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Direct Method to Solve Differential-Algebraic Equations by Using the Operational Matrices of Chebyshev Cardinal Functions

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Abstract. A new and effective direct method to determine the numerical solution of linear and nonlinear differential-algebraic equations (DAEs) is proposed. The method consists of expanding the required approximate solution as the elements of Chebyshev cardinal functions. The operational matrices for the integration and product of the Chebyshev cardinal functions are presented. A general procedure for forming these matrices is given. These matrices play an important role in modelling of problems. By using these operational matrices together, a differential-algebraic equation can be transformed to a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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1. Introduction

Differential-algebraic equations (DAEs) can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical systems, power systems, chemical process simulation, optimal control atc. Also, many important mathematical models can be expressed in terms of differential-algebraic equations. In resent years, much research has been focused on the numerical solution of systems of differential-algebraic equations. The numerical approaches include the backward differentiation formulae (BDF) [6, 8], Runge-Kutta method [2], specialized Runge-Kutta method, which is a modification of the classic Runge-Kutta method to solve index-2 DAEs [17] and Krylov deffered correction (KDC) method [16]. Recently, tau method [21], the Adomian decomposition method [14, 15], the Variational iteration method (VIM) [23] and Homotopy perturbation method (HPM)[22] have been used to solve the linear and nonlinear DAEs. A system of DAEs is characterized by its index, which is the number of differentiations required to convert it into a system of ODEs. DAEs with index $\geqslant 2$ are generally hard to solve and are still under active research. In this paper we apply the Chebyshev cardinal function bases to solve linear and nonlinear differential-algebraic equation. The method consists in reducing the differential-algebraic equation to a set of algebraic equations by expanding the current system as Chebyshev cardinal functions with unknown coefficients. The properties of Chebyshev cardinal functions are then utilized to evaluate the unknown coefficients.

The outline of this paper is as follows. In Section 2., we briefly present the main steps of reducing index method for linear semi-explicit DAEs. In Section 3., we describe the basic properties of the Chebyshev cardinal functions required for our subsequent development. In Section 4., the operational matrices of the integration and the product of Chebyshev cardinal functions are presented. Section 5. is devoted to the solution of differential-algebraic equations. Some numerical illustrations are given in Section 6. to show the efficiency of the proposed method. Finally, a brief conclusion is drawn in Section 7.

2. DAEs and Reducing Index

A system of DAEs is one that consists of ordinary differential equations (ODEs) coupled with purely algebraic equations,on the other hand, DAEs are everywhere singular implicit ODEs. The general form of DAEs is

$$F(x(t), x'(t), t) = 0, \qquad F \in C^1(R^{2m+1}, R^m), \quad t \in [t_0, t_f], \tag{1}$$

where $\partial F/\partial x'$ is singular on R^{2m+1}[24]. Most DAEs arising in applications are in semi-explicit form and many are in the further restricted Hessenberg form[6].

The index-1 semi-explicit DAEs is given by:

$$\begin{cases} x'(t) = f(x(t), y(t), t), & f \in C^1(R^{m+k+1}, R^m), \quad t \in [t_0, t_f], \\ 0 = g(x(t), y(t), t), & g \in C^1(R^{m+k+1}, R^k), \end{cases} \tag{2}$$

where $\partial g/\partial y$ is non-singular.

The index-2 Hessenberg DAEs is given by:

$$\begin{cases} x'(t) = f(x(t), y(t), t), & f \in C^1(R^{m+k+1}, R^m), \quad t \in [t_0, t_f], \\ 0 = g(x(t), t), & g \in C^2(R^{m+1}, R^k), \end{cases} \tag{3}$$

where $(\partial g/\partial x)(\partial f/\partial y)$ is non-singular[24].

Now, we briefly review the reducing index method for semi-explicit DAEs, which mentioned in [4, 13].

Consider a linear (or linearized) semi-explicit DAEs:

$$\begin{cases} X^{(m)} = \sum_{j=1}^{m} A_j X^{(j-1)} + BY + q, \\ 0 = CX + r, \end{cases} \tag{4}$$

where A_j, B and C are smooth functions of t, $t_0 \leqslant t \leqslant t_f$, $A_j(t) \in R^{n\times n}$, $j = 1, 2, \cdots m$, $B(t) \in R^{n\times k}$, $C(t) \in R^{k\times n}$, $n \geqslant 2$, $1 \leqslant k \leqslant n$ and CB is non-singular (DAE has indix $m+1$) except possibly at a finite number of isolated points of t, which in this case, the DAEs (4) have constraint singularity. The inhomogeneities are $q(t) \in R^n$ and $r(t) \in R^k$.

Now suppose that CB is nonsingular, from (4), we can write

$$Y = (CB)^{-1}C\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right], \quad t \in [t_0, t_f]. \tag{5}$$

Substituting (5) into (4) implies that

$$[I - B(CB)^{-1}C]\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0, \tag{6}$$

so, problem (4) transforms to the overdetermined system:

$$\begin{cases} [I - B(CB)^{-1}C]\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0, \quad t \in [t_0, t_f], \\ 0 = CX + r. \end{cases} \tag{7}$$

Now, system (7) can be transformed to a full-rank DAE system with n equations and n unknowns with index m [4, 13]. Here, for simplicity, we consider problem (4) when $m = 1$ (problem has index 2), $n = 2, 3$ and $k = 1, 2$. Also, if we suppose that DAE is nonsingular, i.e.

$$CB(t) \neq 0, \quad t \in [t_0, t_f], \tag{8}$$

then by the following theorems, the given index-2 problem will transform to index-1 DAE. This discussion can be extended to general form of (4).

