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The Submodule-Based Zero-Divisor Graph with Respect to Some Homomorphism

M. Baziar*
Yasouj University

N. Ranjbar
Yasouj University

Abstract. Let M be an R-module and $0 \neq f \in M^* = \text{Hom}(M, R)$. The graph $\Gamma_f(M)$ is a graph with vertices $Z^f(M) = \{x \in M \setminus \{0\} \mid xf(y) = 0$ or $yf(x) = 0$ for some non-zero $y \in M\}$, in which non-zero elements x and y are adjacent provided that $xf(y) = 0$ or $yf(x) = 0$, which introduced and studied in [3]. In this paper we associate an undirected submodule based graph $\Gamma^f_N(M)$ for each submodule N of M with vertices $Z^f_N(M) = \{x \in M \setminus N \mid xf(y) \in N$ or $yf(x) \in N$ for some $y \in M \setminus N\}$, in which non-zero elements x and y are adjacent provided that $xf(y) \in N$ or $yf(x) \in N$. We observe that over a commutative ring R, $\Gamma^f_N(M)$ is connected and $diam(\Gamma^f_N(M)) \leqslant 3$. Also we get some results about clique number and connectivity number of $\Gamma^f_N(M)$

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1. Introduction

All rings in this paper are commutative with identity and modules are unitary right modules. Let M be an R-module, following [9] all R-homomorphism from M to R will be denoted by M^*.
In recent decades, the zero-divisor graphs of commutative rings have been extensively studied by many authors and become a major field of

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*Corresponding author

research. S. P. Redmond replaced zero (ideal) in the definition of zero divisor graph by an arbitrary ideal (see [7]) to get a nice generalization of the zero-divisor graph of a commutative ring. The zero divisor graph for modules over commutative rings, introduced by M. Behboodi in [4], was one of the first attempts to generalize the zero-divisor graphs in module theoretic context. In [2] and [3] the authors gave a new interpretation of zero-divisor graph for modules, which in some cases, coincide with the zero-divisor graph of commutative rings.

In this paper, we extend Redmond's findings to see if additional information about the structure of commutative rings is hidden in ideal-divisor graphs.

Let G be a (undirected) graph. We say that G is *connected* if there is a path between any two distinct vertices. For distinct vertices x and y in G, the *distance* between x and y, denoted by $d(x, y)$, is the length of a shortest path connecting x and y ($d(x, x) = 0$ and $d(x, y) = \infty$ if no such path exists).

The *diameter* of G is $\operatorname{diam}(G) = \sup\{d(x, y) \mid \ x \text{ and } y \text{ are vertices of } G\}$. A *cycle* of length n in G is a path of the form $x_1 - x_2 - x_3 - \cdots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. We define the *girth* of G, denoted by $\operatorname{gr}(G)$, as the length of a shortest cycle in G, provided G contains a cycle; otherwise, $\operatorname{gr}(G) = \infty$. A graph is *complete* if any two distinct vertices are adjacent. By a complete subgraph we mean a subgraph which is complete as a graph. In this article all subgraphs are induced subgraphs, where a subgraph G' of a graph G is an *induced subgraph* of G if two vertices of G' are adjacent in G' if and only if they are adjacent in G. A complete subgraph of G is called a *clique*. The *clique number* of G, denoted by $cl(G) = \sup\{|G'| : \text{ where } G' \text{ is a complete subgraph of } G\}$. Unexplained terminology and standard results may be found in [8] and [9].

2. Definition and Some Preliminary Results

We begin with the definition of the submodule-based zero-divisor graph of modules and then give some clarifications of the relation between this definition and those appeared in the literature.

Definition 2.1. *Let R be a commutative ring, M an R-module, $f \in M^* = \mathrm{Hom}_R(M,R)$ and N a submodule of M. We define an undirected graph with vertices*

$$Z_N^f(M) = \{x \in M \setminus N \mid xf(y) \in N \text{ or } yf(x) \in N \text{ for some } y \in M \setminus N\},$$

where distinct vertices x and y are adjacent if and only if $xf(y) \in N$ or $yf(x) \in N$, and will be denoted by $\Gamma_N^f(M)$.

