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On the Block Coloring of Steiner Triple Systems

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Abstract. A *Steiner triple system* of order v, STS(v), is an ordered pair $S = (V, B)$, where V is a set of size v and B is a collection of triples of V such that every pair of V is contained in exactly one triple of B. A *k-block coloring* is a partitioning of the set B into k color classes such that every two blocks in one color class do not intersect. In this paper, we introduce a construction and use it to show that for every k-block colorable STS(v) and l-block colorable STS(w), there exists a ($k+lv$)-block colorable STS(vw). Moreover, it is shown that for every k-block colorable STS(v), every STS($2v+1$) obtained from the well-known construction is $(k+v)$-block colorable.

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1. Introduction

Let G be a graph. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The *degree* of $v \in V(G)$ is the number of edges of G incident with v. The maximum degree of G is denoted by $\Delta(G)$. A graph G called *strongly* (k, λ, μ)*-regular* if there are parameters k, λ and μ such that G is k-regular, every adjacent pair of vertices have λ common neighbors, and every nonadjacent pair of vertices have μ common neighbors. A *proper vertex coloring* of G is a function $c : V(G) \longrightarrow L$, with this property that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A *vertex k-coloring* is a proper vertex coloring with $|L| = k$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k for which G has a vertex k-coloring.

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Theorem 1.1. *If G is not an odd cycle or a complete graph, then $\chi(G) \leqslant \Delta(G)$.*
A (proper) k-edge coloring of a graph G is a function $f : E(G) \longrightarrow L$, where $|L| = k$ and $f(e_1) \neq f(e_2)$, for every two adjacent edges of G. A matching in a graph is a set of non-adjacent edges. A perfect matching of G is a matching that covers all vertices of G. Given an edge coloring of a graph, a rainbow matching is a matching whose edges have distinct colors.
An $n \times n$ matrix $L = (l_{ij})$ whose entries are taken from a set S of n symbols is called a latin square of order n on S if each symbol appears precisely once in each row and in each column of L. A pair of latin squares $L = (l_{ij})$ and $L' = (l'_{ij})$ are called orthogonal latin squares if and only if the ordered pairs (l_{ij}, l'_{ij}) are distinct for all i and j. Here we say that L is orthogonal to L'. The following theorem states the condition for existence orthogonal latin squares of order n.

Theorem 1.2. ([2]) *For every natural number $n \neq 2, 6$, there is a pair of orthogonal latin squares of order n.*
A Steiner triple system of order v, STS(v), is an ordered pair $S = (V, B)$, where V is a set of size v and B is a set of size b which is a collection of triples of V such that every pair of V is contained in exactly one triple of B. Every triple of STS(v) called a block. The number of times that each $v \in V$ appears in the blocks is denoted by r. One can easily see that for every STS(v), $r = \frac{v-1}{2}$. It is not hard to see that a STS(v) exists if and only if the edges of the complete graph K_v partitions into triangles. It is well known that a necessary and sufficient condition for existing a STS(v), $v \geqslant 3$ is $v \equiv 1$ or $3 \pmod 6$ (see [1]). Such a v is said to be admissible.
A Steiner triple system (V, B) is called resolvable if the triples of B can be partitioned into $\frac{b}{r}$ classes, where each class is a partition of V. By Lemma 9.1.1 *of [2], a resolvable STS(v) can exist only if $v \equiv 3, \pmod 6$.*
Let $S = (V, B)$ be a Steiner triple system. A color class is a system of pairwise disjoint triples. A k-block coloring is a partitioning of the set B into k color classes. Here we say that (V, B) is k-block colorable. The chromatic index, $\chi'(S)$, of a Steiner triple system S is the least k for which a k-block coloring exists. We say two blocks are adjacent if they have an element of S in common. A block intersection graph of a Steiner triple system $S = (V, B)$, denoted by G_S, is a graph with the vertex set B; the vertices are adjacent if and only if the respective blocks are adjacent. Moreover, it is not hard to see that G_S is a strongly $(3r-3, r+2, 9)$-regular graph. So, by Theorem 1.1, $\chi'(S) \leqslant 3r-3$ for $v > 7$. Also, since the the clique number of G_S is r, $\chi'(S) \geqslant r$ if $v \equiv 3 \pmod 6$. The following well known theorem states that in what conditions $\chi'(S) = r$.

