrak{v}) \neq 0$, it follows that

$$
\begin{aligned}
|\mathcal{L}'\mathfrak{u}(t) - \mathcal{L}'\mathfrak{v}(t)| &\leq \frac{\Xi\|\mathfrak{u}-\mathfrak{v}\|_*}{\Gamma(\alpha)\Gamma(\mathfrak{r}-1)}\sum_{j=1}^{k^*}(q_j+\epsilon_1)\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}\\
&= \Bigg[\frac{\Xi}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Bigg(|\Delta|(\mathfrak{r}-1)\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}\Bigg)\\
&+\frac{\Xi\epsilon_0}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Bigg(|\Delta|(\mathfrak{r}-1)\sum_{j=1}^{k^*}\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}\Bigg)\Bigg]\|\mathfrak{u}-\mathfrak{v}\|_*\\
&\leq \lambda\|\mathfrak{u}-\mathfrak{v}\|_*.
\end{aligned}
$$

So $\|\mathcal{L}'\mathfrak{u} - \mathcal{L}'\mathfrak{v}\| \leq \lambda\|\mathfrak{u}-\mathfrak{v}\|_*$. and $\|\mathcal{L}\mathfrak{u} - \mathcal{L}\mathfrak{v}\|_* \leq \lambda\|\mathfrak{u}-\mathfrak{v}\|_*$. Define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \lambda t$, then $\sum_{i=1}^{\infty}\psi^i(t) = \frac{\lambda}{1-\lambda}t < \infty$ for all $t \in [0,\infty)$, so $\psi \in \Psi$. Therefore we have proved $\mathfrak{u},\mathfrak{v} \in X$ in which $\mathcal{A}(\mathfrak{u},\mathfrak{v}) \neq 0$, $\mathcal{A}(\mathfrak{u},\mathfrak{v})d(\mathcal{L}\mathfrak{u},\mathcal{L}\mathfrak{v}) \leq \psi(d(\mathfrak{u},\mathfrak{v}))$. In the case $\mathcal{A}(\mathfrak{u},\mathfrak{v}) = 0$, the inequality is obvious. So for all $\mathfrak{u},\mathfrak{v} \in X$, the inequality $\mathcal{A}(\mathfrak{u},\mathfrak{v})d(\mathcal{L}\mathfrak{u},\mathcal{L}\mathfrak{v}) \leq \psi(d(\mathfrak{u},\mathfrak{v}))$ is held. Now, regarding lemma (1.2), $\mathcal{L} : X \to X$ has a fixed point in X, so the singular problem (1) has a solution. □

The following example demonstrates the main result.

Example 2.3. Let

$$
c(t) = \begin{cases} 0 & t \in [0,1]\cap Q \\ \\ 1 & t \in (0,1)\cap Q^c. \end{cases}
$$

and

$$\Theta(t, x_1, x_2, x_3) = \frac{1}{c(t)}(\|x_1\| + \|x_2\| + \|x_3\|).$$

Consider the following pointwise defined bi-singular equation

$$\mathcal{D}^{\frac{3}{2}}\left(3\sqrt{t}\mathcal{D}^{\frac{5}{2}}\mathfrak{u}(t)\right) = \Theta(t, \mathfrak{u}(t), \mathfrak{u}'(t), \mathcal{D}^{\frac{1}{2}}\mathfrak{u}(t)) \tag{6}$$

with boundary condition $\mathcal{D}^{\frac{5}{2}+j}\mathfrak{u}(0) = \mathfrak{u}'(0) = 0$ for $0 \leq j \leq 2$ and $\mathfrak{u}'(\frac{1}{2}) = 2\mathfrak{u}(\frac{1}{2})$. Put $k_0 = 1$, $\gamma_1 = \frac{1}{2}$, $\eta = \frac{1}{2}$, $g(t) = 3\sqrt{t}$ and $\phi_{\mathfrak{u}}(t) = \mathcal{D}^{\frac{1}{2}}\mathfrak{u}(t)$, then

$$\|\phi_{\mathfrak{u}} - \phi_{\mathfrak{v}}\| \leq \frac{1}{\Gamma(2-\frac{1}{2})}\|\mathfrak{u}' - \mathfrak{v}'\| = \frac{2}{\sqrt{\pi}}\|\mathfrak{u}' - \mathfrak{v}'\|,$$

so $\omega_1 = 0$, $\omega_2 = \frac{2}{\sqrt{\pi}}$ and $\Xi = \max\{1, \omega_1 + \omega_2\} = 1$. Regarding lemma (1.3), it is resulted in

$$\begin{aligned}\|\bar{g}_{\alpha,\mathfrak{r}-1}\| = \int_0^1 \frac{(1-\zeta)^{\frac{1}{2}}\zeta^{\frac{3}{2}}}{3\sqrt{\zeta}} = \frac{1}{3}\int_0^1 (1-\zeta)^{\frac{1}{2}}\zeta = \frac{1}{3}\mathcal{B}(2, \frac{3}{2})\\ = \frac{\Gamma(\frac{3}{2})\Gamma(2)}{3\Gamma(\frac{5}{2})} = \frac{2}{9} < \infty.\end{aligned}$$