The following proposition shows that the graph in Definition 2.1 is a direct generalization of ideal-based zero-divisor graph introduced in [6].

Proposition 2.2. *Let M be an R-module and N be a submodule of M.*
(a) If $N = (0)$, then $\Gamma_N^f(M) = \Gamma_f(M)$
(b) If N is a non-zero proper submodule of M then $\Gamma_N^f(M) = \emptyset$ if and only if N is a prime submodule of M and $Z_N^f(M) \neq M \setminus N$

Proof. (a) It is clear.
(b) Let N be a prime submodule of M and $Z_N^f(M) \neq M \setminus N$. If $\Gamma_N^f(M)$ $\neq \emptyset$ then there exists $x \in Z_N^f(M)$ and $y \in Z_N^f(M)$ such that $xf(y) \in N$ or $yf(x) \in N$, since $x \notin N$ an N is a prim submodule, $Mf(y) \subseteq N$ and so for all $m \in M \setminus N$, $mf(y) \in N$ and hence $Z_N^f(M) = M \setminus N$, a contradiction.
Conversely, suppose that $mr \in N$ and $m \notin N$, for some $r \in R$ and $m \in M$. If $Ma \nsubseteq N$, then there exists $m_0 \in M$ such that $m_0a \notin N$ and hence $mf(m_0a) \in N$. Thus $\Gamma_N^f(M) \neq \emptyset$, a contradiction. □

Theorem 2.3. *Let N be a submodule of M. Then $\Gamma_N^f(M)$ is a connected graph with $diam(\Gamma_N^f(M)) \leqslant 3$.*

Proof. Let x and y be distinct vertices of $\Gamma_N^f(M)$. The following cases may be hold.
Case 1. If $xf(y) \in N$ or $yf(x) \in N$, then $x - y$ is a path of $\Gamma_N^f(M)$.
Case 2. If $xf(y) \notin N$ but $xf(x) \in N$ and $yf(y) \in N$, then $x - xf(y) - y$ is a path of $\Gamma_N^f(M)$ since $xf(xf(y)) = xf(x)f(y) \in N$ and $yf(xf(y)) = y(f(x)f(y)) \in N$.
Case 3. If $xf(y) \notin N$ and $yf(y) \notin N$ but $xf(x) \in N$, then there exists $z \in \mathbb{Z}$ such that $yf(z) \in N$ or $zf(y) \in N$. If $yf(z) \in N$, then

$x - xf(z) - y$ is a path of $\Gamma_N^f(M)$. If $zf(y) \in N$, then $x - zf(x) - y$ is a path of $\Gamma_N^f(M)$.
Case 4. If $xf(y) \notin N$ and $xf(x) \notin N$ but $yf(y) \in N$, then the proof is the same as case 3.
Case 5. If $xf(y) \notin N$ and $yf(y) \notin N$ and $xf(x) \notin N$, then there exists $z, t \in \mathbb{Z}$ such that $xf(z) \in N$ or $zf(x) \in N$ and $yf(t) \in N$ or $tf(y) \in N$. If $tf(z) \in N$ or $zf(t) \in N$ then $x - z - t - y$ is a path in $\Gamma_N^f(M)$ else $tf(z) \notin N$ and $zf(t) \notin N$. If $xf(z) \in N$ then $x - tf(z) - y$ is a path of $\Gamma_N^f(M)$. If $zf(x) \in N$ and $yf(t) \in N$ then $x - zf(t) - y$ is a path in $\Gamma_N^f(M)$. If $zf(x) \in N$ and $tf(y) \in N$ then $x - zf(t) - tf(x) - y$ is a path in $\Gamma_N^f(M)$ thus $\Gamma_N^f(M)$ is connected and $diam(\Gamma_N^f(M)) \leqslant 3$. $\square$

For an R-module M the graph $\Gamma_{\{0\}}^f(M)$ will be denoted by $\Gamma^f(M)$. The next results show the relation between $\Gamma_N^f(M)$ and $\Gamma^f(\frac{M}{N})$.