Theorem 1.3. *Let S be a STS(v). Then $\chi'(S) = r$ if and only if S is resolvable.*
Now, By Theorem 1.1 and Theorem 1.3, we conclude that $r \leqslant \chi'(S) \leqslant 3r - 3$

if $v \equiv 3 \pmod 6$ and $r+1 \leqslant \chi'(S) \leqslant 3r-3$ if $v \equiv 1 \pmod 6$.
The upper bound $\chi'(S) \leqslant 3r-3$ seems to be weak in general. In fact, using probabilistic methods Pippenger and Spencer in [7] proved that $\chi'(STS(v))$ is asymptotic to $\frac{v}{2}$. For more information on the chromatic index of Steiner triple systems the reader is referred to Chapter 18 of [4]. For some classes of STS(v) the upper bound was improved. In particular, Colbourn in [3] improved it for cyclic STS(v) by proving $\chi'(STS(v)) \leqslant v$. Block coloring of Steiner triple systems studied by several authors (For more information see [4, 5, 6]).
In this paper, we introduce a construction and use it to show that for every k-block colorable STS(v) and l-block colorable STS(w), there exists a $(k+lv)$-block colorable STS(vw). Moreover, it is shown that for every k-block colorable STS(v), every STS(2v+1) obtained from the well-known construction is $(k+v)$-block colorable.

2. Block Coloring of Steiner Triple Systems of Order vw

In this section, we introduce a block coloring of a Steiner triple system STS(vw) obtained from two Steiner triple systems STS(v) and STS(w). For this purpose, first we establish the following Construction. In particular, we introduce some resolvable STS(vw) of STS(v) and STS(w).

Construction 2.1.
Let (V, B) be a STS(v) on the set $V := \{x_1, \ldots, x_v\}$ and (W, B') be a STS(w) on the set $W := \{y_1, \ldots, y_w\}$. Then define (Z, S) as a STS(vw) on the set $Z := \{z_{ij}, 1 \leqslant i \leqslant v, 1 \leqslant j \leqslant w\}$ with two types of blocks as follows:
For every j, $1 \leqslant j \leqslant w$, consider a copy of the complete graph K_v, K_v^j , with the vertex set $\{z_{1j}, \ldots, z_{vj}\}$. Using (V, B) one can partition the edges of each K_v^j, for every $1 \leqslant j \leqslant w$, into triangles. Call the blocks made by these triangles, Type 1. Now, consider the complete graph K_w with the vertex set $\{K_v^j, 1 \leqslant j \leqslant w\}$. Using (W, B') one can partition the edges of K_w into triangles. Let us call this partition by F. For every i, j, $1 \leqslant i, j \leqslant w$, $i \neq j$, join every vertex of K_v^i to every vertex of K_v^j. So, every triangle in the K_w is corresponding to $3v^2$ edges. Now, for each triangle of F such as $\{K_v^p, K_v^s, K_v^t\}$, $1 \leqslant p, s, t \leqslant w$, consider a latin square L of order v on the set $\{z_{1t}, \ldots, z_{vt}\}$ such that the rows and the columns are indexed by $\{z_{1p}, \ldots, z_{vp}\}$ and $\{z_{1s}, \ldots, z_{vs}\}$, respectively. For every z_{ip} and z_{js}, $1 \leqslant i, j \leqslant v$, $\{z_{ip}, z_{js}, L_{z_{ip}z_{js}}\}$ is considered as a block of Type 2. It is not hard to see that all blocks of Type 1 and Type 2 form a STS(vw). Call a Steiner triple Systems obtained from this Construction such that every used latin square has an orthogonal latin square, by OLS(vw).

Note that by Theorem 1.2, OLS$(vw) \neq \emptyset$ for each admissible v and w.

Theorem 2.2. *For every k-block colorable STS(v) and l-block colorable STS(w), there exists a $(k+lv)$-block colorable OLS(vw).*

