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# Parter, Periodic and Coperiodic Functions on Groups and their Characterization

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**Abstract.** Decomposer functional equations were introduced by the author and have been completely solved on arbitrary groups. Their solutions are as decomposer functions and play important role regarding to decomposition (factorization) of groups by their two subsets. In this paper, we introduce an important class of strong decomposer functions, namely *parter (or cyclic decomposer) functions*. As some important applications of this topic, we characterize all periodic , coperiodic functions in arbitrary groups and give general solution of their functional equations: $f(bx) = f(x)$ , $f(xb) = f(x)$, $f(bx) = bf(x)$ and $f(xb) = f(x)b$. Moreover, we characterize all parter functions in arbitrary groups and completely solve the decomposer equation with the condition which its $*$-range is a cyclic subgroup of $G$. Finally, we give some functional characterization for related projections and $b$-parts functions and also, we introduce some uniqueness conditions for $b$-parts of real numbers.

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## 1. Introduction and Preliminaries

In [2] decomposer and strong decomposer functions on groups were introduced and characterized. They have many important properties and so close relations to factorization of groups by their two subsets, and associative functions. Every strong decomposer function $f$ on a group

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$G$ is associative and induces another binary operation $\cdot_f$ namely *$f$-multiplication* such that $(G, \cdot_f)$ is a grouplike (a new algebraic structure that is something between semigroup and group, see [5,6]). Here, we introduce and study an important special case of decomposer functions, namely *parter (or cyclic decomposer)* functions. On the other hand, $b$-parts of real numbers were introduced in [8] and some of their properties have been studied in [8,4]. Here, we see that the $b$-decimal part function is a real parter function and we consider it as our first idea for introducing the topic.
Let $(G, .)$ be a group with the identity element $e$. If $f, g$ are functions from $G$ to $G$, then define the functions $f.g$ and $f^-$ by

$$f.g(x) = f(x)g(x) \quad , \quad f^-(x) = [f(x)]^{-1} \ : \qquad \forall x \in G.$$

Note that the notation $fg$ is used for the composition of $f$ and $g$ ($fog$), also $f^{-1}$ is the inverse function of $f$. We denote the identity function on $G$ by $\iota_G$ or simply by $\iota$ and constant function $f(x) = c$ by $c$ (especially when $c = e$).
For every $f : G \to G$ , we put $f^* = \iota.f^-$ , $f_* = f^-.\iota$ and call $f^*$ [resp. $f_*$] *left $*$-conjugate of $f$* [resp. *right $*$-conjugate of $f$*]. They are also called *$*$-conjugates of $f$*. Clearly $\iota^* = \iota_* = e$, $e^* = e_* = \iota$, $f = (f^*)_* = (f_*)^*$. Also, the identity $(fg)^- = f^- g$ implies $f^- f = (f^2)^-$, $(fg)^* = g^*.f^* g$ and $(fg)_* = f_* g.g_*$ . Note that $f$ is idempotent if and only if $f^* f = e$ (or $f_* f = e$).

**Additive notations.** If $(G, +)$ is additive group, then the notations $e$, $f^-$, $f.g$, $f.g^-$ are replaced by $0$, $-f$, $f + g$, $f - g$ and we have $f^* = f_* = \iota - f$.

**Example 1.1.** Consider the additive group $\mathbb{R}$ and fix $b \in \mathbb{R} \setminus \{0\}$. For each real number $a$ denote by $[a]$ the largest integer not exceeding $a$ and put $(a) = a - [a]$ (the decimal part of $a$). Now, set

$$[a]_b = b[\frac{a}{b}] \ \ , \ \ (a)_b = b(\frac{a}{b}).$$

We call $[a]_b$ *$b$-integer part of $a$* and $(a)_b$ *$b$-decimal part of $a$*. Also, $[\ ]_b$, $(\ )_b$ are called *$b$-decimal part function and $b$-integer part function*, respectively. Clearly $(\ )_b^* = [\ ]_b$ and $[\ ]_b^* = (\ )_b$, both are idempotent, so their compositions are zero and $(\mathbb{R})_b = \mathbb{R}_b = b[0,1) = \{bd|0 \leqslant d < 1\}$ and $[\mathbb{R}]_b = b\mathbb{Z} = \langle b\rangle$. Here, the $b$-parts functions are our idea for introducing $b$-parter functions as an important class of strong decomposer, $b$-periodic and $b$-coperiodic functions.