Let $k^* = 1$, $\theta_1(t) = \frac{1}{c(t)}$, $\mathcal{N}_i(t, x_i) = \frac{1}{c(t)}x_i$, and $\Lambda_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, then

$$|\Theta(t, \omega_1, \omega_2, \omega_3) - \Theta(t, z_1, z_2, z_3)| \leq \theta_1(t)\Lambda_1(|\omega_1 - z_1|, |\omega_2 - z_2|, |\omega_3 - z_3|),$$

Λ_1 is nondecreasing with respect to all their components,

$$q_i := \lim_{z\to 0^+}\frac{\Lambda(z, z, z)}{z} = 3 \in [0, \infty),$$

$$\hat{\theta}_1^{\alpha,\mathfrak{r}}(t, \xi) = \int_0^\xi \frac{(t-\zeta)^2}{c(\zeta)}d\zeta = \frac{1}{3}[t^3 - (t-\xi)^3],$$

$$\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]} := \frac{1}{3}\int_0^1 [1 - (1-\xi)^3]\xi^{\frac{-1}{2}}d\xi = \frac{38}{105},$$

$|\Theta(t,x_1,x_2,x_3)| \leq \sum_{i=11}^{3}\mathcal{N}_i(t,x_i)$, $\mathcal{N}_i : [0,1]\times X \to [0,\infty)$ for each $1 \leq i \leq 3$ is nondecreasing with respect to its second component, $\mathcal{V}_i(t) = \lim_{z\to 0^+} \frac{\mathcal{N}_i(t,z)}{z} = \frac{1}{c(t)}$,

$$\hat{\mathcal{V}}_i^{\alpha,\mathfrak{r}-1} = \int_0^\xi (t-s)^{\alpha+\mathfrak{r}-3}\mathcal{V}_i(s)ds = \frac{1}{2}[t^2-(t-\xi)^2]$$

and

$$\|\hat{\mathcal{V}}_i^{\alpha,\mathfrak{r}-1}\|_{[0,1]} = \frac{1}{2}\int_0^1 [1-(1-\xi)^2]\xi^{\frac{-1}{2}}d\xi = \frac{7}{15}.$$

It is easy to see the other properties in Theorem (2.2) are held and

$$\sum_{j=1}^{3}\Big(|\Delta|(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]} + (\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}}\|_{[0,\eta]} + \sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}}\|_{[0,\gamma_i]}\Big) \in [0, \frac{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}{\Xi})$$

and

$$\frac{\Xi}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Big(|\Delta|\sum_{j=1}^{k^*}q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*}q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]} + \sum_{j=1}^{k^*}\sum_{i=1}^{k_0}q_j|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\Big) < 1.$$

Therefore, by using Theorem (2.2) , the bi-singular problem (6) has a solution.

Example 2.4. Consider the singular problem

$$\mathcal{D}^{\frac{3}{2}}\Big(5t\mathcal{D}^{\frac{3}{2}}\mathfrak{u}(t)\Big) = \frac{|\mathfrak{u}(t)|}{1+|\mathfrak{u}(t)|} + \mathfrak{u}'(t) + |\int_0^t \mathfrak{u}(s)ds| \tag{7}$$

with boundary condition $\mathcal{D}^{\frac{3}{2}+j}\mathfrak{u}(0) = \mathfrak{u}'(0) = 0$ for $0 \leq j \leq 2$ and $\mathfrak{u}'(\frac{1}{2}) = \mathfrak{u}(\frac{1}{2})$. Put $k_0 = 1$, $\gamma_1 = 1$, $\Delta = \lambda_1 = 1$, $\eta = \frac{1}{2}$, $g(t) = 5t$, $\phi_{\mathfrak{u}}(t) = \mathcal{D}^{\frac{1}{2}}\mathfrak{u}(t)$ and

$$\Theta(t,x_1,x_2,x_3) = \|x_1\| + \|x_2\| + \|x_3\|.$$

then

$$\begin{aligned}\|\phi_{\mathfrak{u}}-\phi_{\mathfrak{v}}\| &\leq |\int_0^t \mathfrak{u}(s)ds-\int_0^t \mathfrak{v}(s)ds| \leq \int_0^t |\mathfrak{u}(s)-\mathfrak{v}(s)|ds \\ &\leq \|\mathfrak{u}-\mathfrak{v}\|\int_0^t ds \leq \|\mathfrak{u}-\mathfrak{v}\|,\end{aligned}$$