Theorem 2.4. *Let M be an R-module, N a submodule of M and f a non-zero homomorphism in $(\frac{M}{N})^*$.*
(a) *If $x+N-y+N$ in $\Gamma^f(\frac{M}{N})$, then $x-y$ in $\Gamma_N^{f\pi}(M)$ (where $\pi : M \to M/N$ is the natural projection map).*
(b) *If $x-y$ in $\Gamma^g(M)$ for some non-zero homomorphism $g \in M^*$, $x+N \neq y+N$ and $N \subseteq ker(g)$, then $x+N-y+N$ in $\Gamma^{\bar{g}}(\frac{M}{N})$ for some $\bar{g}$ in $(\frac{M}{N})^*$.*
(c) *If $x-y$ in $\Gamma^g(M)$ for some non-zero homomorphism $g \in M^*$ and $x+N = y+N$, then $xg(x) \in N$ or $yg(y) \in N$.*

Proof. (a) Since $x + N - y + N$, then $(x + N)f(y + N) = 0_{\frac{M}{N}}$ or $(y+N)f(x+N) = 0_{\frac{M}{N}}$. Suppose $(x+N)f(y+N) = 0_{\frac{M}{N}} = N$, therefore $xf\pi(y) \in N$ and hence $x - y$ in $\Gamma_N^{f\pi}(M)$.
(b) Let $x - y$ in $\Gamma^g(M)$ for some non-zero homomorphism $g \in M^*$ and $x + N \neq y + N$ and $N \subseteq ker(g)$. Now suppose $xg(y) \in N$. Then for all $n_1, n_2 \in N$ we have $(x + n_1)g(y + n_2) = xg(y) + n_1 g(y + n_2) + xg(n_2) \in N$, which implies that $(x+N)\bar{g}(y+N) = 0$, for $\bar{g} \in (\frac{M}{N})^*$ with $\bar{g}(m + N) = g(m)$.
(c) Let $x - y$ and $xg(y) \in N$. Since $x + N = y + N$ thus $xg(y) + N = yg(y) + N$ and so we have $yg(y) \in N$. In the same way we can show if

$yg(x) \in N$ then $xg(x) \in N$. $\square$

Using the notation as in the above construction, we call the subset $x+N$ a *column* of $\Gamma^f_N(M)$. If $xf(x) \in N$, then we call $x+N$ a *connected column* of $\Gamma^f_N(M)$.

Corollary 2.5. *Let M be an R-module, N a submodule of M and $f \in M^*$. If $x - y$ in $\Gamma^f_N(M)$ and $N \subseteq ker(f)$, then all(distinct) elements of $x + N$ and $y + N$ are adjacent in $\Gamma^f_N(M)$.*
If $xf(x) \in N$, then all distinct elements of $x+N$ are adjacent in $\Gamma^f_N(M)$. For a graph G, we say $\{G_\delta\}_{\delta\in\Delta}$ is a collection of disjoint subgraphs of G if all the vertices and edges of G_δ are contained in G and no two of these G_δ contain a common vertex.

Proposition 2.6. *Let N be a submodule of R-module M. Then the graph $\Gamma^f_N(M)$ contains $|N|$ disjoint subgraphs isomorphic to $\Gamma^f(\frac{M}{N})$.*

Proof. Let $\{m_\alpha\}_\alpha$ be a set of coset representatives of the vertices of $\Gamma^f(\frac{M}{N})$ that is, $\bigcup_\alpha(m_\alpha + N) = Z(\frac{M}{N})$, and if $\alpha \neq \beta$, then $m_\alpha+N \neq m_\beta+N$. For each $n \in N$, define a graph G_n with vertices $\{m_\alpha + n \mid n \in N\}$, where $m_\alpha+n$ is adjacent to $m_\beta+n$ in G_n whenever $m_\alpha+N$ is adjacent to $m_\beta + N$ in $\Gamma^f(\frac{M}{N})$; i.e., whenever $m_\alpha f(m_\beta) \in I$. By the above theorem, G_n is a subgraph of $\Gamma^f_N(M)$. Also, for each $n, n_1, n_2 \in N$, $G_n \cong \Gamma^f(\frac{M}{N})$, and G_{n_1} and G_{n_2} contain no common vertices if $n_1 \neq n_2$. $\square$

3. Cut-Points and Clique Number

A vertex x of a connected graph is a cut-point of G if there are vertices u and w of G such that $x \neq u$ *and* $x \neq w$ and x lies in every path from u to w.

