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Original Research Paper

On the Pseudo Almost Automorphic Solutions for Liénard -Type Systems with Multiple Delays

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Abstract. In this study, pseudo almost automorphic(PAA) solutions of a Liénard-type system with multiple delays are considered. By applying the main features of PAA, Banach fixed point theorem and some differential inequalities, sufficient conditions for the existence and uniqueness of such solutions are obtained . Since the PAA is more general than the almost periodicity(AP) and pseudo almost periodicity(PAP), this work is a new and complementary compared to previous studies. In addition, an example is given to show the correctness of the created conditions.

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Keywords and Phrases: Banach fixed point theorem, Liénard-type system, Pseudo almost automorphic.

1 Introduction

It is known that Liénard differential equation systems have wide application properties in many scientific fields such as physics, engineering and mechanics [5, 9, 10, 21]. Therefore, it is very important to obtain results regarding the qualitative behavior of the solutions of such equation systems. In the literature, qualitative features such as oscillation, periodicity, AP, and PAP of the solutions of such equations are quite

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common [6, 8, 11, 12, 13, 14, 15]. Gao and Liu [8] studied the following AP solutions of the following multiple-delay Liénard equation:

$$z(t)+g\left(z(t)\right)z'(t)+h_0\left(z(t)\right)+\sum_{i=1}^{m}h_i\left(z(t-\sigma_i(t))\right)=q(t). \qquad (1)$$

By applying some analysis techniques and taking an appropriate Lyapunov function, the authors obtained some adequate conditions that guarantee the existence and exponential stability of AP solutions of (1). Liu [13] created some conditions for a Liénard-type system to obtain AP solutions and exponential stability of these solutions by applying mathematical analysis techniques. Xu and Liao [18] discussed following Liénard type system

$$\begin{aligned}
&{z'}_1(t)=-\kappa(t)z_1(t)+z_2(t)+A_1(t), \qquad (2)\\
&{z'}_2(t)=\kappa(t)z_2(t)-\kappa^2(t)z_1(t)-g(z_1(t))\times\left[z_2(t)-\kappa(t)z_1(t)+A_1(t)\right]\\
&-h_0(z_1(t))-\sum_{i=1}^{m}h_i\left(z_1(t-\sigma_i(t))\right)+A_2(t),
\end{aligned}$$

where these authors got some conclusions about the PAP solutions of system (2). Bochner in [4] introduced the theory of AA to the literature. Additionally, Xiao et al. [16] introduced the concept of PAA, which is a generalization of the notions of AP, AA, and PAP to the literature. Due to their application in mathematical biology, mechanics and physics, important results such as the existence of AA and PAA solutions and the stability of these solutions are the most interesting subjects in the qualitative theory. Recently, important findings such as the existence and uniqueness of AA and PAA solutions have been widely discussed in many types of differential equations [1, 2, 3, 15, 16, 19, 20, 22]. As far as we know from the literature, there is no study related to the PAA solutions of the system (2). Xu and Liao [18] obtained some conditions for the existence of PAP solutions of the system (2). Since the PAA function class is more general than the PAP class, it is quite important to consider the PAA class. In this study, we obtain sufficient conditions for the existence and uniqueness of PAA solutions of system (2). The results obtained here are new and complementary previous studies.

2 Preliminary Results

Define the following notations:

$\{z_i(t)\} = (z_1(t), z_2(t)) \in \mathbb{R}^2$, $|z| = \{|z_i(t)|\}$ and $\|z(t)\| = \max\limits_{1\leq i\leq 2} \{|z_i(t)|\}$.

Let $BC(\mathbb{R},\mathbb{R})$ denotes collection of bounded continuous functions. $BC(\mathbb{R},\mathbb{R})$ is Banach space with norm $\|z\|_\infty = \sup_{t\in\mathbb{R}} |z(t)|$. Also we use the notations

$$\widehat{z} = \sup_{t\in\mathbb{R}} |z(t)|\,, \underset{\smile}{z} = \inf_{t\in\mathbb{R}} |z(t)|\,,$$

where $z(t) \in BC(R,R)$.