Theorem 2.1. *Consider problem (4) with index-2, $n = 2$ and $k = 1$. This problem is equivalent to the following index-1 DAE system:*

$$E_1 X' + E_0 X = \hat{q}, \tag{9}$$

such that

$$E_0 = \begin{bmatrix} b_1(t)a_{21}(t) - b_2(t)a_{11}(t) & b_1(t)a_{22}(t) - b_2(t)a_{12}(t) \\ c_1(t) & c_2(t) \end{bmatrix},$$

$$E_1 = \begin{bmatrix} b_2(t) & -b_1(t) \\ 0 & 0 \end{bmatrix}, \quad \hat{q} = \begin{bmatrix} b_2(t)q_1(t) - b_1(t)q_2(t) \\ -r(t) \end{bmatrix}, \tag{10}$$

and

$$y = (CB)^{-1}C[X' - AX - q]. \tag{11}$$

proof. this theorem is presented in [4]. □

Theorem 2.2. *Consider problem (4) with index-2, $n = 3$ and $k = 2$. This problem is equivalent to the following index-1 DAE system:*

$$\begin{bmatrix} M \\ 0 \end{bmatrix} X' + \begin{bmatrix} -MA \\ C \end{bmatrix} X = \begin{bmatrix} Mq \\ -r \end{bmatrix}, \tag{12}$$

such that

$$M = [b_{21}(t)b_{32}(t) - b_{22}(t)b_{31}(t)b_{12}(t)b_{31}(t) - b_{11}(t)b_{32}(t)b_{11}(t)b_{22}(t)]$$

$$[-b_{12}(t)b_{21}(t)]_{1\times 3}, \tag{13}$$

and

$$Y = (CB)^{-1}C[X' - AX - q]. \tag{14}$$

proof. It is presented in [13]. □

3. Chebyshev Cardinal Functions

Chebyshev cardinal functions of order N in $[-1, 1]$ are defined as [5]

$$C_j(x) = \frac{T_{N+1}(x)}{T_{N+1,x}(x_j)(x - x_j)}, \quad j = 1, 2, \ldots, N+1, \tag{15}$$

where $T_{N+1}(x)$ is the first kind Chebyshev function of order $N + 1$ in $[-1, 1]$ defined by

$$T_{N+1}(x) = \cos((N + 1)\arccos(x)), \tag{16}$$

subscript x denotes x-differentiation and x_j, $j = 1, 2, \ldots, N + 1$, are the zeros of $T_{N+1}(x)$ defined by $\cos(\frac{(2j-1)\pi}{2N+2})$, $j = 1, 2, \ldots, N + 1$.

Lemma 3.1. *The functions* $C_j(x),\ j = 1, 2, \cdots, N+1$ *are orthogonal with respect to* $w(x) = \frac{1}{\sqrt{1-x^2}}$ *on* $[-1, 1]$ *and satisfy the orthogonality condition*

$$\langle C_i(x), C_j(x)\rangle_w = \int_{-1}^{1} \frac{C_i(x)C_j(x)}{\sqrt{1-x^2}}dx = \begin{cases} \frac{\pi}{N+1}, & if\ j = i, \\ 0, & if\ j \neq i. \end{cases} \tag{17}$$

proof. See [19]. □

We change the variable $t = \frac{t_f - t_0}{2}x + \frac{t_f + t_0}{2}$ to use these functions on $[t_0, t_f]$. Now any function $g(t)$ on $[a, b]$ can be approximated as

$$g(t) \simeq \sum_{j=1}^{N+1} g(t_j)C_j(t) = G^T\Theta_N(t), \tag{18}$$

where t_j, $j = 1, 2, \ldots, N+1$, are the shifted points of x_j, $j = 1, 2, \ldots, N+1$, by transforming $t = \frac{t_f - t_0}{2}x + \frac{t_f + t_0}{2}$ (here we choose t_j so that, $t_1 < t_2 < \cdots < t_{N+1}$),

$$G = [g(t_1), g(t_2), \ldots, g(t_{N+1})]^T, \tag{19}$$

and

$$\Theta_N(t) = [C_1(t), C_2(t), \ldots, C_{N+1}(t)]^T. \tag{20}$$

The differentiation of vector $\Theta_N(t)$ defined in (20) can be expressed as [18]

$$\Theta'_N(t) = D\Theta_N(t), \tag{21}$$

where D is the $(N+1) \times (N+1)$ operational matrix of derivative for Chebyshev cardinal functions and given by

$$D_N = \begin{bmatrix} C'_1(t_1) & \cdots & C'_1(t_{N+1}) \\ \vdots & & \vdots \\ C'_{N+1}(t_1) & \cdots & C'_{N+1}(t_{N+1}) \end{bmatrix}, \tag{22}$$

where

$$C'_j(t_k) = \begin{cases} \sum_{i=1, i\neq j}^{N+1} \frac{1}{t_j - t_i}, & j = k, \\ \frac{\beta}{T_{N+1,t}(t_j)} \prod_{l=1, l\neq k,j}^{N+1} (t_k - t_l), & j \neq k, \end{cases} \tag{23}$$

and $\beta = \frac{2^{2N+1}}{(t_f - t_0)^{N+1}}$.

4. Operational Matrices of Chebyshev Cardinal Functions

Recently Heydari et al. [11, 12] derived the operational matrices of Chebyshev cardinal functions. In this section, we review the operational matrices of integration and product which mentioned in [11, 12].