Theorem 3.1. *Let N be a non-zero proper submodule of an R-module M. Then the graph $\Gamma^f_N(M)$ has no cut-point.*

Proof. Let x be a cut-point of $\Gamma^f_N(M)$. Then there exist u and w in $\Gamma^f_N(M)$ such that x lies in every path from u to w. By Theorem 2.4, the shortest path from u to w is of length 2 or 3.

Case 1. The path from u to w is of length 2. Let $u - x - w$ is the shortest path from u to w, then we have $uf(x) \in N$ or $xf(u) \in N$ and $xf(w) \in N$ or $wf(x) \in N$ and these cases may compose in 4 way,

1. $xf(u) \in N$ and $xf(w) \in N$;
2. $xf(u) \in N$ and $wf(x) \in N$;
3. $uf(x) \in N$ and $xf(w) \in N$;
4. $uf(x) \in N$ and $wf(x) \in N$.

If all elements $xf(u)$, $xf(w)$, $wf(x)$ and $uf(x)$ are zero, then $u - x + n - w$, which n is a non-zero element of N, a contradiction.
If y is one of non-zero element $xf(u)$, $xf(w)$, $wf(x)$ or $uf(x)$, then there exists a path $u - x + y - w$, a contradiction.

Case 2. The path from u to w is of length 3. Let $u - x - y - w$ be the shortest path from u to w, which implies that shortest path from u to y be the path $u - x - y$. Now similar case 1 we can find another path from u to y, which shows that x is not cut-point. □

The connectivity of a graph G, denoted by $k(G)$, is defined to be the minimum number of vertices which is necessary to remove from G in order to produce a disconnected graph. We provide bounds on $k(\Gamma^f_N(M))$ for a given module M, non-zero homomorphism f in M^* and submodule N of M. Recall that if M is the regular module R, f a monomorphism in M^* and $N = I$, then $\Gamma^f_N(M) = \Gamma_I(R)$.

Theorem 3.2. *Let N be a proper submodule of M, $f \in (\frac{M}{N})^*$ and $\pi : M \to \frac{M}{N}$ be the natural projection map and $\bar{f} = f\pi$.*
(a) *If $\Gamma^f(\frac{M}{N})$ is the graph on one vertex, then $k(\Gamma^{\bar{f}}_N(M)) = |N| - 1$;*
(b) *If $\Gamma^f(\frac{M}{N})$ has at least two vertices, then $2 \leqslant k(\Gamma^{\bar{f}}_N(M)) \leqslant |N|.k(\Gamma^f(\frac{M}{N}))$;*
(c) $|N| - 1 \leqslant k(\Gamma^{\bar{f}}_N(M))$

Proof. (a) If $\Gamma^f(\frac{M}{N})$ has only one vertex. Then $x+N$ is connect to itself, thus for all $n_1, n_2 \in N$, $(x+n_1)\bar{f}(x+n_2) = x\bar{f}(x)+n_1\bar{f}(x+n_2)+x\bar{f}(n_2) \in N$. So that $x\bar{f}(x) \in N$ and hence $\Gamma^{\bar{f}}_N(M)$ completed graph with $|N|$

vertices. Thus $k(\Gamma_N^{\bar{f}}(M)) = |N| - 1$.