Definition 2.1. (See [10]) *Let $f \in C(\mathbb{R},\mathbb{R})$. For f to be $AP(\mathbb{R},\mathbb{R})$ it is necessary and sufficient that the familiy of functions $H = \left\{f^h\right\} = \{f(t+h)\}$, $-\infty < h < \infty$, is compact in $C(\mathbb{R})$.*

Definition 2.2. (See [2]) *$z_1 \in C(\mathbb{R},\mathbb{R})$ is called $AA(\mathbb{R},\mathbb{R})$ if for every real sequence (υ'_n), there exists a subsequence (υ_n) such that*

$$\lim_{n\to\infty} z_1(t+\upsilon_n) = z_2(t)$$

and

$$\lim_{n\to\infty} z_2(t-\upsilon_n) = z_1(t)$$

for each $t \in \mathbb{R}$.

Example 2.3. The function $\psi(t) = \cos(\frac{1}{2+\sin t+\sin\sqrt{2}t}) \in AA(\mathbb{R},\mathbb{R})$ but $\psi(t)$ is not $AP(\mathbb{R},\mathbb{R})$, because $\psi(t) \notin BC(\mathbb{R})$.

Definition 2.4. (See [3]) *$f \in C(\mathbb{R},\mathbb{R})$ is called $PAA(\mathbb{R},\mathbb{R})$ if it can be noted as*

$$f = f_1 + f_2,$$

with $f_1 \in AA(\mathbb{R},\mathbb{R})$ and $f_2 \in PAA_0(\mathbb{R})$ where space $PAA_0(\mathbb{R},\mathbb{R})$ is defined by

$$PAA_0(\mathbb{R}) := \left\{ f_2 \in BC(\mathbb{R},\mathbb{R}) | \lim_{q\to\infty} \frac{1}{2q} \int_{-q}^{q} \|f_2(t)\|\, dt = 0 \right\}.$$

Lemma 2.5. (See [16]) *$PAA(\mathbb{R},\mathbb{R})$ is a Banach space with norm $\|z\|_\infty = \sup_{t\in\mathbb{R}} |z(t)|$.*

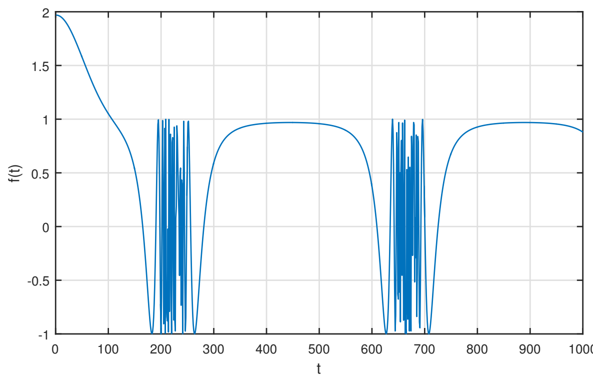

Figure 1: Graphic of $f(t)$.

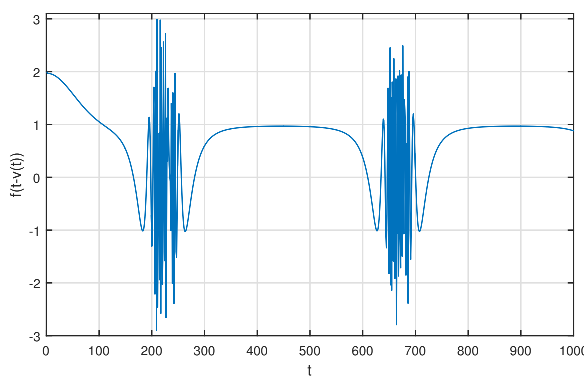

Figure 2: Graphic of $f(t - 0.001 \cos t)$.

It is clear that $AA(\mathbb{R}, \mathbb{R}) \subset PAA(\mathbb{R}, \mathbb{R}) \subset BC(\mathbb{R}, \mathbb{R})$.

Lemma 2.6. (See [3]) *If* $v(t) \in AA(\mathbb{R}, \mathbb{R})$ *and* $S(t), R(t) \in PAA(\mathbb{R}, \mathbb{R})$, *then* $S(t-k)$,$S(t) + R(t), S(t) \times R(t), S(t - v(t)) \in PAA(\mathbb{R}, \mathbb{R})$.

