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Original Research Paper

Investigation of a Common Solution for a Multi-Singular Fractional System by Using Control Functions Method

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Abstract. In this article, first of all, we investigate a pointwise defined multi-singular fractional differential equation. Using control functions method, existence a solution for the problem, will be proved. In the following, we determine some conditions to prove the existence of a common solution for two multi-singular fractional differential equations with integral boundary conditions. To this purpose, we use inequalities, control functions and fixed point method. Finally, an example will illustrate our main results.

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1 Introduction

Besides the fact that fractional calculus had been dated back to the last three centuries, it is of high significance among the recent researchers

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and academians (see, for instance, [1]- [7]), that sometimes are singular at some points (see [8]- [13]). Sometimes, considering a mathematical model of a sceintific phenomena, leads to a fractional differential equation, therefore many application in fractional calculus can be seen (see [14]- [20]).

In [21], the authors investigated the fractional equation ${}^c\mathcal{D}^{\sigma}\nu(t)+y(t,\nu(t)) = 0$ with initial conditions $\nu(0) = \nu''(0) = 0$ and $\nu(1) = \tau\int_0^1 \nu(s)ds$, where $0 < t < 1$, $2 < \sigma < 3$, $0 < \tau < 2$, ${}^c\mathcal{D}^{\sigma}$ is the Caputo fractional derivative and $y : [0,1]\times[0,\infty) \to [0,\infty)$ is a continuous function.

In 2013, the fractional problem ${}^c\mathcal{D}^{r}\nu(\xi) + y(t,\nu(\xi)) = 0$ with boundary conditions $\nu'(0) = \nu''(0) = \cdots = \nu^{(k_0-1)}(0) = 0$ and $\nu(1) = \int_0^1 \nu(s)d\gamma(s)$ was investigated, where $0 < \xi < 1$, $n \geq 2$, $r \in (k_0-1, k_0)$, $\gamma(s)$ is a function of bounded variation, y may have singularity at $\xi = 1$ and $\int_0^1 d\gamma(s) < 1$ ([22]).

In 2015, the fractional problem ${}^c\mathcal{D}^{\rho}y(t) = \psi(t, y(t), {}^c\mathcal{D}^{\sigma}y(t))$ with boundary conditions $y(0)+y'(0) = g(x)$, $\int_0^1 y(t)dt = m_0$ and $y''(0) = y^{(3)}(0) = \cdots = y^{(n_\rho-1)}(0) = 0$ was studied where, $0 < t < 1$, m_0 is a real number, $n_\rho \geq 2$, $\rho \in (n_\rho-1, n_\rho)$, $0 < \sigma < 1$, ${}^c\mathcal{D}^{\rho}$ and ${}^cD^{\sigma}$ is the Caputo fractional derivatives, $g \in C([0,1],\mathbb{R}) \to \mathbb{R}$ and $\psi : (0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous with $\psi(t,u,v)$ that may be singular at $t = 0$ ([23]).

In 2018, the existence of a solution for the following three steps crisis problem was investigated:

$$ {}^c\mathcal{D}^{\eta}z(t) + \psi(t, z(t), z'(t), {}^c\mathcal{D}^{\sigma}z(t), \int_0^t \Omega(\xi)z(\xi)d\xi, \omega(x(t))) = 0 $$

with boundary conditions $z(1) = z(0) = z''(0) = z^{n_\eta}(0) = 0$, where $\eta \geq 2$, $\lambda, \mu, \sigma \in (0,1)$, $\Omega \in L^1[0,1]$, $\omega : C^1[0,1] \to C^1[0,1]$ is a mapping such that $\|\omega(x_1) - \omega(x_2)\| \leq \iota_0\|x_1 - x_2\| + \iota_1\|x_1' - x_2'\|$ for some non-negative real numbers ι_0 and $\iota_1 \in [0,\infty)$ and all $x_1, x_2 \in C^1[0,1]$, ${}^c\mathcal{D}^{\eta}$ is the η-order Caputo fractional derivative, $\psi(t, z_1(t), ..., z_5(t)) = \psi_1(t, z_1(t), ..., z_5(t))$ for all $t \in [0,\lambda)$, $\psi(t, z_1(t), ..., z_5(t)) = \psi_2(t, z_1(t), ..., z_5(t))$ for all $t \in [\lambda,\mu]$ and $\psi(t, z_1(t), ..., z_5(t)) = \psi_3(t, z_1(t), ..., z_5(t))$ for all $t \in (\mu, 1]$, $\psi_1(t, ., ., ., ., .)$ and $\psi_3(t, ., ., ., ., .)$ are continuous on $[0,\lambda)$ and $(\mu, 1]$ and $\psi_2(t, ., ., ., ., .)$ is multi-singular ([24]).

In 2019, the existence and uniqueness of solutions were discussed for the following class of boundary value problem of nonlinear fractional differential equations depending with non-separated type integral boundary conditions

$$^c\mathcal{D}^q z(t) = \Psi(t, z(t), ^c\mathcal{D}^r z(t))$$

with the conditions $z(0) - \iota_1 z(\tau) = \kappa_1 \int_0^\tau U(s, z(s))ds$ and $z'(0) - \iota_2 z'(\tau) = \kappa_2 \int_0^\tau V(s, z(s))ds$, where $t \in [0, \tau]$, $t > 0$, $1 < q \leq 2$, $0 < r \leq 1$, $^c\mathcal{D}^q$ is the q-th order of the Caputo fractional derivative, $\Psi \in C([0, \tau] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), U, V : [0, \tau] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\iota_1, \iota_2, \kappa_1, \kappa_2 \in \mathbb{R}$ with $\iota_1 \neq 1$ and $\iota_2 \neq 1$ ([25]).