Proof. Let (V,B) be a k-block colorable STS(v) and $f : B \longrightarrow \{1,\ldots,k\}$ be a such coloring and (W,B') be a l-block colorable STS(w) with the function $f' : B' \longrightarrow \{1,\ldots,l\}$. Moreover, let (Z,S) be an OLS(vw). Now, define $c : S \longrightarrow \{1,\ldots,lv+k\}$ as follows. First, we color the blocks of Type 1. For all $1 \leqslant j \leqslant w$ and $\{x_m,x_n,x_p\} \in B$, let $c(\{z_{mj},z_{nj},z_{pj}\}) = f(\{x_m,x_n,x_p\})$. Next, to color the blocks of Type 2, consider a triangle $t = \{K_v^p,K_v^s,K_v^t\}$ of partition F in Construction **??**. Note that t is corresponding to a block of (W,B'), say b. Let L be a latin square used to partition the edges of t and L' be a latin square on the set $\{k+(f'(b)-1)v+1,\ldots,k+f'(b)v\}$ orthogonal to L. Then, let $c(\{z_{ip},z_{js},L_{z_{ip}z_{js}}\}) = L'_{z_{ip}z_{js}}$ and repeat this procedure for every triangle of F. We show that c is a $(k+lv)$-block coloring of (Z,S). First, note that if two adjacent blocks call b_1 and b_2 have the same color, since the set of colors used to color the blocks of Type 1 and Type 2 have no color in common, then b_1 and b_2 belong to the same type. First, suppose that b_1 and b_2 are the blocks of Type 1. Since f is a k-block coloring of (V,B), $c(b_1) \neq c(b_2)$. Now, suppose that b_1 and b_2 are the blocks of Type 2. Two cases may be assumed. Suppose that $b_1 = \{m,n,L_{mn}\}$ and $b_2 = \{p,q,L_{pq}\}$ are obtained from the same triangle of partition F. If $m=p$ or $n=q$, since L' is a latin square, then $c(b_1) \neq c(b_2)$. If $L_{mn} = L_{pq}$, since L and L' are orthogonal latin squares, then $L'_{mn} \neq L'_{pq}$. So, in this case $c(b_1) \neq c(b_2)$. Now, assume that b_1 and b_2 belong to different triangles, call t_1 and t_2. The adjacency of b_1 and b_2 concludes the adjacency of t_1 and t_2. So, t_1 and t_2 are corresponding to two adjacent blocks in B'. Since f' is a block coloring of B', the sets of colors used to color the blocks obtained from t_1 and t_2 have no color in common. Thus $c(b_1) \neq c(b_2)$ and the proof is complete. $\square$

Corollary 2.3. *If there exists a resolvable STS(v) and a resolvable STS(w), then there exists a resolvable STS(vw).*

Proof. Let (V,B) and (W,B') be a resolvable STS(v) and a resolvable STS(w). Moreover, let (Z,S) be an OLS(vw). By Theorem 2.2, (Z,S) is $(\frac{vw-1}{2})$-block colorable. Since the chromatic index of (Z,S) is at least $\frac{vw-1}{2}$, by Theorem 1.3 we are done. $\square$

Theorem 2.4. *If (Z,S) is a resolvable STS(vw) obtained from Construction 2.1, then (Z,S) is an OLS(vw).*

Proof. Let $f : Z \longrightarrow \{1,\ldots,\frac{vw-1}{2}\}$ be a $(\frac{vw-1}{2})$-block coloring of (Z,S).

First we claim that exactly v colors are appeared in the coloring of the blocks obtained from each used latin square. Note that for every $x \in Z$, x appears in $\frac{v-1}{2}$ blocks of Type 1 and $\frac{v(w-1)}{2}$ blocks of Type 2. Call $q = \frac{w-1}{2}$ latin square used in Construction 2.1, by $L_1, \ldots, L_q$. Note that x appears in v blocks obtained from L_i for eah $1 \leqslant i \leqslant q$. Thus for every $1 \leqslant i, j \leqslant q$, $i \neq j$, there are v colors appeared in the blocks obtained from L_i not in the blocks obtained from L_j. Moreover, since f is a $(\frac{vw-1}{2})$-block coloring, for each latin square exactly v colors are used in f. Now, for each latin square L define L' be a square of size v such that $L'_{ij} = f(\{i, j, L_{ij}\})$. The properties of block coloring conclude that L' is orthogonal to L and the proof is complete. □

3. Block Coloring of Steiner Triple Systems of Order $2v + 1$

In this section, we study the block chromatic index of STS$(2v + 1)$. We show that there exists $(k+v)$-block colorable STS$(2v+1)$ for every k-block colorable STS(v). Before stating the main result, we need the following definition and theorems.