Example 2.7. The function f defined by $f(t) = e^{-2t^2} + \psi(t)$ is $PAA(\mathbb{R}, \mathbb{R})$ function but it is not $AA(\mathbb{R}, \mathbb{R})$. The graphs of the $f(t)$ and $f(t - 0.001cost)$ functions are in Figure 1 and Figure 2.

The following conditions are given for our main results:

(Ξ_1) h_i are global Lipschitz with Lipschitz constants L_i^h and there exists positive ξ such that

$$|h_i(\varsigma_1) - h_i(\varsigma_2)| \leq L_i^h |\varsigma_1 - \varsigma_2|$$

for all $\varsigma_1, \varsigma_2 \in \mathbb{R}$, $|g(\varsigma)| \leq \xi$, $h_i(0) = h_0(0) = 0$.

(Ξ_2) $\sigma_i(t), \kappa(t), A_1(t), A_2(t)$, $q(t) \in PAA(\mathbb{R}, \mathbb{R})$, $\inf \kappa(t) > 0$, for all $t \in \mathbb{R}$, $i = 1, 2, ..., m$.

(Ξ_3)

$$i)\ \gamma = \left(\underset{\smile}{\kappa}\right)^{-1} \max\left\{\left|\widehat{A}_1\right|, \left|\widehat{A}_2\right|\right\},$$

$$j)\ \nu = \left(\underset{\smile}{\kappa}\right)^{-1} \max\left\{1, \left(\widehat{\kappa}^2 + \xi[1 + \widehat{\kappa} + \widehat{A}_1] + L_0^h + \sum_{i=1}^{m} L_i^h\right)\right\},$$

$$k)\ \pi = \left(\underset{\smile}{\kappa}\right)^{-1} \max\left(1, \left[\widehat{\kappa}^2 + \left|\widehat{\kappa}\right| \xi + \xi(\left|\widehat{A}_1\right| + 2) + L_0^h + \sum\nolimits_{i=1}^{m} h_i^h\right]\right).$$

3 PAA solutions

Theorem 3.1. *Suppose that $(\Xi_1) - (\Xi_3)$ hold. Define a nonlinear operator G for each $\varphi = (\varphi_1, \varphi_2) \in PAA(\mathbb{R}, \mathbb{R}^2)$, $(G\varphi) := z_\varphi(t)$ where*

$$z_\varphi(t) = \left(\int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} \gamma_1(t)dt, -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} \gamma_2(t)dt\right),$$

where

$$\begin{aligned}
&\gamma_1(t) = \varphi_2(t) + A_1(t) \\
&\gamma_2(t) = -\kappa^2(t)\varphi_1(t) - g(\varphi_1(t))[\varphi_2(t) - \kappa(t)\varphi_1(t) + A_1(t)] \\
&- h_0(\varphi_1(t)) - \sum_{i=1}^{n} h_i\left(\varphi_1(t - \sigma_i(t))\right) + A_2(t).
\end{aligned}$$

Then $G\varphi \in PAA(\mathbb{R}, \mathbb{R}^2)$.

Proof. Noting that $G\varphi \in PAA(\mathbb{R}, \mathbb{R}^2)$then it follows from Definition 2.4 such that

$$\begin{pmatrix} \gamma_1(s) \\ \gamma_2(s) \end{pmatrix} = \begin{pmatrix} \gamma_{11}(s) + \gamma_{12}(s) \\ \gamma_{21}(s) + \gamma_{22}(s) \end{pmatrix}.$$

And so

$$
\begin{aligned}
(G\varphi) &= \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du}(\gamma_{11}(s)+\gamma_{12}(s))ds \\ -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du}(\gamma_{21}(s)+\gamma_{22}(s))ds \end{pmatrix} \\
&= \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du}\gamma_{11}(s)ds \\ -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du}\gamma_{21}(s)ds \end{pmatrix} + \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du}\gamma_{12}(s)ds \\ -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du}\gamma_{22}(s)ds \end{pmatrix} \\
&= (G_1\varphi) + (G_2\varphi)\,.
\end{aligned}
$$

Firstly, we prove that $(G_1\varphi) \in AA(\mathbb{R}, \mathbb{R}^2)$.