In 2020, the existence of solutions were examined for the following nonlinear differential pointwise defined system:

$$\begin{cases} ^c\mathcal{D}^{\alpha_1}\nu_1(t) = h_1(t, \nu_1(t), \nu_1'(t), ^c\mathcal{D}^{\beta_1}\nu_1(t), I^{p_1}\nu_1(t), \\ ..., \nu_m(t), \nu_m'(t), ^c\mathcal{D}^{\beta_m}\nu_m(t), I^{p_m}\nu_m(t)), \\ . \\ . & , \qquad t \in [0,1] \\ . \\ ^c\mathcal{D}^{\alpha_m}\nu_m(t) = h_m(t, \nu_1(t), \nu_1'(t), ^c\mathcal{D}^{\beta_1}\nu_1(t), I^{p_1}\nu_1(t), \\ ..., \nu_m(t), \nu_m'(t), ^c\mathcal{D}^{\beta_m}\nu_m(t), I^{p_m}\nu_m(t)), \end{cases}$$

with boundary value conditions $\nu_k^{(j)}(0) = 0$ for $2 \leq j \leq n_k - 1$ and $k = 1, \ldots, m$,

$$\nu_k(\theta_k) = \sum_{i=1}^{n_0} \lambda_{i,k} {^c\mathcal{D}^{\mu_{i,k}}} \nu_k(\gamma_{i,k})$$

and $\nu_k'(0) = \nu_k(\eta_k)$ for all $k = 1, 2, ..., m$, where $\lambda_{i,k} \geq 0$, $\beta_k, \gamma_{i,k}, \mu_{i,k}, \theta_k, \eta_k \in (0, 1)$, $p_k > 0$, $m, n_0 \in \mathbb{N}$, $k = 1, 2, ..., m$, $i = 1, 2, ..., n_0$, $^c\mathcal{D}^{\alpha_k}$ is the Caputo fractional derivative of order $\alpha_k \geq 2$, $n_k = [\alpha_k] + 1$, and $h_k : [0, 1] \times X^{4m} \to \mathbb{R}$, is singular at some points $[0, 1]$, where $X = C^1[0, 1]$ ([26]).

Regarding the main ideas of above papers, we investigate the non-controlled multi-singular fractional differential pointwisly defined equation

$$^c\mathcal{D}^\sigma w(t) + \mathcal{U}(t, w(t), w'(t), ^c\mathcal{D}^\beta w(t), \phi(w(t))) = 0 \tag{1}$$

with boundary conditions $w(0) = 0$ for $\sigma \in [2, 3)$ and $w(0) = w''(0) = w^{(n_0)}(0) = 0$ where $n_0 = [\sigma] - 1$ for $\sigma \in [3, \infty)$ and also $w(\eta) +$

$\int_0^1 w(s)ds = 0$ where $\sigma \geq 2$, $\eta, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that for all $w_1, w_2 \in X$, $\|\phi(w_1) - \phi(w_2)\| \leq a_0\|w_1 - w_2\| + a_1\|w_1' - w_2'\|$ for some $a_0, a_1 \in [0,\infty)$, ${}^c\mathcal{D}^\sigma$ is the Caputo fractional derivative of order σ and $\mathcal{U} : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a function such that $\mathcal{U}(t,.,.,.,.)$ is singular at some points $t \in [0,1]$. In fact, $\mathcal{U}$ is stated to be multi-sigular when it is singular at more than one point t (see Example 2.1 and 2.2). Likewise, ${}^c\mathcal{D}^\alpha w(t) + \mathcal{U}(t) = 0$ is pointwise defined equation on $[0,1]$ if there is the set $E \subset [0,1]$ such that its measure of complenment E^c is zero and equation on E is being hold. It's obvious that every equation is a pointwisly defined equation. In this paper, we use $\|.\|_1$ as the norm of $L^1[0,1]$, $\|.\|$ as the sup norm $Y = C[0,1]$ and $\|w\|_* = \max\{\|w\|, \|w'\|\}$ as the norm of $X = C^1[0,1]$.

2 Preliminaries

In this section, we introduce some notations and basic facts which are used throughout the paper. The Riemann-Liouville integral of order r with the lower limit $\mathfrak{b} \geq 0$ for a function $y : (\mathfrak{b},\infty) \to \mathbb{R}$ is defined by $\mathcal{I}^r_{\mathfrak{b}+}y(t) = \frac{1}{\Gamma(r)}\int_{\mathfrak{b}}^t (t-s)^{r-1}y(s)ds$ provided that the right-hand side is pointwise defined on $(\mathfrak{b},\infty)$. we denote $\mathcal{I}^r y(t)$ for $\mathcal{I}^r_{0+}y(t)$. Also, The Caputo fractional derivative of order $r > 0$ of an absolutely continuous function $y : (0,\infty) \to \mathbb{R}$ is defined by ${}^c\mathcal{D}^r y(t) = \frac{1}{\Gamma(n-r)}\displaystyle\int_0^t \frac{y^n(s)}{(t-s)^{r+1-n}}ds$, where $n = [r] + 1$ ([27]).
Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$ ([28]). One can check that $\psi(t) < t$ for all $t > 0$ ([28]). Let $\mathcal{T} : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two maps. Then $\mathcal{T}$ is called an α-admissible map whenever $\alpha(x,y) \geq 1$ implies $\alpha(\mathcal{T}x, \mathcal{T}y) \geq 1$ ([29]). Let (X,d) be a complete metric space, $\psi \in \Psi$ and $\alpha : X \times X \to [0,\infty)$ a map. A self-map $\mathcal{T} : X \to X$ is called an α-ψ-contraction whenever $\alpha(x,y)d(\mathcal{T}x,\mathcal{T}y) \leq \psi(d(x,y))$ for all $x, y \in X$ ([29]). We need the following results.