Lemma 4.1. *The integration of the vector $\Theta_N(t)$ defined in (20) can be approximated as*

$$\int_{t_0}^{t} \Theta_N(s)ds \simeq P_N \Theta_N(t), \tag{24}$$

where P_N is the $(N+1)\times(N+1)$ operational matrix of integration as follows

$$P_N = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1(N+1)} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2(N+1)} \\ \vdots & \vdots & & \vdots \\ \alpha_{(N+1)1} & \alpha_{(N+1)2} & \cdots & \alpha_{(N+1)(N+1)} \end{bmatrix}, \tag{25}$$

where

$$\alpha_{jk} = \int_{t_0}^{t_k} C_j(s)ds = \frac{\beta}{T_{N+1,s}(t_j)} \int_{t_0}^{t_k} \prod_{i=1, i\neq j}^{N+1} (s-t_i)ds, \tag{26}$$

$$j, k = 1, 2, \ldots, N+1.$$

Proof. See [11, 12]. □

Lemma 4.2. *Assume $\Theta_N(t)$ in (20) and $F = [f_1, f_2, \ldots, f_{N+1}]^T$ as the column vectors, then*

$$\Theta_N(t)\Theta_N^T(t)F \simeq \widetilde{F}_N \Theta_N(t), \tag{27}$$

where $\widetilde{F}_N$ is a $(N+1)\times(N+1)$ product operational matrix as follows

$$\widetilde{F}_N = \begin{bmatrix} f_1 & 0 & \cdots & 0 \\ 0 & f_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{N+1} \end{bmatrix}. \tag{28}$$

proof. See [11, 12]. □

5. Direct Method to Solve DAEs

In this section, by using results obtained in previous section about Chebyshev cardinal functions, an effective and accurate direct method for solving linear and nonlinear differential-algebraic equations is presented. Here, the implementation of this method is presented for DAEs system (4), when $m = 1, k = 1$ and $n = 2$. This discussion can simply be extended to general form (4) (with and without singularities). Now consider the DAEs system,

$$\begin{cases} X' = AX + By + q, \quad t_0 \leqslant t \leqslant t_f, \\ 0 = CX + r(t), \end{cases} \tag{29}$$

where

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, \quad B = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix},$$

$$q = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}, \quad C = \begin{bmatrix} c_1(t) & c_2(t) \end{bmatrix},$$

with initial condition,

$$x_1(t_0) = \alpha_1, \qquad x_2(t_0) = \alpha_2. \tag{30}$$

Let

$$x_1'(t) = X_1^T \Theta_N(t), \tag{31}$$

$$x_2'(t) = X_2^T \Theta_N(t), \tag{32}$$

$$y(t) = Y^T \Theta_N(t), \tag{33}$$

where $\Theta_N(t)$ is defined in (20), and X_1, X_2 and Y are vectors with $N+1$ unknowns as follows

$$X_1 = [x_{11}, x_{12}, \ldots, x_{1(N+1)}]^T, \tag{34}$$

$$X_2 = [x_{21}, x_{22}, \ldots, x_{2(N+1)}]^T, \tag{35}$$

$$Y = [y_1, y_2, \ldots, y_{N+1}]^T. \tag{36}$$

By expanding $x_1(a)$ and $x_2(a)$ in terms of Chebyshev cardinal functions we get

$$x_1(t_0) = [\alpha_1, \alpha_1, \ldots, \alpha_1]\Theta_N(t) = e_1^T \Theta_N(t), \tag{37}$$
$$x_2(t_0) = [\alpha_2, \alpha_2, \ldots, \alpha_2]\Theta_N(t) = e_2^T \Theta_N(t). \tag{38}$$

Integrating (31) and (32) from a to t and using (37) and (38), we obtain

$$x_1(t) \simeq (X_1^T P_N + e_1^T)\Theta_N(t) = E_1^T \Theta_N(t), \tag{39}$$
$$x_2(t) \simeq (X_2^T P_N + e_2^T)\Theta_N(t) = E_2^T \Theta_N(t), \tag{40}$$

where P_N is the operational matrix of integration given in (24). Also using (18) the functions $a_{ij}(t), b_j(t), q_j(t), c_i(t)$ and $r(t)$, $i, j = 1, 2$ can be expanded as:

$$a_{ij}(t) \simeq \sum_{k=1}^{N+1} a_{ij}(t_k) C_k(t) = A_{ij}^T \Theta_N(t), \quad i, j = 1, 2, \tag{41}$$

$$b_j(t) \simeq \sum_{k=1}^{N+1} b_j(t_k) C_k(t) = B_j^T \Theta_N(t), \quad j = 1, 2, \tag{42}$$

$$q_j(t) \simeq \sum_{k=1}^{N+1} q_j(t_k) C_k(t) = Q_j^T \Theta_N(t), \quad j = 1, 2, \tag{43}$$

$$c_i(t) \simeq \sum_{k=1}^{N+1} c_i(t_k) C_k(t) = C_i^T \Theta_N(t), \quad i = 1, 2, \tag{44}$$

$$r(t) \simeq \sum_{k=1}^{N+1} r(t_k) C_k(t) = R^T \Theta_N(t), \tag{45}$$

where

$$
\begin{aligned}
A_{ij} &= [a_{ij}(t_1), a_{ij}(t_2), \ldots, a_{ij}(t_{N+1})]^T, \quad i,j=1,2,\\
B_j &= [b_j(t_1), b_j(t_2), \ldots, b_j(t_{N+1})]^T, \quad j=1,2,\\
Q_j &= [q_j(t_1), q_j(t_2), \ldots, q_j(t_{N+1})]^T, \quad j=1,2,\\
C_i &= [c_i(t_1), c_i(t_2), \ldots, c_i(t_{N+1})]^T, \quad i=1,2,\\
R &= [r(t_1), r(t_2), \ldots, r(t_{N+1})]^T.
\end{aligned}
$$