Let $(a'_n) \subset R$ be a sequence. It can be extracted a subsequence (a_n) of (a'_n) such that

$$
\begin{pmatrix} \lim\limits_{n\to+\infty} \gamma_{11}(s+a_n) \\ \lim\limits_{n\to+\infty} \gamma_{21}(s+a_n) \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_{11}(t) \\ \tilde{\gamma}_{21}(t) \end{pmatrix}, \begin{pmatrix} \lim\limits_{n\to+\infty} \tilde{\gamma}_{11}(s-a_n) \\ \lim\limits_{n\to+\infty} \tilde{\gamma}_{21}(s-a_n) \end{pmatrix} = \begin{pmatrix} \gamma_{11}(s) \\ \gamma_{21}(s) \end{pmatrix}
$$

for all $t \in R$.

Define

$$
\left(\tilde{G}_1\varphi\right)(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du}\tilde{\gamma}_{11}(s)ds \\ -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du}\tilde{\gamma}_{21}(s)ds \end{pmatrix}.
$$

Then we have

$$
\begin{aligned}
(G_1\varphi)(t+a_n) - \left(\tilde{G}_1\varphi\right)(t) &= \begin{pmatrix} \int_{-\infty}^{t+a_n} e^{-\int_s^{t+a_n} \kappa(u)du}\gamma_{11}(s)ds \\ -\int_{t+a_n}^{+\infty} e^{-\int_s^{t+a_n} \kappa(u)du}\gamma_{21}(s)ds \end{pmatrix} \\
&\quad - \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t \tilde{\kappa}(u)du}\tilde{\gamma}_{11}(s)ds \\ -\int_{t}^{+\infty} e^{-\int_t^s \tilde{\kappa}(u)du}\tilde{\gamma}_{21}(s)ds \end{pmatrix} \\
&= \begin{pmatrix} \int_{-\infty}^{t+a_n} e^{-\int_{s-a_n}^{t} \kappa(\tau+a_n)d\tau}\gamma_{11}(s)ds \\ -\int_{t+a_n}^{+\infty} e^{-\int_{s-a_n}^{t} \kappa(\tau+a_n)d\tau}\gamma_{21}(s)ds \end{pmatrix} \\
&\quad - \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_s^t \kappa(u)du}\tilde{\gamma}_{11}(s)ds \\ -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du}\tilde{\gamma}_{21}(s)ds \end{pmatrix}
\end{aligned}
$$

$$
= \begin{pmatrix} \int_{-\infty}^{t} e^{-\int_u^t \kappa(\tau+a_n)d\tau}\gamma_{11}(u+a_n)du - \int_{-\infty}^{t} e^{-\int_u^t \kappa(\tau+a_n)d\tau}\tilde{\gamma}_{11}(u)du \\ -\int_{t}^{+\infty} e^{-\int_u^t \kappa(\tau+a_n)d\tau}\gamma_{21}(u+a_n)ds + \int_{t}^{+\infty} e^{-\int_u^t \kappa(\tau+a_n)d\tau}\tilde{\gamma}_{21}(u)ds \end{pmatrix}
$$