## 2. Decomposer Type Functions on Groups

If $f$ is an arbitrary function from $G$ to $G$ and $f(x) = f(y)$, then $x = f^*(x)f(y) = f(y)f_*(x)$. The converse is valid if $f$ is decomposer and we have the following definition (see [2,3]).

**Definition 2.1.** *Let $f$ be a function from $G$ to $G$. We call $f$:*
(a) *right [resp. left] strong decomposer* if

$$f(f^*(x)y) = f(y) \quad [\text{resp. } f(xf_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(b) *right [resp. left] semi-strong decomposer* if

$$f(f^*(x)y) = f(f^*(e)y) \quad [\text{resp. } f(xf_*(y)) = f(xf_*(e))] \quad : \forall x, y \in G.$$

(c) *right [resp. left] decomposer* if

$$f(f^*(x)f(y)) = f(y) \quad [\text{resp. } f(f(x)f_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(d) *right [resp. left] weak decomposer* if

$$f(f^*(e)f(x)) = f(x)\ ,\ f(f^*(x)f(e)) = f(e) \quad : \forall x \in G.$$

$$[\text{resp. } f(f(x)f_*(e)) = f(x)\ ,\ f(f(e)f_*(x)) = f(e) \quad : \forall x \in G].$$

(e) *right [resp. left] separator* if $f^*(G) \cap f(G) = \{f(e)\}$ [resp. $f(G) \cap f_*(G) = \{f(e)\}$].

*We call $f$* decomposer *or* two-sided decomposer *[resp. separator] if it is left and right decomposer [resp. separator].*

**Note:** In each parts of the above and other definitions if $\underline{f(e) = e}$, then we will add the word *standard* to the titles. For example "$f$ is standard right separator" means $f^*(G) \cap f(G) = \{e\}$.

Fix $c \in G$. The functions $c$ and $cx$ [resp. $c$ and $x$] are decomposer [resp. strong decomposer], we call them *trivial decomposers* [resp. *trivial strong decomposers*].
In the additive real numbers group, $|x|$ is standard separator (and idempotent) but it is not (standard) weak decomposer. The function $f(x) = \max\{x, 0\}$ is standard weak decomposer but it is not (standard) decomposer. Finally $[\ ]_b$ is standard decomposer but it is not (standard) strong decomposer.

**Theorem 2.2.** *Let $f : G \to G$.*
*(a) $f$ is right strong decomposer $\Rightarrow$ $f$ is right semi-strong decomposer $\Rightarrow$ $f$ is right decomposer $\Rightarrow$ $f$ is right weak decomposer.*
*(b) $f$ is standard right strong decomposer $\Leftrightarrow$ $f$ is standard right semi-strong decomposer $\Rightarrow$ $f$ is standard right decomposer $\Rightarrow$ $f$ is standard right weak decomposer $\Rightarrow$ $f$ is standard right separator.*
*(c) If $f$ is right strong decomposer, then $f$ is right separator, idempotent, $ff^* = f(e)$ and*

$$f^*(e).ff^* = f^*f = e \quad , \quad \langle f(e)\rangle \leqslant f^*(G) \leqslant G.$$

*(d) If $f$ is right decomposer and $f^*(G) \leqslant G$, then $f$ is right strong decomposer (and visa versa). We have similar theorem for left decomposer type functions.*

**Proof**. See [2,3]. $\square$

**Example 2.3.** Consider $G = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\} \cong D_4$ ($a^4 = b^2 = 1, bab = a^{-1} = a^3$). Put $\Omega = \{1, ba, ba^2, ba^3\}$ and

$$f(x) = \begin{cases} x & x \in \Omega \\ bx & x \notin \Omega \end{cases}$$

Considering the relation $x \notin \Omega \Leftrightarrow bx \in \Omega$, it can be seen that $f$ is (standard) right strong decomposer.