Definition 3.1. ([2]) *Let S be a set of $n + 1$ elements (symbols). A Room square of side n (on symbol set S), $RS(n)$, is an $n \times n$ array, F, that satisfies the following properties:*
(1) *every cell of F either is empty or contains an unordered pair of symbols from S.*
(2) *Each symbol of S occurs once in each row and column of F.*
(3) *Every unordered pair of symbols occurs in precisely one cell of F.*

Theorem 3.2. ([2]) *A Room square of side n exists if and only if n is odd and $n \neq 3, 5$.*

Corollary 3.3. *Let n be an even integer where $n \neq 4, 6$. Then there exists a $(n-1)$-edge coloring of K_n such that partition the edges of K_n to $(n-1)$ rainbow perfect matchings.*

Proof. Let $S = V(K_n) = \{v_1, \ldots, v_n\}$. Since $n \neq 4, 6$ is an even integer, by Theorem 3.2, there exists a room square F of side $n - 1$ on S. Note that each unordered pair appeared in each cell of F is corresponding to an edge of K_n. Moreover, by Part (2) of Definition 3.1, the union of edges appeared in each row or column is a perfect matching of K_n. Now, assign color i to all edges appeared in cells of row i in F, for every i, $1 \leqslant i \leqslant n-1$. Note that since every unordered pair of symbols occur in precisely one cell of F, we obtain a

$(n-1)$-edge coloring of K_n. Now, the columns of F partition the edges of K_n to $(n-1)$ rainbow perfect matchings. $\square$

In the following, we introduce a construction of STS$(2v+1)$ obtained from a STS(v) and use it to find $(k+v)$-block colorable STS$(2v+1)$ for every k-block colorable STS(v).

Construction 3.4. Let (V, B) be a STS(v). Define a STS$(2v+1)$, (W, B'), on the set $W = \{x_1, \ldots, x_{2v+1}\}$ with two types of blocks as follows. The blocks of Type 1 are the blocks of B on $\{x_1, \ldots, x_v\}$. Now, consider the complete graph K_{v+1} with the vertex set $\{x_{v+1}, \ldots, x_{2v+1}\}$. Since v is odd, the edges of K_{v+1} can be partitioned to v perfect matchings $F_1, \ldots, F_v$. Now, every triangle obtained from vertex x_i, $1 \leqslant i \leqslant v$, and two vertices of every edge of F_i introduce a block. These blocks are blocks of Type 2.

Theorem 3.5. *For every k-block colorable STS(v), every STS$(2v+1)$ obtained from Construction 3.4 is $(k+v)$-block colorable.*

Proof. Let (V, B) be a k-block colorable STS(v) and $f : B \to \{1, \ldots, k\}$ be a such coloring. Moreover, let (W, B') be a STS$(2v+1)$ obtained from Construction 3.4. Define $c : B' \to \{1, \ldots, k+v\}$ as follows. For $1 \leqslant i, j, t \leqslant v$, let $c(\{x_i, x_j, x_t\}) = f(\{x_i, x_j, x_t\})$. Since $v+1$ is even, by Theorem 3.3, there exists a v-edge coloring ϕ of K_{v+1} on the vertices $\{x_{v+1}, \ldots, x_{2v+1}\}$ such that all edges partitions into v rainbow perfect matchings $F_1, \ldots, F_v$. Now, for every block $\{x_i, x_j, x_t\}$, $1 \leqslant i \leqslant v$ and $v+1 \leqslant j, t \leqslant 2v+1$, let $c(\{x_i, x_j, x_t\}) = \phi(x_j x_t)$. We claim that c is a $(k+v)$-block coloring. Notice that if two adjacent blocks call b_1 and b_2 have the same color, since the set of colors used to color the blocks of Type 1 and Type 2 have no color in common, then b_1 and b_2 belong to the same type. So, two cases may be considered. First, suppose that b_1 and b_2 are the blocks of Type 1. Since f is a k-block coloring of (V, B), $c(b_1) \neq c(b_2)$. Otherwise, two cases may be considered. First assume that $b_1 \cap b_2 = \{x_i\}$ such that $1 \leqslant i \leqslant v$. Since every perfect matching F_p, $1 \leqslant p \leqslant v$ is rainbow, $c(b_1) \neq c(b_2)$. Now, suppose that $b_1 \cap b_2 = \{x_i\}$, $v+1 \leqslant i \leqslant 2v+1$. Since ϕ is a proper edge coloring, we are done and the proof is complete. $\square$

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