Using (31)-(33),(39),(40) and (41)-(45) in (29), we get

$$
\begin{cases}
X_1^T\Theta_N(t) - A_{11}^T\Theta_N(t)\Theta_N^T(t)E_1 - A_{12}^T\Theta_N(t)\Theta_N^T(t)E_2 \\
\qquad -B_1^T\Theta_N(t)\Theta_N^T(t)Y - Q_1^T\Theta_N(t) = 0,\\
X_2^T\Theta_N(t) - A_{21}^T\Theta_N(t)\Theta_N^T(t)E_1 - A_{22}^T\Theta_N(t)\Theta_N^T(t)E_2 \\
\qquad -B_2^T\Theta_N(t)\Theta_N^T(t)Y - Q_2^T\Theta_N(t) = 0,\\
C_1^T\Theta_N(t)\Theta_N^T(t)E_1 + C_2^T\Theta_N(t)\Theta_N^T(t)E_2 + R^T\Theta_N(t) = 0.
\end{cases} \tag{46}
$$

Using (27) we have

$$
\Theta_N(t)\Theta_N^T(t)E_j \simeq \widetilde{E}_j\Theta_N(t), \quad j=1,2, \tag{47}
$$

$$
\Theta_N(t)\Theta_N^T(t)Y \simeq \widetilde{Y}\Theta_N(t). \tag{48}
$$

where $\widetilde{E}_1, \widetilde{E}_2$ and $\widetilde{Y}$ can be calculated similar to matrix $\widetilde{F}_N$ in (27). From Equations (46), (47) and (48) we obtain

$$
\begin{cases}
X_1^T - A_{11}^T\widetilde{E}_1 - A_{12}^T\widetilde{E}_2 - B_1^T\widetilde{Y} - Q_1^T = 0,\\
X_2^T - A_{21}^T\widetilde{E}_1 - A_{22}^T\widetilde{E}_2 - B_2^T\widetilde{Y} - Q_2^T = 0,\\
C_1^T\widetilde{E}_1 + C_2^T\widetilde{E}_2 + R^T = 0.
\end{cases} \tag{49}
$$

This is a system of algebraic equations with $3N+3$ unknowns and $3N+3$ equations, which can be solved by Newton's iteration method to obtain the unknown vectors X_1, X_2 and Y .

Remark 5.1. *In case $F(x(t), x'(t), t)$ in (1) is strongly nonlinear, the Taylor series for several variables can be used to approximate $F(x(t), x'(t), t)$ as a polynomial in $x(t)$ and $x'(t)$. Then the above method can be applied easily by using operational matrices of integration and product.*

6. Numerical Examples

The direct method, presented in this article, is applied to five examples. These examples are selected from different references, so the numerical results obtained here can be compared with both the exact solution and other numerical results. The computations associated with the examples were performed using MAPLE 13 with 64 digits precision on a personal computer.

Example 6.1. Consider the linear index-2 semi-explicit DAE problem [23, 22]:

$$\begin{cases} X' = AX + By + q, & 0 \leqslant t \leqslant 20, \\ 0 = CX + r, \end{cases} \tag{50}$$

where

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1+2t \end{bmatrix} \quad q = \begin{bmatrix} -\sin(t) \\ 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and $r(t) = -(e^{-t} + \sin(t))$ with $x_1(0) = 1$ and $x_2(0) = 0$. The exact solutions of this problem are

$$x_1(t) = e^{-t}, \quad x_2(t) = \sin(t), \quad y(t) = \frac{\cos(t)}{1+2t}.$$

From Theorem 2.1, problem (50) can be converted to the index-1 DAE:

$$\begin{cases} x_2 = -x_1 + e^{-t} + \sin(t), \\ x_1' = x_2 - x_1 - \sin(t), \end{cases} \tag{51}$$

with $x_1(0) = 1$ and $x_2(0) = 0$. By solving this problem and using (11), we can obtain $y(t)$.
Figures 1 and 2 show the plot of error with $N = 25$ using presented method For this example without and with index reduction, respectively. From the above example it is evident that in the solution of the problem without index reduction a system of order $3N+3$ of equations is obtained, but by using index reduction we obtain a system of equations

of order $2N+2$ in which the results for $x_1(x)$ and $x_2(x)$ are as well as before, but the result for $y(t)$ from (11) is much better than before.

Example 6.2. Consider the linear index-2 problem [23, 22]:

$$\begin{cases} X' = AX + By + q, \qquad 0 \leqslant t \leqslant 14, \\ 0 = CX + r, \end{cases} \tag{52}$$

where

$$A = \begin{bmatrix} 2 & t \\ 0 & \frac{1}{1+t} \end{bmatrix}, B = \begin{bmatrix} 1 \\ t \end{bmatrix} q = \begin{bmatrix} e^t\left(1 - t - t\sin(t) - \frac{1}{1+t}\right) \\ e^t\left(\sin(t) + \cos(t) - \frac{\sin(t)+t}{1+t}\right) \end{bmatrix}, C^T = \begin{bmatrix} 1 \\ t \end{bmatrix},$$

and $r(t) = -te^t(1+\sin(t))$ with $x_1(0) = 0$ and $x_2(0) = 0$. The exact solutions of this problem are