$$+\begin{pmatrix} \int_{-\infty}^{t} e^{-\int_u^t \kappa(\tau+a_n)d\tau}\tilde{\gamma}_{11}(u)du - \int_{-\infty}^{t} e^{-\int_s^t \kappa(\tau)d\tau}\tilde{\gamma}_{11}(u)dt \\ -\int_t^{+\infty} e^{-\int_u^t \kappa(\tau+\kappa_n)d\tau}\tilde{\gamma}_{21}(u)ds + \int_t^{+\infty} e^{-\int_t^s \kappa(\tau)d\tau}\tilde{\gamma}_{21}(u)du \end{pmatrix}$$
$$=\begin{pmatrix} \int_{-\infty}^{t} e^{-\int_u^t \kappa(\tau+a_n)d\tau}[\gamma_{11}(u+a_n)-\tilde{\gamma}_{11}(u)]du \\ -\int_t^{+\infty} e^{-\int_u^t \kappa(\tau+a_n)d\tau}[\gamma_{21}(u+a_n)-\tilde{\gamma}_{21}(u)]ds \end{pmatrix}$$
$$+\begin{pmatrix} \int_{-\infty}^{t} [e^{-\int_u^t \kappa(\tau+a_n)d\tau} - e^{-\int_s^t a(\tau)d\tau}]\tilde{\gamma}_{11}(u)du \\ -\int_t^{+\infty} [e^{-\int_u^t \kappa(\tau+a_n)d\tau} - e^{-\int_s^t \kappa(\tau)d\tau}]\tilde{\gamma}_{21}(u)ds \end{pmatrix}.$$

Using Theorem 1.80 in [7], we get $\lim_{n\to\infty} (G_1\varphi)(t+a_n) = \left(\tilde{G}_1\varphi\right)(t)$. Similarly, we can prove that $\lim_{n\to\infty} \left(\tilde{G}_1\varphi\right)(t-a_n) = (G_1\varphi)(t)$ which implies that the function $(G_1\varphi) \in AA(\mathbb{R},\mathbb{R}^2)$.

Secondly, we can get

$$\begin{aligned}
\lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} |(G_2\varphi)(t)|\,dt &= (\lim_{T\to\infty} \frac{1}{2q}\int_{-q}^{q} \left|\int_{-\infty}^{t} e^{-\int_s^t \kappa(u)du}\gamma_{12}(s)ds\right|, \\
&- \lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \left|\int_{t}^{+\infty} e^{-\int_s^t \kappa(u)du}\gamma_{22}(s)ds\right|) \\
&= \lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \left|\int_{-\infty}^{t} e^{-\int_s^t \kappa(u)du}\gamma_{12}(s)ds\right|dt \\
&\le \lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \left|\int_{-q}^{t} e^{-\int_s^t \kappa(u)du}\gamma_{12}(s)ds\right|dt \\
&+ \lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \left|\int_{-\infty}^{-q} e^{-\int_s^t \kappa(u)du}\gamma_{12}(s)ds\right|dt \\
&\le \lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \|\gamma_{12}(s)\|\,dt \int_{-q}^{t} e^{-\underset{\smile}{\kappa}(t-s)}ds \\
&+ \lim_{q\to\infty} \frac{\sup|\gamma_{12}(t)|}{2q}\int_{-q}^{q} dt\int_{-\infty}^{-q} |e^{-\underset{\smile}{\kappa}(t-s)}|ds \\
&= \lim_{q\to\infty} \frac{\sup|\gamma_{12}(t)|}{2q(\underset{\smile}{\kappa})^2}(1-e^{-2q}) = 0.
\end{aligned}$$

Similarly, we can write

$$\lim_{q\to\infty} \frac{1}{2q}\int_{-q}^{q} \left|\int_{t}^{+\infty} e^{-\int_s^t \kappa(u)du}\gamma_{22}(s)ds\right| = 0.$$

Then

$$\lim_{q\to\infty} \frac{1}{2q} \int_{-q}^{q} |(G_2\varphi)(t)|\, dt = 0,$$

which implies that $(G_2\varphi) \in AA_0(\mathbb{R}, \mathbb{R}^2)$. Thus, $G\varphi \in PAA(\mathbb{R}, \mathbb{R}^2)$. $\square$

Theorem 3.2. *Suppose that assumptions* $(\Xi_1)-(\Xi_3)$ *hold. Then system (2) has a unique PAA solution in the region*

$$U = \left\{ \varphi | \, \|\varphi - \varphi_0\| \leq \frac{\chi\nu}{1-\nu}, \varphi \in BUC(\mathbb{R}, \mathbb{R}^2) \cap PAA(\mathbb{R}, \mathbb{R}^2) \right\},$$

where

$$\varphi_0 = \left(\int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} A_1(t)dt, \int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} A_2(t)dt \right).$$