(b) Since $\Gamma_N^{\bar{f}}(M)$ is connected and by Theorem 3.1 has no cut-point, so that $2 \leqslant k(\Gamma_N^{\bar{f}}(M))$. Let $k_0 = k(\Gamma^f(\frac{M}{N}))$ and let $a_1 + N, \ldots, a_{k_0} + N$ be vertices of $\Gamma^f(\frac{M}{N})$ witch, once removed, give a disconnect graph and let G be the graph obtained from $\Gamma_N^{\bar{f}}(M)$ by removing columns corresponding to $a_1 + N, \ldots, a_{k_0} + N$ (this means to remove $k_0.|N|$ vertices). If we show that G is disconnected, it means $k(\Gamma_N^{\bar{f}}(M)) \leqslant |N|.k(\Gamma^f(\frac{M}{N}))$. By the choice of $a_1 + N, \ldots, a_{k_0} + N$, there exist vertices $b + N$ and $c + N$ of $\Gamma^f(\frac{M}{N})$ such that are not connected by removing $a_1 + N, \ldots, a_{k_0} + N$, then b and c are vertices of G. Suppose $b - x_1 - \ldots - x_m - c$ is a path in G. Without loss of generality, $x_j + N \neq x_{j+1} + N$, for $1 \leqslant j \leqslant m$. Therefor, $b + N - x_1 + N - \ldots - x_m + N - c + N$ is a path in $\Gamma(M)$ after $a_1 + N, \ldots, a_k + N$ have been removed, this contradicts with the hypothesis $b + N$ and $c + N$ are disconnected in $\Gamma(M)$. Hence G must be disconnected.

(c) We define a number d such that $d = |N| - 1$ if $|N| < \infty$ and d is any positive integer if $|N| = \infty$. Let $a_1, \ldots, a_d$ be some arbitrary vertices of $\Gamma_N^{\bar{f}}(M)$ and G be the graph obtained by removing $a_1, \ldots, a_d$ from $\Gamma_N^{\bar{f}}(M)$. We show that G is a connected graph. Let x and y be vertices of G. If $x - y$ there is nothing to proof. We know that $diam(\Gamma_N^{\bar{f}}(M)) \leqslant 3$ thus we have two cases.

Case 1. Let $x - w - y$ be the shortest path from x to y in $\Gamma_N^{\bar{f}}(M)$. If $w \neq a_j$ for $j = 1, \ldots, d$, then $x - w - y$ in G. Supposes there exists $1 \leqslant j \leqslant d$ such that $w = a_j$. The column $a_j + N$ contains $|N|$ elements, so we can choose $v \in a_j + N$ such that $v \neq a_i$ for $1 \leqslant i \leqslant d$. Since we have choose $d < |N|$ and x and y are adjacent to w, we have x and y are adjacent to v and v is in G, thus $x - v - y$ is a path in G.

Case 2. Let $x - w - v - y$ be the shortest path from x to y in $\Gamma_N^{\bar{f}}(M)$. If $w \in G$ or $v \in G$ then by above proof we have a path from x to y in G. Suppose $w \notin G$ and $v \notin G$. We can choose $a \in w + N$ (so $a = w + n_1$ for some $n_1 \in N$) such that $a \neq a_j$ for $1 \leqslant i \leqslant d$. Since x is adjacent to w, we have $x - a$ (if $x\bar{f}(w) \in N$ then $x\bar{f}(a) = x\bar{f}(w + n_1) = x\bar{f}(w) + x\bar{f}(n_1) \in N$ and if $w\bar{f}(x) \in N$ then $a\bar{f}(x) = (w + n)\bar{f}(x) = w\bar{f}(x) + n\bar{f}(x) \in N$) and a is in G, thus $x - a$ is a path in G and like this we can choose

$b \in G$ ($b = v + n_2$ for some $n_2 \in N$) such that $b - y$. Now we prove that $a - b$ is a path. If $w\bar{f}(v) \in N$, then $a\bar{f}(b) = (w + n_1)\bar{f}(v + n_2) = w\bar{f}(v) + w\bar{f}(n_2) + n_1\bar{f}(v + n_2) \in N$ and if $v\bar{f}(w) \in N$ then $b\bar{f}(a) = (v + n_2)\bar{f}(w + n_1) = v\bar{f}(w) + v\bar{f}(n_1) + n_2\bar{f}(w + n_1) \in N$. □

Corollary 3.3. *If N is a proper non-zero submodule of M that is not prime and f and $\bar{f}$ be as above theorem, then $|N| - 1 < k(\Gamma_N^{\bar{f}}(M)) < (|N|.k(\Gamma(M))$. Moreover, if N is infinite, then $k(\Gamma_N^{\bar{f}}(M))) = \infty$*

We recall that for a graph G, a complete subgraph is called a clique. The clique number, $\omega(G)$, is the greatest integer $n \geqslant 1$ such that $K^n \subseteq G$, and $\omega(G) = \infty$ if $K^n \subseteq G$ for all $n \geqslant 1$.