$$x_1(t) = te^t, \qquad x_2(t) = e^t\sin(t), \qquad y(t) = \frac{e^t}{1+t}.$$

By Theorem 2.1, the index-2 DAE (52) transforms to the following index-1 DAE:

$$\begin{cases} x_1 = -tx_2 + g_1(t), \\ x_2' = tx_1' - 2tx_1 + g_2(t)x_2 + g_3(t), \end{cases} \tag{53}$$

with $x_1(0) = x_2(0) = 0$, when $g_1(t) = te^t(1+\sin(t))$, $g_2(t) = \frac{1-t^2-t^3}{1+t}$ and
$g_3(t) = e^t\left(\cos(t) - t + t^2 + \frac{t+t^2+t^3}{1+t}\sin(t)\right)$. Similar to Example 1, by solving this problem and using (11), we can obtain $y(t)$. Figures. 3 and 4 show the plot of error with $N = 36$ using presented method for this example without and with index reduction, respectively. It is easily found that the present approximations with index reduction (proposed in Section 2.) is more efficient.

Example 6.3. Consider the following problem with initial value [3, 1, 20]:

$$\begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix} x'(t) + \begin{bmatrix} 1 & -(1+t) \\ -\mu & 1+\mu t \end{bmatrix} x(t) = \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}, 0 \leqslant t \leqslant 1. \tag{54}$$

The exact solutions are, $x_1(t) = t\sin(t) + (1+\mu t)e^{-t}$ and $x_2(t) = \mu e^{-t} + \sin(t)$. Although, this problem has index-1, but Ascher showed that in 1989, for $\mu \gg 0$ symmetric methods solving numerically encounter with difficulty [1]. In 1994 Amodio solved it by techniques of boundary values [3]. Recently Saravi et al. [20] solved it by pseudo-spectral method. Here we solved it by presented method for $\mu = 200$ and the results are given in Table 1. In this table e_{pm}, e_{ps} and e_A mean maximum error between $x_1(t)$ and $x_2(t)$ using presented method in Section 5., pseudo-spectral method in [20] and Adams method [20], respectively.Table 2 shows the maximum error between $x_1(t)$ and $x_2(t)$ using presented method for different values of N and μ.

Example 6.4. Consider the following problem with initial condition as [8, 3, 9, 7]:

$$\begin{bmatrix} 0 & 0 \\ 1 & \mu t \end{bmatrix} x'(t) + \begin{bmatrix} 1 & \mu t \\ 0 & 1+\mu \end{bmatrix} x(t) = \begin{bmatrix} e^t \\ t^2 \end{bmatrix}, \qquad -\frac{1}{2} \leqslant t \leqslant \frac{1}{2}, \quad (55)$$

with exact solution, $x_1(t) = e^t + \mu t(e^t - t^2)$ and $x_2(t) = t^2 - e^t$. This problem has global index-2 and was considered in several papers such as [8, 3, 9, 7]. Gear and Petzold in 1984 shown that, when $\mu \ll -\frac{1}{2}$, then recurrence Euler method is unable to solve it numerically [8], and in [10], numerical methods based on finite differences encounter with difficulty. In 1994 Amodio [3], solve it by techniques of boundary values, but the rate of convergency for $\mu < -\frac{1}{2}$ is very low. We solved it for $\mu = -2$, and examined it with different values of N. The results are given in Table 3 and Table 4 shows the maximum error between $x_1(t)$ and $x_2(t)$ using presented method for different values of N and μ.

Example 6.5. In this example, consider the nonlinear index-1 semi-explicit DAE problem [23, 22]:

$$\begin{cases} y' = y - zw + g_1(t), \\ z' = tw + y^2 + g_2(t), \qquad 0 \leqslant t \leqslant 1, \\ 0 = y - w + g_3(t), \end{cases} \quad (56)$$

with initial conditions $y(0) = z(0) = w(0) = 0$, when $g_1(t) = \sin(t) + t\cos(t)$, $g_2(t) = \sec^2(t) - t^2(\cos(t) + \sin^2(t))$ and $g_3(t) = t(\cos(t) - \sin(t))$

and the exact solutions are $y(t) = t\sin(t)$, $z(t) = \tan(t)$ and $w(t) = t\cos(t)$. Tables 5-7 show the absolute errors using presented method with $N = 5, 10$ and 15.

7. Conclusion

The Chebyshev cardinal functions and the associated operational matrices of integration P_N and product $\widetilde{F}_N$ were applied to solve the linear and nonlinear differential-algebraic equations. The method is based upon reducing the system into a set of algebraic equations. The obtained results showed that this approach can solve the problem effectively and needs few computations. The merit of this method is that the system of equations obtained for the solution does not need to consider collocation points; this means that the system of equations is obtained directly. In addition this method can be employed over large intervals with sufficient accuracy. The method of Chebyshev cardinal functions proposed in this paper can be extended to solve the linear and nonlinear ordinary differential equations.