Lemma 2.1. *([30]) Assume that* $0 < n-1 \leq r < n$ *and* $v \in C[0,1] \cap L^1[0,1]$. *Then* $\mathcal{I}^r {}^c\mathcal{D}^r v(\xi) = v(\xi) + \sum_{i=0}^{n-1} \iota_i \xi^i$ *for some constants* $\iota_0, \dots, \iota_{n-1} \in \mathbb{R}$.

Lemma 2.2. *([31])Let X is a Banach space and $\mathcal{C} \subseteq X$ is closed and convex. Suppose that Ξ be a relatively open subset of $\mathcal{C}$ with $0 \in \Xi$ and let $\mathcal{T} : \Xi \to \mathcal{C}$ be a continuous and compact mapping. Then either*
i) the mapping $\mathcal{T}$ has a fixed point in $\bar{\Xi}$, or
ii) there exists $w_0 \in \partial\Xi$ and $\gamma \in (0, 1)$ with $w_0 = \gamma\mathcal{T}w_0$.

Lemma 2.3. *([32]) Let (X, d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0, \infty)$ is a map and $\mathcal{S}, \mathcal{T} : X \to X$ are mappings satisfying the following conditions*
i) for $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(\mathcal{S}x, \mathcal{T}y) \geq 1$ or $\alpha(\mathcal{T}x, \mathcal{S}y) \geq 1$,
ii) there exists $x_0 \in X$ such that $\alpha(x_0, \mathcal{S}x_0) \geq 1$,
iii) $\mathcal{S}$ and $\mathcal{T}$ are continuous
iv) for all $x, y \in X$, $\alpha(x, y)d(\mathcal{S}x, \mathcal{T}y) \leq \psi(d(x, y))$ and $\alpha(y, x)d(\mathcal{S}x, \mathcal{T}y) \leq \psi(d(x, y))$.
Then $\mathcal{T}$ and $\mathcal{S}$ have a common fixed point.

3 Main Results

In this section, we declare existence condititions for the problem (6). First of all, we change the differential equation to a integral one, then we prove the exisitence of a solution for the problem (6).

Lemma 3.1. *Let $\sigma \geq 2$, $\eta \in (0, 1)$ and $\mathcal{U} \in L^1[0, 1]$. Then $w(t) = \int_0^1 \kappa(t, s)\mathcal{U}(s)ds$ is a solution for the pointwise defined problem ${}^c\mathcal{D}^\sigma w(t) + \mathcal{U}(t) = 0$ with boundary value conditions $w(0) = 0$ for $\sigma \in [2, 3)$ and $w(0) = w''(0) = w^{(n_0)}(0) = 0$ where $n_0 = [\sigma] - 1$ for $\sigma \in [3, \infty)$ and also $w(\eta) + \int_0^1 w(s)ds = 0$ for all $\sigma \in [2, \infty)$, where*

$$\kappa(t, s) = \begin{cases} \frac{-(t-s)^{\sigma-1}}{\Gamma(\sigma)} + \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} + \frac{2t(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)} & 0 \leq s \leq t \leq 1, \ s \leq \eta \\ \frac{-(t-s)^{\sigma-1}}{\Gamma(\sigma)} + \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} & 0 \leq \eta \leq s \leq t \leq 1 \\ \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} & 0 \leq t \leq s \leq 1, \ \eta \leq s \\ \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} + \frac{2t(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)} & 0 \leq t \leq s \leq \eta \leq 1. \end{cases}$$

Proof. Let for all $t \in E \subset [0,1]$ the equation ${}^{c}\mathcal{D}^{\sigma} w(t) + \mathcal{U}(t) = 0$ is held, where $m(E^c) = 0$ and m is the Lebesgue measure on $\mathbb{R}$. Also let $\mathcal{U}_0 \in L^1[0,1] \cap C[0,1]$ be a function such that $\mathcal{U}_0 = \mathcal{U}$ on E. Note that if this problem has a solution then $\mathcal{U}_0$ exists, because if $w_0 \in C[0,1]$ is a solution for the pointwise defined problem, it is enough to consider $\mathcal{U}_0(t) = -{}^{c}\mathcal{D}^{\sigma} w_0(t)$ for all $t \in [0,1]$, so we have $\mathcal{U}_0 \in L^1[0,1] \cap C[0,1]$ and $\mathcal{U}_0 = \mathcal{U}|_E$. Hence if $t \in E$, we have

$$
\begin{aligned}
&\mathcal{I}^{\sigma}(\mathcal{U}(t)) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \mathcal{U}(s) ds \\
&= \frac{1}{\Gamma(\sigma)} (\int_{[0,t]\cap E} (t-s)^{\sigma-1} \mathcal{U}(s) ds + \int_{[0,t]\cap E^c} (t-s)^{\sigma-1} \mathcal{U}(s) ds) \\
&= \frac{1}{\Gamma(\sigma)} \int_{[0,t]\cap E} (t-s)^{\sigma-1} \mathcal{U}_0(s) ds \\
&= \frac{1}{\Gamma(\sigma)} (\int_{[0,t]\cap E} (t-s)^{\sigma-1} \mathcal{U}_0(s) ds + \int_{[0,t]\cap E^c} (t-s)^{\sigma-1} \mathcal{U}_0(s) ds) \\
&= \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \mathcal{U}_0(s) ds = \mathcal{I}^{\sigma}(\mathcal{U}_0(t)).
\end{aligned}
$$