so $\omega_1 = 1$, $\omega_2 = 0$ and $\Xi = \max\{1, \omega_1+\omega_2\} = 1$. Also we have

$$\|\bar{g}_{\alpha,\mathfrak{r}-1}\| = \frac{1}{5}\int_0^1 \frac{(1-\zeta)^{\frac{-1}{2}}\zeta^{\frac{3}{2}}}{t} = \frac{1}{5}\int_0^1 (1-\zeta)^{\frac{-1}{2}}\zeta^{\frac{1}{2}} = \frac{1}{5} < \infty.$$

Let $k^* = 1$, $\theta_1(t) = 1$, $\mathcal{N}_i(t, x_i) = x_i$, and $\Lambda_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, then

$$\frac{|\mathfrak{u}|}{1+|\mathfrak{u}|}-\frac{|\mathfrak{v}|}{1+|\mathfrak{v}|} = \frac{|\mathfrak{u}|+|\mathfrak{u}||\mathfrak{v}|-|\mathfrak{u}||\mathfrak{v}|-|\mathfrak{v}|}{(1+|\mathfrak{u}|)(1+|\mathfrak{v}|)} \leq \frac{|\mathfrak{u}-\mathfrak{v}|}{(1+|\mathfrak{u}|)(1+|\mathfrak{v}|)} \leq |\mathfrak{u}-\mathfrak{v}|,$$

therefore

$$|\Theta(t, \omega_1, \omega_2, \omega_3) - \Theta(t, z_1, z_2, z_3)| \leq \theta_1(t)\Lambda_1(|\omega_1 - z_1|, |\omega_2 - z_2|, |\omega_3 - z_3|),$$

Λ_1 is nondecreasing with respect to all their components,

$$q_i := \lim_{z\to 0^+} \frac{\Lambda(z, z, z)}{z} \leq 3 \in [0, \infty),$$

$$\hat{\theta}_1^{\alpha,\mathfrak{r}-1}(t, \xi) = \int_0^\xi d\zeta = \xi,$$

$$\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]} := \frac{1}{5}\int_0^1 \frac{\xi}{\xi} d\xi = \frac{1}{5},$$

$|\Theta(t, x_1, x_2, x_3)| \leq \sum_{i=11}^3 \mathcal{N}_i(t, x_i)$, $\mathcal{N}_i : [0, 1] \times X \to [0, \infty)$ for each $1 \leq i \leq 3$ is nondecreasing with respect to its second component, $\mathcal{V}_i(t) = \lim_{z\to 0^+} \frac{\mathcal{N}_i(t,z)}{z} = 1$,

$$\hat{\mathcal{V}}_i^{\alpha,\mathfrak{r}-1} = \int_0^\xi (t-s)^{\alpha+\mathfrak{r}-3}\mathcal{V}_i(s)ds = \xi$$

and

$$\|\hat{\mathcal{V}}_i^{\alpha,\mathfrak{r}-1}\|_{[0,1]} = \frac{1}{5}\int_0^1 \xi^{\frac{-1}{2}} d\xi = \frac{2}{5}.$$

One can see, the following properties are held

$$\sum_{j=1}^{3}\Big(|\Delta|(\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]} + (\mathfrak{r}-1)\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}}\|_{[0,\eta]}$$
$$+\sum_{i=1}^{k_0}|\lambda_i|\|\hat{g}_{\mathcal{V}_j}^{\alpha,\mathfrak{r}}\|_{[0,\gamma_i]}\Big) \in [0, \frac{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}{\Xi})$$

and

$$\frac{\Xi}{|\Delta|\Gamma(\alpha)\Gamma(\mathfrak{r})}\Big(|\Delta|\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]} + (\mathfrak{r}-1)\sum_{j=1}^{k^*} q_j\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}-1}\|_{[0,1]}$$
$$+\sum_{j=1}^{k^*}\sum_{i=1}^{k_0} q_j|\lambda_i|\|\tilde{g}_{\theta_j}^{\alpha,\mathfrak{r}}\|_{[0,1]}\Big) < 1.$$

Thus, by using Theorem (2.2) , the singular problem (7) has a solution.

3 Conclusion

There are no many methods regarding the singular differential equations. Using control functions method causes investigating the multi-singular differential equations with less limited conditions in their properties. The given techniques in this paper can be applied to consider a solution for many other problems, also bi- singular type of the differential equations can be studied. In this article, we introduce bi-singularity concept and consider a bi-singular fractional-order differential equation and prove the existence of a solution for the problem by using inequalities and control functions method. The main result is demonstrated through two examples.

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