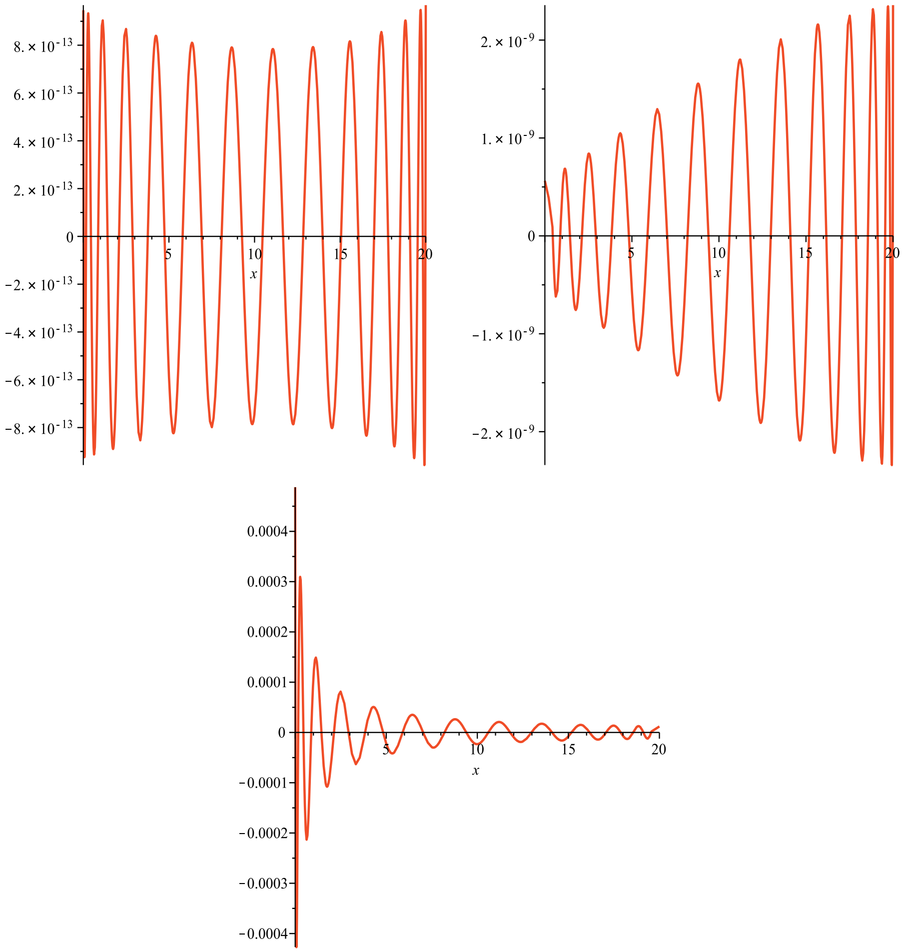

Figure 1: Plot of error for $x_1(t)$(left), $x_2(t)$(right) and $y(t)$(Bottom) with $N = 25$, without index reduction for Example 1.

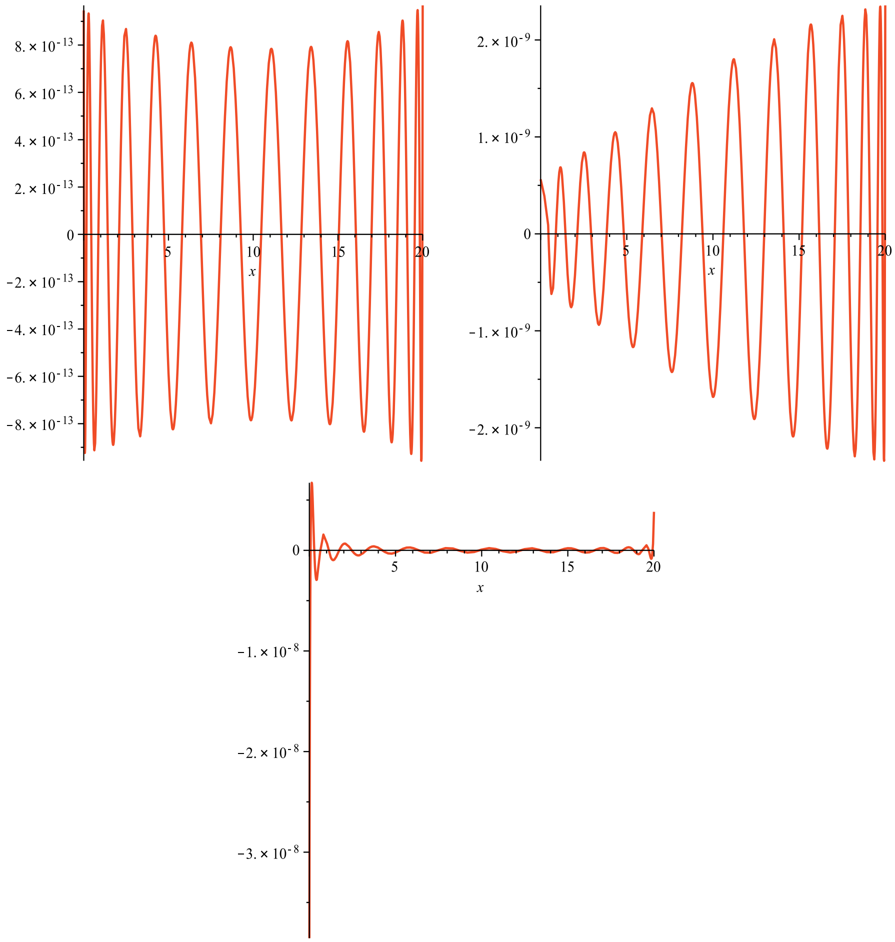

Figure 2: Plot of error for $x_1(t)$(left), $x_2(t)$(right) and $y(t)$(Bottom) with $N = 25$, with index reduction for Example 1.