Proof. Let We can see [12] that U is closed subset. Define a mapping $N : U \to U$, by setting

$$(N\varphi) = \begin{pmatrix} z_{1\varphi} \\ z_{2\varphi} \end{pmatrix},$$

where,

$$z_{1\varphi}(t) = \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} \left[\varphi_2(s) + A_1(s)\right] ds$$

$$z_{2\varphi}(t) = -\int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} \Big[-\kappa^2(s)\varphi_1(s) - g(\varphi_1(s))[\varphi_2(s) - \kappa(s)\varphi_1(s)$$
$$+\varphi_1(s)] - h_0(\varphi_1(s)) - \sum_{i=1}^{n} h_i\left(\varphi_1(s - \sigma_i(s))\right) + A_2(t)\Big] ds.$$

It is clear that

$$\|\varphi_0\| \leq \sup_{t\in\mathbb{R}} \max\left\{ \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} A_1(s)ds, \int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} A_2(s)ds \right\}$$
$$\leq \max\left\{ \frac{\left|\widehat{A}_1\right|}{\underset{\smile}{\kappa}}, \frac{\left|\widehat{A}_2\right|}{\underset{\smile}{\kappa}} \right\} = \gamma < 1.$$

Also

$$\|\varphi\|_\infty \leq \|\varphi - \varphi_0\| + \|\varphi_0\|_\infty \leq \frac{\gamma\nu}{1-\nu} + \gamma = \frac{\lambda}{1-\nu} < 1.$$

Therefore, we can write

$$\|N\varphi - \varphi_0\|_\infty = \left(\left| \int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} \varphi_2(t)dt \right|, \left| \int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} \gamma_2^s(t)dt \right| \right), \tag{3}$$

where

$$\begin{aligned} \gamma_2^s(t) = &- \kappa^2(t)\varphi_1(t) - g(\varphi_1(t))[\varphi_2(t) - \kappa(t)\varphi_1(t) + A_1(t)] - h_0(\varphi_1(t)) \\ &- \sum_{i=1}^{n} h_i\left(\varphi_1(t-\sigma_i(t))\right). \end{aligned}$$

Then, from (3) we get

$$\|N\varphi - \varphi_0\|_\infty = \left(\underset{\smile}{\kappa}\right)^{-1} \max\left\{1, \left(\widehat{\kappa}^2 + \xi[1 + \widehat{\kappa} + \widehat{A}_1] + \sum_{i=0}^{m} L_i^h\right)\right\} \|\varphi\|_\infty$$
$$= \nu\|\varphi\|_\infty \leq \nu\frac{\gamma}{1-\nu}.$$

Also

$$\|N\varphi\|_\infty \leq \|N\varphi - \varphi_0\|_\infty + \|\varphi_0\| \leq \nu\frac{\gamma}{1-\nu} + \gamma = \frac{\gamma}{1-\nu} < 1.$$

Therefore, can conclude from [17] that N is a self mapping from U to U.

$$\|(N\varphi)(t) - (N\eta)(t)\|_\infty = \left(\left|((N\varphi_1)(t) - (N\eta_1)(t))_1\right|, \left|((N\varphi_1)(t) - (N\eta_1)(t))_2\right|\right)^T$$
$$\leq \left(\int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} \left|\varphi_2(s) - \eta_2(s)\right| ds, \int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} \left|\gamma_2(t) - \widetilde{\gamma}_2(t)\right| dt \right)^T$$
$$\leq \left(\int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} ds \sup_{t\in R} |\varphi_2(t) - \eta_2(t)|, \int_{t}^{+\infty} e^{-\int_t^s \kappa(u)du} \right.$$