If $t \in E^c|\{0\}$, then there exists $\{t_n\} \subset E$ such that $t_n \to t^-$ as $n \to \infty$, so

$$
\begin{aligned}
&\mathcal{I}^{\sigma}(\mathcal{U}(t)) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \mathcal{U}(s) ds \\
&= \lim_{n\to\infty} \frac{1}{\Gamma(\sigma)} \int_0^{t_n} (t_n-s)^{\sigma-1} \mathcal{U}(s) ds = \lim_{n\to\infty} \mathcal{I}^{\sigma}(\mathcal{U}(t_n)) \\
&= \lim_{n\to\infty} \mathcal{I}^{\sigma}(\mathcal{U}_0(t_n)) = \lim_{n\to\infty} \frac{1}{\Gamma(\sigma)} \int_0^{t_n} (t_n-s)^{\sigma-1} \mathcal{U}_0(s) ds \\
&= \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \mathcal{U}_0(s) ds = \mathcal{I}^{\sigma}(\mathcal{U}_0(t))
\end{aligned}
$$

and in the case $t = 0 \in E^c$, we have $\mathcal{I}^{\sigma}(\mathcal{U}(t)) = \mathcal{I}^{\sigma}(\mathcal{U}_0(t)) = 0$. So for all $t \in [0,1]$, $\mathcal{I}^{\sigma}(\mathcal{U}(t)) = \mathcal{I}^{\sigma}(\mathcal{U}_0(t))$. Therefore if ${}^{c}\mathcal{D}^{\sigma} w(t) + \mathcal{U}(t) = 0$ for all $t \in E$, then $\mathcal{I}^{\sigma}({}^{c}\mathcal{D}^{\sigma} w(t)) = \mathcal{I}^{\sigma}(-\mathcal{U}(t))$ for all $t \in [0,1]$, consequently $\mathcal{I}^{\sigma}({}^{c}\mathcal{D}^{\sigma} w(t)) = \mathcal{I}^{\sigma}(-\mathcal{U}_0(t))$ on $[0,1]$.

Thus, regarding Lemma (2.1) and the boundary conditions, we obtain

$$w(t) = -\frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \mathcal{U}(s) ds + \iota_1 t.$$

Putting $t = \eta$, we have

$$w(\eta) = -\frac{1}{\Gamma(\sigma)} \int_0^\eta (\eta-s)^{\sigma-1} \mathcal{U}(s) ds + \iota_1 \eta.$$

On the other hand,

$$\begin{aligned}
\int_0^1 w(s)ds &= \int_0^1 w(t)dt = -\frac{1}{\Gamma(\sigma)} \int_0^1 \int_0^t (t-s)^{\sigma-1} \mathcal{U}(s) ds dt + \frac{\iota_1}{2} \\
&= -\frac{1}{\Gamma(\sigma)} \int_0^1 \int_s^1 (t-s)^{\sigma-1} dt \mathcal{U}(s) ds + \frac{\iota_1}{2} \\
&= -\frac{1}{\Gamma(\sigma)} \int_0^1 (\frac{1}{\sigma}(t-s)^\sigma |_s^1) \mathcal{U}(s) ds + \frac{\iota_1}{2} \\
&= -\frac{1}{\Gamma(\sigma+1)} \int_0^1 (1-s)^\sigma \mathcal{U}(s) ds + \frac{\iota_1}{2}.
\end{aligned}$$

By hypothesis $w(\eta) = -\int_0^1 w(s)ds$, so we have

$$-\frac{1}{\Gamma(\sigma)} \int_0^\eta (\eta-s)^{\sigma-1} \mathcal{U}(s) ds + \iota_1 \eta = \frac{1}{\Gamma(\sigma+1)} \int_0^1 (1-s)^\sigma \mathcal{U}(s) ds - \frac{\iota_1}{2},$$

hence,

$$\iota_1 (\eta + \frac{1}{2}) = \frac{1}{\Gamma(\sigma+1)} \int_0^1 (1-s)^\sigma \mathcal{U}(s) ds + \frac{1}{\Gamma(\sigma)} \int_0^\eta (\eta-s)^{\sigma-1} \mathcal{U}(s) ds.$$

Therefore,

$$\iota_1 = \frac{2}{(2\eta+1)} (\frac{1}{\Gamma(\sigma+1)} \int_0^1 (1-s)^\sigma \mathcal{U}(s) ds + \frac{1}{\Gamma(\sigma)} \int_0^\eta (\eta-s)^{\sigma-1} \mathcal{U}(s) ds).$$

So we obtain the following equations

$$
\begin{aligned}
w(t) &= -\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\mathcal{U}(s)ds \\
&+\frac{2t}{2\eta+1}(\frac{1}{\Gamma(\sigma+1)}\int_0^1 (1-s)^{\sigma}\mathcal{U}(s)ds+\frac{1}{\Gamma(\sigma)}\int_0^\eta (\eta-s)^{\sigma-1}\mathcal{U}(s)ds) \\
&= -\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\mathcal{U}(s)ds+\frac{2t}{(2\eta+1)\Gamma(\sigma+1)}\int_0^1 (1-s)^{\sigma}\mathcal{U}(s)ds \\
&+\frac{2t}{(2\eta+1)\Gamma(\sigma)}\int_0^\eta (\eta-s)^{\sigma-1}\mathcal{U}(s)ds.
\end{aligned}
$$