## 3. Parter, Periodic and Coperiodic Functions on Groups

An exact observation of $b$-parts functions shows that their basic properties are $(x+b)_b = (x)_b$ [i.e. $(\ )_b$ is $b$-periodic] and $[\mathbb{R}]_b = \langle b \rangle$ (i.e. the $*$-range of $(\ )_b$ is cyclic real subgroup) and other properties are concluded from them. This fact lead us to an important vast class of (strong) decomposer functions which are the most natural generalization of $b$-parts functions. In this section we study those decomposer functions such that $f^*(G)$ is cyclic subgroup and consider their relations to the periodic and coperiodic functions.

**Definition 3.1.** *We call a right [resp. left] decomposer function $f$ right [resp. left]* parter (or cyclic decomposer) *if $f^*(G)$ [resp. $f_*(G)$] is a cyclic subgroup of $G$, and if this is the case $f^*(G) = \langle b \rangle$ [resp. $f_*(G) = \langle b \rangle$], then we call it right $b$-parter [resp. left $b$-parter]. The function $f$ is called $b$-parter if it is left and right $b$-parter. The identity function is trivial (two-sided) standard $e$-parter. If $G = \langle g \rangle$, then the constant function $e$ is also another trivial $g$-parter function on $G$. Also, $f$ is right parter if and only if $f$ is right strong decomposer and $f^*(G)$ is subset of a cyclic subgroup of $G$ (by Theorem 2.2).*

**Example 3.2.** The function $f$ in Example 2.3 is $b$-parter, because is right decomposer and $f^*(G) = \langle b \rangle = \{1, b\}$. The parter functions have so closed relationship to periodic and coperiodic functions. Recall that if $b \in G$ is fix. then a function $f : G \to G$ is called left [resp. right] $b$-periodic if $f(bx) = f(x)$ [resp. $f(xb) = f(x)$] for all $x \in G$. It is $b$-periodic if is left and right $b$-periodic. In group $G$, every function is trivial (two-sided) $e$-periodic, and the function $e$ is trivial (two-sided) $b$-periodic, for all $b \in G$.

**Definition 3.3.** *We call $f : G \to G$ left [resp. right]* $b$-coperiodic *if $f(bx) = bf(x)$ [resp. $f(xb) = f(x)b$] for all $x \in G$, and it is* $b$-coperiodic *if is left and right $b$-coperiodic. The $b$-integer part function $[\ ]_b$ is standard $b$-coperiodic and the $b$-decimal part function $(\ )_b$ is standard $b$-periodic. In general, $f : G \to G$ is left [resp. right] $b$-coperiodic if and only if $f_*$ [resp. $f^*$] is left [resp. right] $b$-periodic:*

$$f(bx) = bf(x) \Leftrightarrow f_*(bx) = f_*(x), f(xb) = f(x)b \Leftrightarrow f^*(xb) = f^*(x) : \forall x \in G.$$

**Example 3.4.** If $f$ is right [resp. left] strong decomposer and $b \in f^*(G)$ [resp. $b \in f_*(G)$],then $f$ is left [resp. right] $b$-periodic. The function $f$ in Example 2.3 is left $b$-periodic but it is not right $b$-periodic.