$$\times \left|-\kappa^2(s)(\varphi_1(s)-\eta_1(s)) - g(\varphi_1(s))[\varphi_2(s)-\kappa(s)\varphi_1(s)+A_1(s)]\right.$$
$$+ g(\eta_1(s))[\eta_2(s)-\kappa(s)\eta_1(s)+A_1(s)] - (h_0(\varphi_1(s)) - h_0(\eta_1(s)))$$
$$\left.\left. -(\sum_{i=1}^{m} h_i\left(\varphi_1(s-\sigma_i(s))\right) - \sum_{i=1}^{m} h_i\left(\eta_1(t-\sigma_i(t))\right))\right| ds\right)^T$$
$$\le \left(\int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} ds \sup_{t\in R}|\varphi_2(t)-\eta_2(t)|, \int_t^{+\infty} e^{-\int_t^s \kappa(u)du} ds\right.$$
$$\times \left\{\kappa^2(t)\sup_{t\in R}|\varphi_1(t)-\eta_1(t)| + \xi \sup_{t\in R}|\varphi_2(t)-\eta_2(t)| + \right.$$
$$|\kappa(t)|\sup_{t\in R}|\varphi_1(t)-\eta_1(t)| + \xi\,|A_1(t)|\sup_{t\in R}|\varphi_1(t)-\eta_1(t)|$$
$$+ \xi \sup_{t\in R}|\varphi_1(t)-\eta_1(t)| + h_0 \sup_{t\in R}|\varphi_1(t)-\eta_1(t)| +$$
$$\left.\left. + \sum_{i=1}^{m} h_i \sup_{t\in R}|\varphi_1(t-\sigma_i(t)) - \eta_1(t-\sigma_i(t))|\right\}\right)^T$$
$$\le (\int_{-\infty}^{t} e^{-\int_t^s \kappa(u)du} ds, \int_t^{+\infty} e^{-\int_t^s \kappa(u)du} ds$$
$$\times \left[\widehat{\kappa}^2 + \left|\widehat{\kappa}\right|\xi + \xi(\left|\widehat{A}_1\right| + 2) + h_0 + \sum\nolimits_{i=1}^{m} h_i\right])^T \|\varphi-\eta\|_\infty$$
$$\le \left(\underset{\smile}{\kappa}\right)^{-1} \max\left(1, \left[\widehat{\kappa}^2 + \left|\widehat{\kappa}\right|\xi + \xi(\left|\widehat{A}_1\right| + 2) + h_0 + \sum\nolimits_{i=1}^{m} h_i\right]\right) \|\varphi-\eta\|_\infty$$
$$= \pi\|\varphi-\eta\|_\infty.$$

By (Ξ_3), we can conclude that N is a contraction. It follows that N has a unique fixed point $q \in N$ of (2), $Uq = q$. From Theorem 3.1, q is PAA solution. $\square$

Example 3.3. Considering the following Lienard-type system with multiple delays

$$\begin{aligned} z'_1(t) &= -\kappa(t)z_1(t) + z_2(t) + A_1(t), \\ z'_2(t) &= \kappa(t)z_2(t) - \kappa^2(t)z_1(t) - g(z_1(t)) \times [z_2(t) - \kappa(t)z_1(t) + A_1(t)] \\ &- h_0(z_1(t)) - h_1\left(z_1(t-\sigma_1(t))\right) + A_2(t), \end{aligned} \tag{4}$$

where $\kappa(t) = 24 + 2\sin\frac{1}{2+\sin t+\sin\sqrt{2}t}$, $A_1(t) = -2 - 20cos\frac{1}{2+\sin t+\sin\sqrt{2}t} + e^{-t^2}$, $h_0 = h_1 = \frac{1}{2}(|z+1| - |z-1|)$, $A_2(t) = cos\sqrt{5}(\frac{1}{1+\sin t}) + \sin\sqrt{7}t + e^{-t^2}$, $g(z) = \arctan z$, $\sigma_1(t) = \frac{1}{2}|\cos t|$.Then, $L = \xi = 1$, $\underset{\smile}{\kappa} = 22$,

$\nu = \frac{1}{22} < 1$, $\pi = \frac{1}{22} < 1$, $\gamma = \frac{21}{22} < 1$. Then $(\Xi_1) - (\Xi_3)$ in Theorem 3.2 hold, thus system (4) has an unique positive PAA solution.

Remark 3.4. In this study, some important results were obtained concerning the PAA solutions of the Lienard type system discussed. As seen in Example 2.7, the set of PAA functions is more extensive than the known sets of AP, PAP and AA. Therefore, the main results of the study are complementary to the new and previous studies. In addition, an example is given that confirms the results.

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