If $\eta \geq t$, then

$$
\begin{aligned}
w(t) = & - \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\mathcal{U}(s)ds \\
& + \frac{2t}{(2\eta+1)\Gamma(\sigma+1)}(\int_0^t+\int_t^\eta+\int_\eta^1)(1-s)^{\sigma}\mathcal{U}(s)ds \\
& + \frac{2t}{(2\eta+1)\Gamma(\sigma)}(\int_0^t+\int_t^\eta)(\eta-s)^{\sigma-1}\mathcal{U}(s)ds.
\end{aligned}
$$

If $\eta \leq t$ then

$$
\begin{aligned}
w(t) = & - \frac{1}{\Gamma(\sigma)}(\int_0^\eta+\int_\eta^t)(t-s)^{\sigma-1}\mathcal{U}(s)ds \\
& + \frac{2t}{(2\eta+1)\Gamma(\sigma+1)}(\int_0^\eta+\int_\eta^t+\int_t^1)(1-s)^{\sigma}\mathcal{U}(s)ds \\
& + \frac{2t}{(2\eta+1)\Gamma(\sigma)}\int_0^\eta (\eta-s)^{\sigma-1}\mathcal{U}(s)ds.
\end{aligned}
$$

So $w(t) = \int_0^1 \kappa(t,s)\mathcal{U}(s)ds$ can be written, where

$$\kappa(t,s)=\begin{cases}\frac{-(t-s)^{\sigma-1}}{\Gamma(\sigma)}+\frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}+\frac{2t(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)} & 0\le s\le t\le 1,\ s\le\eta\\ \frac{-(t-s)^{\sigma-1}}{\Gamma(\sigma)}+\frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} & 0\le \eta\le s\le t\le 1\\ \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} & 0\le t\le s\le 1,\ \eta\le s\\ \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}+\frac{2t(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)} & 0\le t\le s\le\eta\le 1.\end{cases}$$

□

Lemma 3.2. *Let $\kappa(t,s)$ be given in Lemma (3.1). Then for all $t,s\in[0,1]$, $\kappa(t,s)$ has the following properties*
i) $|\kappa(t,s)|\le A_{\sigma,\eta}t(1-t)^{\sigma-1}$,
ii) $|\frac{\partial\kappa(t,s)}{\partial t}|\le A_{\sigma,\eta}(1-t)^{\alpha-1}$,
where $A_{\sigma,\eta}=\frac{2(1+\sigma)}{(2\eta+1)\Gamma(\sigma+1)}$.

Proof. i) For all $t,s\in[0,1]$ we have

$$\begin{aligned}|\kappa(t,s)| &\le \frac{2t(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}+\frac{2t(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)}\\ &= \frac{2t(1-s)^{\sigma}+2t\sigma(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma+1)}\le\frac{2t(1-s)^{\sigma}+2t\sigma(1-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma+1)}\\ &= \frac{2t(1-s)^{\sigma-1}(1-s+\sigma)}{(2\eta+1)\Gamma(\sigma+1)}\le\frac{2t(1-t)^{\sigma-1}(1+\sigma)}{(2\eta+1)\Gamma(\sigma+1)}\\ &= A_{\sigma,\eta}t(1-t)^{\sigma-1}.\end{aligned}$$

ii) By differentiating from the $\kappa(t,s)$ with respect to t, it is deduced that

$$\frac{\partial\kappa}{\partial t}(t,s)=\frac{-(\sigma-1)(t-s)^{\sigma-2}}{\Gamma(\sigma)}+\frac{2(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}+\frac{2(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)}$$

for $0\le s<t<1$ and $s\le\eta$,

$$\frac{\partial\kappa}{\partial t}(t,s)=\frac{-(\sigma-1)(t-s)^{\sigma-2}}{\Gamma(\sigma)}+\frac{2(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}$$

for $0 \le \eta \le s < t < 1$,

$$\frac{\partial \kappa}{\partial t}(t,s) = \frac{-(\sigma-1)(t-s)^{\sigma-2}}{\Gamma(\sigma)} + \frac{2(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}$$

for $0 \le \eta \le s < t < 1$,

$$\frac{\partial \kappa}{\partial t}(t,s) = \frac{2(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)}$$

for $0 < t < s \le 1$ and $\eta \le s$, and finally

$$\frac{\partial \kappa}{\partial t}(t,s) = \frac{2(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} + \frac{2(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)}$$

for $0 < t < s \le \eta \le 1$, hence

$$\begin{aligned}
|\frac{\partial \kappa(t,s)}{\partial t}| &\le \frac{2(1-s)^{\sigma}}{(2\eta+1)\Gamma(\sigma+1)} + \frac{2(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma)} \\
&= \frac{2(1-s)^{\sigma} + 2\sigma(\eta-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma+1)} \le \frac{2(1-s)^{\sigma} + 2\sigma(1-s)^{\sigma-1}}{(2\eta+1)\Gamma(\sigma+1)} \\
&= \frac{2(1-s)^{\sigma-1}(1-s+\sigma)}{(2\eta+1)\Gamma(\sigma+1)} \le \frac{2(1-t)^{\sigma-1}(1+\sigma)}{(2\eta+1)\Gamma(\sigma+1)} \\
&= A_{\sigma,\eta}(1-t)^{\sigma-1},
\end{aligned}$$

for all $t, s \in [0,1]$ that $t \neq s, t \neq 0$ and $t \neq 1$. In the case $t = s, t = 0$ or $t = 1$, the same result is obtained. □