**Definition 3.5.** *If $\Delta$ and $\Omega$ are subsets of $G$, then the notation $A = \Delta\cdot\Omega$ means $A = \Delta\Omega$ and if $\delta_1\omega_1 = \delta_2\omega_2$ where $\delta_1, \delta_2 \in \Delta$, $\omega_1, \omega_2 \in \Omega$, then $\delta_1 = \delta_2$ and $\omega_1 = \omega_2$. If $A = \Delta \cdot \Omega$, then we say $A$ is* direct product of (subsets) $\Delta$ and $\Omega$. *By the notation $A = \Delta : \Omega$ we mean $A = \Delta \cdot \Omega$ and $\Delta\cap\Omega = \{e\}$ and say $A$ is* standard direct product of $\Delta$ and $\Omega$. *Note that additive notations are $\Delta\dot{+}\Omega$ (direct sum of subsets) and $\Delta\ddot{+}\Omega$ (standard direct sum of subsets). For example $\mathbb{R} = b\mathbb{Z}\ddot{+}b[0, 1) = \langle b\rangle\ddot{+}\mathbb{R}_b$.*

Clearly if $\Delta\Omega = \Delta\cdot\Omega$, then $|\Delta\Omega| = |\Delta||\Omega|$. Also if $\Delta$ and $\Omega$ are nonempty subsets of $G$, then $\Delta\Omega = \Delta \cdot \Omega$ if and only if $(\Delta^{-1}\Delta) \cap (\Omega\Omega^{-1}) = \{e\}$ (in additive notation $(\Delta - \Delta) \cap (\Omega - \Omega) = \{0\}$).

Let $G = \Delta\cdot\Omega$. Define the functions $P_\Delta,\ P_\Omega$, from $G$ to $G$, by $P_\Delta(x) = \delta$, $P_\Omega(x) = \omega$, where $x = \delta\omega, \delta \in \Delta, \omega \in \Omega$. Clearly, they are well-defined and $P_\Delta(G) = \Delta,\ P_\Omega(G) = \Omega,\ P^*_\Omega = P_\Delta$. We call $P_\Omega$, [resp. $P_\Delta$] *right [resp. left] projection* of the direct decomposition $G = \Delta\cdot\Omega$. The $b$-parts functions are projections of the direct decomposition $\mathbb{R} =< b > \ddot{+}\mathbb{R}_b$.

Now, we can prove some equivalent conditions for a function to be left $b$-parter and state its relations to $b$-periodic, $b$-coperiodic functions and factorization of $G$ by its subsets. Similar theorem can be stated for the right case.

**Theorem 3.6.** *Fix $b \in G$. The followings are equivalent for every $f : G \to G$:*
*(a) $f$ is right $b$-parter,*
*(b) $f$ is right strong decomposer and $b \in f^*(G) \subseteq \langle b\rangle$,*
*(c) $f$ is right semi-strong decomposer and $f^*(G) = \langle b\rangle$,*
*(d) $f$ is left $b$-periodic and $f^*(G) = \langle b\rangle$,*
*(e) $f^*$ is left $b$-coperiodic and $f^*(G) = \langle b\rangle$,*
*(f) $G = \langle b\rangle \cdot f(G)$ and $b \in f^*(G) \subseteq \langle b\rangle$.*

**Proof**. It is clear that $(a) \Rightarrow (b) \Rightarrow (c)$, by applying Theorem 2.2 and the definitions.
Now if (c) holds, then there exist $x_0, y_0 \in G$ such that $f^*(x_0) = e$, $f^*(y_0) = b$ and

$$f(bx) = f(f^*(y_0)x) = f(f^*(e)x) = f(f^*(x_0)x) = f(ex) = f(x);\ \forall x \in G.$$

So, we arrive at (d).