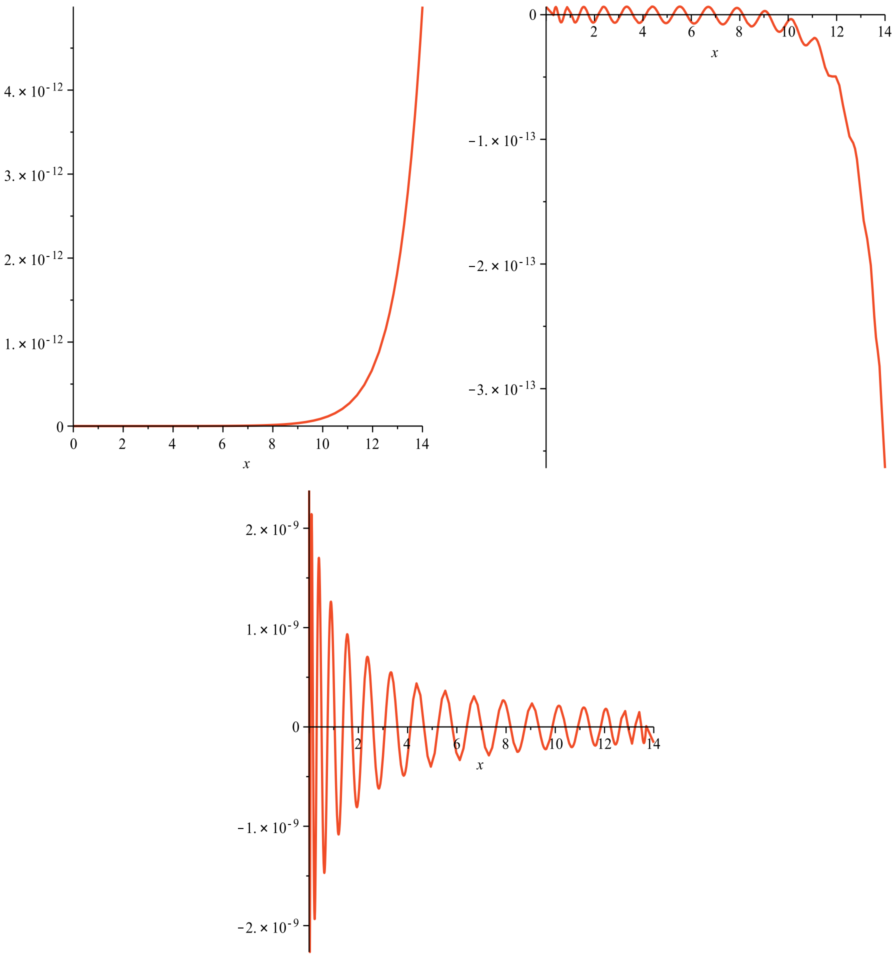

Figure 3: Plot of error for $x_1(t)$(left), $x_2(t)$(right) and $y(t)$(Bottom) with $N = 36$, without index reduction for Example 2.

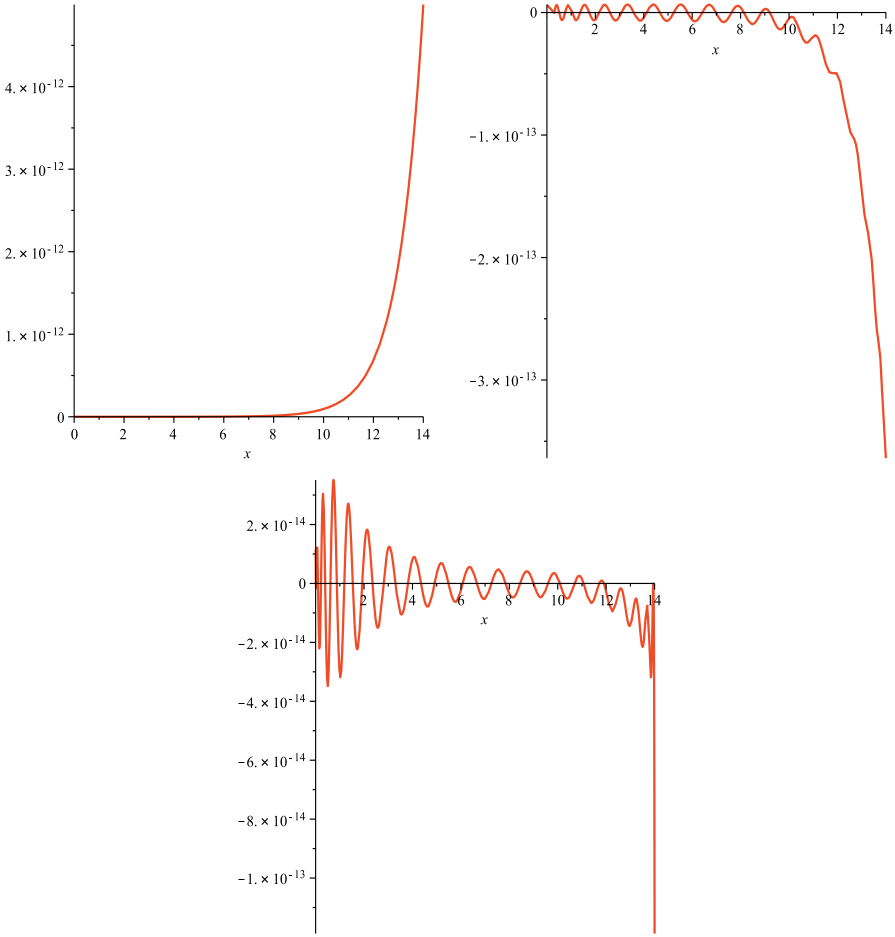

Figure 4: Plot of error for $x_1(t)$(left), $x_2(t)$(right) and $y(t)$(Bottom) with $N = 36$, with index reduction for Example 2.