Now, let $\mathcal{F} : X \to X$ be defined as

$$\begin{aligned}
&\mathcal{F}w(t) = \int_0^1 \kappa(t,s)\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^{\beta}w(s), \phi(w(s)))ds \\
&= -\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^{\beta}w(s), \phi(w(s)))ds \\
&+\frac{2t}{(2\eta+1)\Gamma(\sigma+1)}\int_0^1 (1-s)^{\sigma}\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^{\beta}w(s), \phi(w(s)))ds \\
&+\frac{2t}{(2\eta+1)\Gamma(\sigma)}\int_0^{\eta} (\eta-s)^{\sigma-1}\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^{\beta}w(s), \phi(w(s)))ds,
\end{aligned}$$

where $0 < \beta < 1$ and $\phi : X \to X$ is a mapping such that

$$\|\phi(w_1) - \phi(w_2)\| \leq a_0\|w_1 - w_2\| + a_1\|w_1' - w_2'\|,$$

for all $w_1, w_2 \in X$ and some $a_0, a_1 \in [0, \infty)$. By taking $l_0 = a_0 + a_1$, it can be seen that $\|\phi(w_1) - \phi(w_2)\| \leq l_0\|w_1 - w_2\|_*$, for all $w_1, w_2 \in X$. According to the defintion of Caputo derivative, for all $t \in [0, 1]$ and $w_1, w_2 \in X$ it follows

$$\begin{aligned} |{}^c\mathcal{D}^\beta w_1(t) -{}^c \mathcal{D}^\beta w_2(t)| \quad &\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} |w_1'(s) - w_2'(s)| ds \\ &\leq \frac{\|w_1' - w_2'\|}{\Gamma(2-\beta)} t^{1-\beta}, \end{aligned}$$

so

$$\|{}^c\mathcal{D}^\beta w_1 -{}^c \mathcal{D}^\beta w_2\| \leq \frac{\|w_1' - w_1'\|}{\Gamma(2-\beta)} \leq \frac{\|w_1 - w_2|_*}{\Gamma(2-\beta)}.$$

Now, we consider $\mathcal{F} : X \to X$, to prove that the pointwise problem (1) has a solution in X. For this, by lemma (3.1), we indicate that $\mathcal{F}$ has a fixed point in X. In the next results, by using some functions which are called control functions, we will control the singularity and then, investigate the existence of a sloution for the singular fractional differential problem.

Theorem 3.3. *Let $\mathcal{U} : [0, 1] \times (C[0, 1])^4 \to \mathbb{R}$ be a singular function at some points $t \in [0, 1]$ such that $\mathcal{U}(t, \mathcal{O}, \mathcal{O}, \mathcal{O}, \mathcal{O}) \in L^1[0, 1]$ where $\mathcal{O}$ is the zero function on $[0, 1]$, i.e for all $s \in [0, 1]$, $\mathcal{O}(s) = 0$. Assume that there exists a nondecreaing mapping $\Lambda : X^4 \to \mathbb{R}^+ := [0, \infty)$ such that $\frac{\Lambda(z,z,z,z)}{z} \to q_0 < \infty$ as $z \to 0^+$ and $\frac{\Lambda(z,z,z,z)}{z} \to 0$ as $z \to \infty$. If the inequality*

$$\begin{aligned} &|\mathcal{U}(t, w_1, w_2, w_3, w_4) - \mathcal{U}(t, z_1, z_2, z_3, z_4)| \\ &\leq b(t)\Lambda(w_1 - z_1, w_2 - z_2, w_3 - z_3, w_4 - z_4), \end{aligned}$$

be established for almost all $t \in [0, 1]$, all $(w_1, w_2, w_3, w_4), (z_1, z_2, z_3, z_4) \in X^4$ and some $b \in L^1[0, 1]$, then the poinwise defined problem (1) has a solution.

Proof. Let ϵ be arbitary. Regardig to the properties $\lim_{z\to 0^+}\frac{\Lambda(z,z,z,z)}{z} = q_0 < \infty$, there exists $0 < \delta(\epsilon) \leq \epsilon$ such that for all $z \in (0, \delta(\epsilon)]$, $\frac{\Lambda(z,z,z,z)}{z} < q_0 + \epsilon$, and so $\Lambda(z, z, z, z) < (q_0 + \epsilon)z$. Hence taking $z = \delta(\epsilon) := \delta$, we have

$$\Lambda(\delta, \delta, \delta, \delta) < (q_0 + \epsilon)\delta < (q_0 + \epsilon)\epsilon. \tag{2}$$