Considering the relation

$$f(bx) = f(x) \Leftrightarrow f^*(x) = bf^*(x) \quad : \quad \forall x \in G,$$

(d) implies (e).
Suppose (e) holds. Then $f$ is left $b$-periodic (because of the above relation) and $G = f^*(G)f(G) = \langle b \rangle f(G)$. We claim that $f$ is idempotent and this product is direct. For if $x \in G$, then $f^*(x) = b^{n_x}$ for some $n_x \in \mathbb{Z}$ and

$$f(f(x)) = f(b^{n_x} f(x)) = f(f^*(x)f(x)) = f(x),$$

also if $x, y \in G$, $m, n \in \mathbb{Z}$ and $b^m f(x) = b^n f(y)$, then

$$f(x) = f^2(x) = f(b^m f(x)) = f(b^n f(y)) = f(y),$$

therefore $b^m = b^n$ and the claim has been proved and we obtain (f).
Finally, if (f) holds and $x, y \in G$, then there exist $n_x, n_y \in \mathbb{Z}$ such that $f^*(x) = b^{n_x}$, $f^*(y) = b^{n_y}$ and so

$$f(f^*(x)y) = f(f^*(x)f^*(y)f(y)) = f(b^{n_x+n_y} f(y)) = f(y),$$

where the last equation is concluded from $G = \langle b \rangle \cdot f(G)$ and the identity $b^{n_x+n_y} f(y) = f^*(b^{n_x+n_y} f(y)) f(b^{n_x+n_y} f(y))$. Therefore, $f$ is right strong decomposer thus $f^*(G) \leqslant G$ and so $f^*(G) = \langle b \rangle$. Therefore, the proof is complete. $\square$

**Remark 3.7.** *The conditions (a)-(f) are not equivalent to the statement "$f$ is right decomposer and $b \in f^*(G) \subseteq \langle b \rangle$" (that is gotten from (a) if replace $f^*(G) = \langle b \rangle$ by $b \in f^*(G) \subseteq \langle b \rangle$). Because, putting*

$$f(x) = \begin{cases} x & \textit{if } [x] \textit{ is even} \\ x - 1 & \textit{if } [x] \textit{ is odd} \end{cases}$$

*$f$ is decomposer and $1 \in f^*(\mathbb{R}) \subseteq \langle 1 \rangle = \mathbb{Z}$, but $f$ is not $1$-parter ($f^*(\mathbb{R}) = \{0, 1\}$).*

Up to now we have studied the properties of parter functions. Now, we show that how we can construct them and prove their existence,

in arbitrary groups. Considering Remark 2.7, Corollary 2.9 of [2] (and p448 of [7]), if $H \leqslant G$ [resp. $H \trianglelefteq G$], then $H$ is a left and right factor [resp. two-sided factor] of $G$ as the sense in factorization of a group by its subsets, i.e. there exist subsets $\Omega$ and $\Delta$ of $G$ such that $G = H \cdot \Omega$ and $G = \Delta \cdot H$ [resp. $G = H \cdot \Omega = \Omega \cdot H$]. Moreover, we can find such these subsets in which $G = H : \Omega$ and $G = \Delta : H$ [resp. $G = H : \Omega = \Omega : H$] that means $H$ is a standard left and right factor [resp. standard two-sided factor] of $G$. Also, if $G \neq 0$ is a group (finite or infinite) for which $|G|$ is not a prime number and $H$ is a non-trivial subgroup [resp. normal subgroup], then we can find such subsets with cardinality$> 1$ (so the factorization is non-trivial). This fact implies existence of left and right [resp. two-sided] strong decomposer functions in groups $G \neq 0$ (finite or infinite) for which $|G|$ is not a prime number [resp. non-simple groups] (see [2;Corollary 2.9]).
Also, Theorem 2.5,3.5 and Corollary 2.2,3.6 of [2] state that:

**Theorem 3.8.** *In every group $G$*
*(i) General form of all right [resp. left] decomposer functions is*

$$f = P_\Omega \quad ; \quad \textit{for all representation}\, G = \Delta \cdot \Omega.$$

$$[f = P_\Delta \quad ; \quad \textit{for all representation}\, G = \Delta \cdot \Omega.]$$