Table 1: Maximum error between x_1 and x_2 for Example 3 with $\mu = 200$.

N	e_{pm}	e_{ps}[21]	h	e_A [21]
6	2.16×10^{-5}	1.73×10^{-4}	2.0×10^{-2}	1.22×10^{-5}
10	1.64×10^{-11}	1.29×10^{-10}	5.0×10^{-3}	1.92×10^{-7}
14	2.64×10^{-18}	6.21×10^{-14}	2.5×10^{-3}	2.41×10^{-8}

Table 2: Maximum error between x_1 and x_2 for Example 3 with different values of N and μ.

μ	$N = 5$	$N = 10$	$N = 15$	$N = 20$
500	1.3×10^{-3}	4.11×10^{-11}	1.05×10^{-19}	5.52×10^{-29}
1000	2.62×10^{-3}	8.22×10^{-11}	2.20×10^{-19}	1.15×10^{-28}
5000	1.31×10^{-2}	4.11×10^{-10}	1.10×10^{-18}	5.75×10^{-28}
10000	2.63×10^{-2}	8.22×10^{-10}	2.21×10^{-18}	1.15×10^{-27}

Table 3: Maximum error between x_1 and x_2 for Example 4 with different values of N and μ.

N	e_{pm}	e_{ps}[21]	h	e_A [21]
6	6.33×10^{-7}	2.58×10^{-6}	1.0×10^{-1}	7.06×10^{-6}
10	5.04×10^{-13}	1.95×10^{-12}	1.25×10^{-2}	1.30×10^{-7}
14	8.29×10^{-20}	8.23×10^{-17}	6.25×10^{-3}	1.66×10^{-8}

Table 4: Maximum error between x_1 and x_2 for Example 4 with different values of N and μ.

μ	$N = 5$	$N = 10$	$N = 15$	$N = 20$
-10	7.75×10^{-5}	2.56×10^{-12}	7.01×10^{-21}	3.69×10^{-30}
-50	3.91×10^{-4}	1.29×10^{-11}	3.51×10^{-20}	1.85×10^{-29}
-100	7.82×10^{-4}	2.57×10^{-11}	7.03×10^{-20}	3.70×10^{-29}
-500	3.91×10^{-3}	1.29×10^{-10}	3.51×10^{-19}	1.85×10^{-28}

Table 5: Absolute values of errors for $y(t)$.

t	$N = 5$	$N = 10$	$N = 15$
0.1	6.43×10^{-6}	2.92×10^{-9}	2.04×10^{-12}
0.2	5.53×10^{-6}	4.05×10^{-9}	5.25×10^{-13}
0.3	2.86×10^{-6}	2.47×10^{-10}	1.00×10^{-12}
0.4	8.79×10^{-6}	1.85×10^{-9}	9.20×10^{-15}
0.5	5.08×10^{-6}	5.57×10^{-9}	2.84×10^{-12}
0.6	7.50×10^{-6}	4.64×10^{-9}	4.08×10^{-13}
0.7	2.05×10^{-5}	7.16×10^{-10}	4.59×10^{-12}
0.8	2.29×10^{-5}	5.57×10^{-9}	4.22×10^{-13}
0.9	1.10×10^{-5}	6.59×10^{-9}	4.16×10^{-12}
1.0	2.54×10^{-6}	8.81×10^{-9}	1.37×10^{-12}

Table 6: Absolute values of errors for $z(t)$.

t	$N = 5$	$N = 10$	$N = 15$
0.1	6.89×10^{-4}	3.86×10^{-7}	3.59×10^{-10}
0.2	2.53×10^{-4}	4.49×10^{-7}	1.90×10^{-10}
0.3	4.98×10^{-4}	4.81×10^{-7}	3.59×10^{-10}
0.4	8.35×10^{-6}	2.66×10^{-7}	4.27×10^{-10}
0.5	5.40×10^{-4}	2.45×10^{-7}	4.25×10^{-10}
0.6	4.82×10^{-4}	5.11×10^{-7}	4.57×10^{-10}
0.7	1.67×10^{-4}	4.17×10^{-7}	4.73×10^{-10}
0.8	5.50×10^{-4}	1.24×10^{-7}	4.65×10^{-10}
0.9	1.69×10^{-4}	1.51×10^{-7}	1.14×10^{-10}
1.0	3.61×10^{-4}	4.24×10^{-7}	3.95×10^{-10}

Table 7: Absolute values of errors for $w(t)$.

t	$N = 5$	$N = 10$	$N = 15$
0.1	1.06×10^{-5}	2.92×10^{-9}	2.04×10^{-12}
0.2	1.37×10^{-6}	4.05×10^{-9}	5.25×10^{-13}
0.3	7.21×10^{-6}	2.47×10^{-10}	1.00×10^{-12}
0.4	6.80×10^{-6}	1.85×10^{-9}	9.20×10^{-15}
0.5	7.26×10^{-7}	5.57×10^{-9}	2.84×10^{-12}
0.6	9.50×10^{-6}	4.64×10^{-9}	4.08×10^{-13}
0.7	1.61×10^{-5}	7.16×10^{-10}	4.59×10^{-12}
0.8	1.86×10^{-5}	5.57×10^{-9}	4.22×10^{-13}
0.9	1.53×10^{-5}	6.59×10^{-9}	4.16×10^{-12}
1.0	3.14×10^{-6}	8.81×10^{-9}	1.37×10^{-12}

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