Now, let $\{w_n\}_{n\geq 1}$ be a sequence such that $w_n \to w$ in X as $n \to \infty$. So $\|w_n - w\|_* \to 0$ as $n \to \infty$. Therefore, there exists $m \in \mathbb{N}$ such that $n \geq m$ implies

$$\|w_n - w\|_* = max\{\|w_n - w\|, \|w_n' - w'\|\} < \frac{\delta}{l_1},$$

where $l_1 := max\{1, \frac{1}{\Gamma(2-\beta)}, a_0 + a_1\}$. So it is concluded that $\|w_n - w\| < \frac{\delta}{l_1}$ and $\|w_n' - w'\| < \frac{\delta}{l_1}$, for all $n \geq m$. Hence for all $t \in [0, 1]$ and $n \geq m$, we have

$$\begin{aligned}
&|\mathcal{F}w_n(t) - \mathcal{F}w(t)| \\
\leq\ & \int_0^1 |\kappa(t,s)|\Big|\mathcal{U}(s, w_n(s), w_n'(s), {}^c\mathcal{D}^\beta w_n(s), \phi(w_n(s))) \\
&-\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s)))\Big| ds \\
\leq\ & \int_0^1 A_{\sigma,\eta} t(1-t)^{\sigma-1}\Big|\mathcal{U}(s, w_n(s), w_n'(s), {}^c\mathcal{D}^\beta w_n(s), \phi(w_n(s))) \\
&-\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s)))\Big| ds \\
\leq\ & \int_0^1 A_{\sigma,\eta} t(1-t)^{\sigma-1} b(s)\Lambda((w_n - x)(s), (w_n' - w')(s), \\
&({}^c\mathcal{D}^\beta w_n - {}^c\mathcal{D}^\beta w)(s), \phi(w_n(s)) - \phi(w(s)))ds \\
\leq\ & A_{\sigma,\eta} t(1-t)^{\sigma-1} \int_0^1 b(s)\Lambda(\|w_n - w\|, \|w_n' - w'\|, \frac{\|w_n' - w'\|}{\Gamma(2-\beta)}, \\
&a_0\|w_n - w\| + a_1\|w_n' - w'\|)ds
\end{aligned}$$

$$\begin{aligned}
&\leq A_{\sigma,\eta}t(1-t)^{\sigma-1}\Lambda(\frac{\delta}{l_1},\frac{\delta}{l_1},\frac{\delta}{l_1\Gamma(2-\beta)},(a_0+a_1)\frac{\delta}{l_1})\int_0^1 b(s)ds\\
&\leq m_1A_{\sigma,\eta}t(1-t)^{\sigma-1}\Lambda(l_1\frac{\delta}{l_1},l_1\frac{\delta}{l_1},l_1\frac{\delta}{l_1},l_1\frac{\delta}{l_1})\\
&= m_1A_{\sigma,\eta}t(1-t)^{\sigma-1}\Lambda(\delta,\delta,\delta,\delta)\leq m_1A_{\sigma,\eta}t(1-t)^{\sigma-1}(q_0+\epsilon)\epsilon,
\end{aligned}$$

where $m_1=\int_0^1 b(s)ds$. So $\|\mathcal{F}w_n-\mathcal{F}_w\|\leq m_1A_{\sigma,\eta}(q_0+\epsilon)\epsilon$, for all $n\geq m$. In a similar mannner for all $t\in[0,1]$ and $n\geq m$, it is resulted that

$$\begin{aligned}
&|\mathcal{F}'w_n(t)-\mathcal{F}'w(t)|\\
\leq\ &\int_0^1|\frac{\partial\kappa(t,s)}{\partial t}|\Big|\mathcal{U}(s,w_n(s),w_n'(s),{}^c\mathcal{D}^\beta w_n(s),\phi(w_n(s)))\\
&-\mathcal{U}(s,w(s),w'(s),{}^c\mathcal{D}^\beta w(s),\phi(w(s)))\Big|ds\\
\leq\ &m_1A_{\sigma,\eta}(1-t)^{\sigma-1}(q_0+\epsilon)\epsilon.
\end{aligned}$$

Hence $\|\mathcal{F}'w_n-\mathcal{F}'w\|\leq m_1A_{\sigma,\eta}(q_0+\epsilon)\epsilon$, for all $n\geq m$. Using the above inequalities as well as $*-$norm definition, we conclude that

$$\|\mathcal{F}w_n-\mathcal{F}w\|_*=max\{\|\mathcal{F}w_n-\mathcal{F}w\|,\|F'w_n-\mathcal{F}'w\|\}\leq m_1A_{\sigma,\eta}(q_0+\epsilon)\epsilon$$

for all $n\geq m$, and since $\epsilon>0$ is arbitary, it is deduced that $\mathcal{F}w_n\to\mathcal{F}w$ in X as $w_n\to w$ in X, so $\mathcal{F}$ is a continuous mapping on X. Now, put $m_2=\int_0^1|\mathcal{U}(s,\mathcal{O},\mathcal{O},\mathcal{O},\mathcal{O})|ds$. Since $\lim_{z\to\infty}\frac{\Lambda(z,z,z,z)}{z}=0$, therefore

$$\lim_{z\to\infty}\frac{m_2+m_1\Lambda(z,z,z,z)}{z}=0.$$

So for $\epsilon>0$, there exists $r(\epsilon)>0$ such that $z\geq r(\epsilon)$ implies that

$$\frac{m_2+m_1\Lambda(z,z,z,z)}{z}<\epsilon.$$

Thus, for all $z\geq r(\epsilon)$, we have $m_2+m_1\Lambda(z,z,z,z)<\epsilon z$. Choose an $\epsilon_0>0$ such that $0<\epsilon_0<\frac{1}{A_{\sigma,\eta}l_1}$ and let $r_0:=r(\epsilon_0)$, then, for all $z\geq r_0$ the following inequality is held:

$$m_2+m_1\Lambda(z,z,z,z)<\epsilon_0 z,$$

By putting $z = r_0 l_1$, in the above inequality, we have

$$m_2 + m_1 \Lambda(r_0 l_1, r_0 l_1, r_0 l_1, r_0 l_1) < \epsilon_0 r_0 l_1 < \frac{r_0}{A_{\sigma,\eta}}.$$