*(ii) General form of all right [resp. left] strong decomposer functions is*

$$f = P_\Omega \quad ; \quad \textit{for all representation}\, G = \Delta \cdot \Omega \textit{ with } \Delta \leqslant G.$$

$$[f = P_\Delta \quad ; \quad \textit{for all representation}\, G = \Delta \cdot \Omega \textit{ with } \Omega \leqslant G.]$$

*(iii) General form of all strong decomposer functions is*

$$f = P_\Omega \quad ; \quad \textit{for all representation}\, G = \Delta \cdot \Omega \textit{ with } \Delta \trianglelefteq G.$$

*For the standard case of the every above functions, $\Delta \cdot \Omega$ should be replaced by $\Delta : \Omega$.*

**Remark 3.9.** *Therefore, putting $H = \langle b \rangle$, there exists a class of subsets $\Omega$ and $\Delta$ of $G$ such that $G = \langle b \rangle \cdot \Omega$ and $G = \Delta \cdot \langle b \rangle$. We fix one of such these $\Omega$ in which $G = \langle b \rangle : \Omega$ and denote it by $\Omega_b$. Hence, in this paper* *we consider $G = \langle b\rangle : \Omega_b$ as the fixed standard direct decomposition by $b$ (i.e. $\langle b\rangle$). For example, for $b\mathbb{Z} = \langle b\rangle \leqslant \mathbb{R}$, there exists an infinite class of real subsets $\Omega$ containing all half open intervals $b[n, n+1)$ such that $\mathbb{R} = b\mathbb{Z}\dot{+}\Omega$. Here, we fix $\Omega_b = b[0,1) = \mathbb{R}_b$ and we have $\mathbb{R} =< b > \ddot{+}\mathbb{R}_b$. Hence, we observe that $P_{b\mathbb{Z}} = [\ ]_b$ and $P_{\mathbb{R}_b} = (\ )_b$ which are the same $b$-parts functions. Now, we are ready to prove existence of a vast class of decomposer type functions.*

**Lemma 3.10.** *Let $G \neq 0$ be a group (finite or infinite).*
*(a) If $|G|$ is not a prime number, then nontrivial (standard) right and left parter, periodic and coperiodic functions exist (and vice versa).*
*(b) If $G$ is not cyclic, then for every $b \in G \setminus \{e\}$ nontrivial (standard) right and left $b$-parter, $b$-periodic and $b$-coperiodic functions exist.*
*(c) If $\langle b\rangle$ is nontrivial normal subgroup of $G$, then nontrivial standard (two-sided) $b$-parter, $b$-periodic and $b$-coperiodic functions exist (and vice versa).*

**Proof**. Suppose that $\langle b\rangle$ is a non-trivial subgroup of $G$. Then, Remark 3.6 gives us a non-singleton and proper subset $\Omega_b$ of $G$ such that $G = \langle b\rangle : \Omega_b$. Putting $f = P_{\Omega_b}$, we have $f^*(G) = P_{\langle b\rangle}(G) = \langle b\rangle$, and so Theorem 3.5 implies $f$ is nontrivial standard right parter function on $G$. Therefore, nontrivial standard right $b$-parter, left $b$-periodic and left $b$-coperiodic functions exist (analogously for the left parter function). This fact proves (a) and (b), clearly.
Now, if $\langle b\rangle$ is nontrivial normal subgroup of $G$, then $f = P_{\Omega_b}$ is (two-sided) standard strong decomposer, by Theorem 2.2 (c), and also Corollary 3.6 of [2] implies $f^*(G) = f_*(G) = P_{\langle b\rangle}(G) = \langle b\rangle$ so we arrive at (c), by Theorem 3.6. The converse is also valid, by Theorem 2.2 and Theorem 3.6. $\square$

## 4. General Form of Parter Functions and General Solution of the Periodic and Coperiodic Functional Equations

Up to now, one see that every right $b$-parter function is left $b$-periodic and left $*$-conjugate of every $b$-periodic function is $b$-coperiodic. Hence, considering the fact that left composition of every arbitrary function on $G$ with all $b$-periodic functions are also $b$-periodic, we arrive at the following characterizing theorem.