Proposition 3.4. *Let N be a submodule of an R-module M and $f \in M^*$. If $\Gamma_N^f(M)$ has a connected column, then $\omega(\Gamma_N^f(M)) \geqslant |N|$.*

Proof. If $a+N$ is a connected column of $\Gamma_N^f(M)$, then $a+N$ a complete subgraph of $\Gamma_N^f(M)$. □

Corollary 3.5. *Let N be a proper submodule of M, $f \in (\frac{M}{N})^*$ and $\bar{f} = f\pi$, where $\pi : M \to \frac{M}{N}$ is the natural projection map. If $\Gamma(M)$ consists of only one vertex, then $\omega(\Gamma_N^{\bar{f}}(M)) = |N|$. If $N \neq 0$, then $\omega(\Gamma^f(\frac{M}{N})) < \omega(\Gamma_N^{\bar{f}}(M))$.*

Corollary 3.6. *If $M, N, f, \bar{f}$ are as above, $\Gamma^f(\frac{M}{N})$ has at least two vertices and $\Gamma_N^{\bar{f}}(M)$ has a connected column, then $\omega(\Gamma_N^{\bar{f}}(M)) \geqslant |N| + 1$*

Proof. Let $a + N$ be a connected column of $\Gamma_N^{\bar{f}}(M)$. There exists $b \in M \setminus N$ such that $a + N \neq b + N$ and $a + N$ is adjacent to $b + N$ in $\Gamma^f(\frac{M}{N})$. We know that each element of connected column $a + N$ is adjacent to b and so $\{a + N\} \bigcup \{b\}$ forms a complete subgraph of $\Gamma_N^{\bar{f}}(M)$. □

Theorem 3.7. *Let N be a proper submodule of M, $f \in \frac{M}{N}^*$ and $\bar{f} = f\pi$ where $\pi : M \to \frac{M}{N}$ is the natural projection map. If $\Gamma_N^f(M)$ has no connected column, then $\omega(\Gamma^f(\frac{M}{N})) = \omega(\Gamma_N^{\bar{f}}(M))$*

Proof. By Corollary 3.5, we observed that $\omega(\Gamma^f(\frac{M}{N})) < \omega(\Gamma_N^{\bar{f}}(M))$. Thus it is sufficient to consider the case where $\omega(\Gamma^f(\frac{M}{N})) = d < \infty$. We assume G is the complete subgraph of $\Gamma_N^{\bar{f}}(M)$ on $d+1$ vertices and provide a contradiction. Let $a_1, \ldots, a_{d+1}$ be the set of (distinct) vertices of G. Consider the subgraph G^* of $\Gamma(M)$ on vertices $a_1+N, \ldots, a_{d+1}+N$. G^* is a complete graph (since G is a complete graph), and since we assumed that $\omega(\Gamma^f(\frac{M}{N})) = d$, thus we must have $a_j + N = a_k + N$ for some $j \neq k$. Therefore $a_k = a_j + n$ for some $n \in N$. Since G is complete, a_j is adjacent to a_k in $\Gamma_N^{\bar{f}}(M)$. It follows that either $a_j\bar{f}(a_k) \in N$ or $a_k\bar{f}(a_j) \in N$. Let $a_j\bar{f}(a_k) \in N$. Then $a_j\bar{f}(a_j) + a_j\bar{f}(n) = a_j\bar{f}(a_j+n) = a_j\bar{f}(k) \in N$ and therefore $a_j\bar{f}(a_j) \in N$ (since $N \subseteq ker(f)$). Therefore a_j+N is a connected column in $\Gamma_N^{\bar{f}}(M)$. This contradicts with Corollary 3.6. $\square$

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Mohammad Baziar
Department of Mathematics
Assistant Professor of Mathematics
Yasouj University
Yasouj, Iran
E-mail: mbaziar@yu.ac.ir

Nooshin Ranjbar
Department of Mathematics
M.Sc of Mathematics
Faculty of Mathematical Science
Yasouj University
Yasouj, Iran
E-mail: n.Ranjbar@gmail.com