**Theorem 4.1.** *(Characterization of parter, periodic and coperiodic functions). Fix $b \in G$.*
*(i) General form of all right [resp. left] $b$-parter functions is*

$$f = P_\Omega \quad ; \quad \textit{for all representation } G = \langle b \rangle \cdot \Omega.$$

$$[f = P_\Delta \quad ; \quad \textit{for all representation } G = \Delta \cdot \langle b \rangle.]$$

*(ii) General form of all left [resp. right] $b$-periodic functions is*

$$f = \mu P_\Omega \; ; \textit{for all representation } G = \langle b \rangle \cdot \Omega \textit{and all functions } \mu : \Omega \to G.$$

$$[f = \mu P_\Delta ; \textit{for all representation } G = \Delta \cdot \langle b \rangle \textit{ and all functions } \mu : \Delta \to G.]$$

*(iii) General form of all left [resp. right] $b$-coperiodic functions is*

$$f = P_{\langle b \rangle} . \mu P_\Omega \; ; \textit{for all representation} G = \langle b \rangle \cdot \Omega \textit{ and all functions} \mu : \Omega \to G.$$

$$[f = \mu P_\Delta . P_{\langle b \rangle} ; \textit{for all representation} G = \Delta \cdot \langle b \rangle \textit{and all functions} \mu : \Delta \to G.]$$

**Proof**. Part(i) is concluded from Theorem 2.2 and Theorem 3.6. Now, let $f$ be left $b$-periodic and consider the standard factorization $G = \langle b \rangle : \Omega_b$. Then $f(\beta x) = f(x)$ for all $\beta \in \langle b \rangle$, $x \in G$ and so

$$f(x) = f(P^*_{\Omega_b}(x)^{-1} x) = f(P_{\Omega_b}(x)) \quad : \forall x \in G,$$

because $P^*_{\Omega_b}(G) = \langle b \rangle$. Now putting $\mu = f|_{\Omega_b}$ we have $f = \mu P_{\Omega_b}$ and $\mu : \Omega_b \to G$. Conversely, it is clear that if $f$ has the form (ii), then $f$ is left $b$-periodic.
For the last part, if $f = P_{\langle b \rangle} . \mu P_\Omega$, then

$$f(bx) = P_{\langle b \rangle}(bx) \mu(P_\Omega(bx)) = b P_{\langle b \rangle}(x) \mu(P_\Omega(x)) = b f(x).$$

Conversely, if $f$ is left $b$-coperiodic, then $f_*$ is left $b$-periodic and the part (ii) implies $f_* = \lambda P_\Omega$ where $\lambda, P_\Omega$ are gotten from the general solution (ii) for $f_*$. So,

$$f = (f_*)^* = (\lambda P_\Omega)^* = P^*_\Omega . \lambda^* P_\Omega = P_{\langle b \rangle} . \lambda^* P_\Omega$$

Therefore,
putting $\mu = \lambda^*$ we obtain $\mu : \Omega \to G$ such that $f = P_{\langle b \rangle}.\mu P_\Omega$. $\square$

**Remark 4.2.** *The above theorem completely solve the functional equations $f(bx) = f(x)$, $f(bx) = bf(x)$, the left strong decomposer equation with the condition $f^*(G) = \langle b \rangle$ and the other mentioned functional equations (right cases). Also, it states an important fact that all left b-periodic [resp. b-coperiodic] functions $f$ are gotten from the composition of right b-parter functions $\varepsilon_b$ and arbitrary functions $\mu$ defined on its image, i.e. $f = \mu\varepsilon_b$ [resp. $f = \varepsilon_b^*.\mu\varepsilon_b$]. Moreover, for every left b-periodic function $f$ we can find an standard left b-parter function $\varepsilon_b$ (obtained from the fixed standard direct decomposition $G = \langle b \rangle : \Omega_b$ where $\varepsilon_b = P_{\Omega_b}$). Analogously, we have similar properties for the right cases and b-coperiodic functions. Therefore, we can say the fixed standard b-parter function $\varepsilon_b$ is essential and basic b-periodic function and it generates others (by composition with arbitrary functions). For instance, the above statement says the b-decimal part function $\varepsilon_b = (\ )_b$ is the essential b-periodic real function and other periodic real functions are generated by it, i.e. $f = \mu(\ )_b$ where $\mu : b[0,1) \to \mathbb{R}$. Also, the general solution of the real functional equation $f(b + x) = b + f(x)$ is $f = [\ ]_b + \mu(\ )_b = [\ ]_b + \mu[\ ]_b^*$ that is general form of all real b-coperiodic functions, and says $[\ ]_b$ is the essential b-coperiodic function.*
*Considering the interesting fact that $P_{b\mathbb{Z}} = [\ ]_b$ and $P_{b[0,1)} = (\ )_b$ (corresponding to the standard direct decomposition $\mathbb{R} = b\mathbb{Z}\dot{+}b[0,1)$) we can give a uniqueness conditions for them, by the following lemma.*

**Lemma 4.3.** *Let $G = \Delta \cdot \Omega$ and $f : G \to G$.*
*(i) If $f$ is a right [resp. left] decomposer function such that $f(G) \subseteq \Omega$ and $f^*(G) \subseteq \Delta$ [resp. $f(G) \subseteq \Delta$ and $f_*(G) \subseteq \Omega$], then $(f^*, f) = (P_\Delta, P_\Omega)$ [resp. $(f, f_*) = (P_\Delta, P_\Omega)$].*
*(ii) If $f$ is a right [resp. left] decomposer and $g : G \to G$ is a function such that $g(G) \subseteq f(G)$ and $g^*(G) \subseteq f^*(G)$ [resp. $g(G) \subseteq f(G)$ and $g_*(G) \subseteq f_*(G)$], then $g = f$.*
*(iii) $(P_\Delta, P_\Omega)$ is the unique solution of functional equation $f(f^*(x)f(y)) = f(y)$ [resp. $f(f(x)f_*(y)) = f(x)$] with the conditions $f(G) = \Omega$ and $f^*(G) = \Delta$ [resp. $f(G) = \Delta$ and $f_*(G) = \Omega$].*

**Proof**. If $f(G) \subseteq \Omega$ , $f^*(G) \subseteq \Delta$ and $x \in G$, then there exist $x', x'' \in G$ such that $f(x) = P_\Omega(x')$, $f^*(x) = P_\Delta(x'')$ and

$$P_\Omega(x) = P_\Omega(f^*(x)f(x)) = P_\Omega(P_\Delta(x'')P_\Omega(x')) = P_\Omega(x') = f(x).$$

So (i) is proved and (ii) is a result of (i). Also, (iii) is concluded from Theorem 2.2 and the part (i). □

**Corollary 4.4.** *(Functional characterizations of b-parts of real numbers)*
*(i) b-decimal part function is the only b-parter function for which its range is* $b[0, 1)$.
*(ii) b-decimal part function is the unique solution of the functional equation*
$f(x - f(x) + f(y)) = f(y)$ *such that* $f(\mathbb{R}) = b[0, 1)$ *and* $f^*(\mathbb{R}) = b\mathbb{Z}$.
*(iii) b-decimal part function is the unique solution of the functional equation*
$f(x - f(x) + y) = f(y)$ *such that* $f(\mathbb{R}) = b[0, 1)$ *and* $f^*(\mathbb{R}) = b\mathbb{Z}$.
*(iv) b-integer part function is the unique solution of the functional equation*
$f(x - f(x) + f(y)) = f(y)$ *such that* $f(\mathbb{R}) = b\mathbb{Z}$ *and* $f^*(\mathbb{R}) = b[0, 1)$.

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