Now, let $\Xi = \{w \in X : \|w\|_* < r_0\}$, $\lambda \in (0,1)$ and $w_0 \in \partial\Xi$ be such that $w_0 = \lambda \mathcal{F} w_0$, then for all $t \in [0,1]$, we have

$$\begin{aligned}
&|w_0(t)| = |\lambda \mathcal{F} w_0(t)| \leq \int_0^1 |\kappa(t,s)| \\
&\times \Big| \mathcal{U}(s, w_0(s), w_0'(s), {}^c\mathcal{D}^\beta w_0(s), \phi(w_0(s))) \Big| ds \\
&\leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \bigg(\int_0^1 \Big| \mathcal{U}(s, w_0(s), w_0'(s), {}^c\mathcal{D}^\beta w_0(s), \phi(w_0(s))) \\
&- \mathcal{U}(s, \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s)) \Big| ds \\
&+ \int_0^1 |\mathcal{U}(s, \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s))| ds \bigg) \leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \\
&\times \bigg(\int_0^1 b(s) \Lambda(x_0(s), w_0'(s), {}^c\mathcal{D}^\beta w_0(s), \phi(w_0(s))) ds + m_2 \bigg) \\
&\leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \bigg(\Lambda(\|w_0\|, \|w_0'\|, \|{}^c\mathcal{D}^\beta w_0\|, \|\phi(w_0(s))\|) \\
&\times \int_0^1 b(s) ds + m_2 \bigg) \leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \\
&\times \bigg(\Lambda(l_1 \|w_0\|_*, l_1 \|w_0\|_*, l_1 \|w_0\|_*, l_1 \|w_0\|_*) m_1 + m_2 \bigg),
\end{aligned}$$

conseqently

$$\begin{aligned}
\|w_0\| = \lambda \|\mathcal{F} w_0\| &\leq A_{\sigma,\eta} \bigg(\Lambda(l_1 r_0, l_1 r_0, l_1 r_0, l_1 r_0) m_1 + m_2 \bigg) \\
&< A_{\sigma,\eta} \frac{r_0}{A_{\sigma,\eta}} = r_0.
\end{aligned}$$

Likewise, for all $t \in [0,1]$, it is infered that

$$
\begin{aligned}
&|w_0'(t)| = |\lambda \mathcal{F}' w_0(t)| \\
&\leq \int_0^1 |\frac{\partial \kappa(t,s)}{\partial t}| \Big| \mathcal{U}(s, w_0(s), w_0'(s), {}^c\mathcal{D}^\beta w_0(s), \phi(w_0(s))) \Big| ds \\
&\leq A_{\sigma,\eta}(1-t)^{\sigma-1} \Bigg(\int_0^1 \Big| \mathcal{U}(s, w_0(s), w_0'(s), {}^c\mathcal{D}^\beta w_0(s), \phi(w_0(s))) \\
&- \mathcal{U}(s, \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s)) \Big| ds \\
&+ \int_0^1 |\mathcal{U}(s, \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s), \mathcal{O}(s))| ds \Bigg) \leq A_{\sigma,\eta}(1-t)^{\sigma-1} \\
&\times \Bigg(\int_0^1 b(s) \Lambda(w_0(s), w_0'(s), {}^c\mathcal{D}^\beta w_0(s), \phi(w_0(s))) ds + m_2 \Bigg) \\
\leq \quad & A_{\sigma,\eta}(1-t)^{\alpha-1} \Bigg(\Lambda(\|w_0\|, \|w_0'\|, \|{}^c\mathcal{D}^\beta w_0\|, \|\phi(w_0(s))\|) \\
&\times \int_0^1 b(s) ds + m_2 \Bigg) \leq A_{\sigma,\eta}(1-t)^{\sigma-1} \\
&\times \Bigg(\Lambda(l_1\|w_0\|_*, l_1\|w_0\|_*, l_1\|w_0\|_*, l_1\|w_0\|_*) m_1 + m_2 \Bigg),
\end{aligned}
$$

so

$$
\begin{aligned}
\|w_0'\| = \lambda \|\mathcal{F}' w_0\| \quad &\leq \quad A_{\sigma,\eta} \Bigg(\Lambda(l_1 r_0, l_1 r_0, l_1 r_0, l_1 r_0) m_1 + m_2 \Bigg) \\
&< \quad A_{\sigma,\eta} \frac{r_0}{A_{\sigma,\eta}} = r_0.
\end{aligned}
$$

Hence, $r_0 = \|w_0\|_* = \max\{\|w_0\|, \|w_0'\|\} < r_0$ which is a contradiction. Therefore, regarding to theorem (2.2), $\mathcal{F} : X \to X$ has a fixed point in X, so the pointwise defined fractional differential equation (1) has a solution. □

The final result is illustrated by the following example.

Example 3.4. Let $\sigma_1, ..., \sigma_n \in (0,1)$ such that $\Sigma_{i=1}^n \sigma_i < 1$, $\delta_1, ..., \delta_n \in [0,1]$,

$$
d(t) = \frac{1}{(t-\delta_1)^{\sigma_1}(t-\delta_2)^{\sigma_2} ... (t-\delta_n)^{\sigma_n}},
$$

$$c(t) = \begin{cases} 0 & t \in [0,1] \cap Q \\ \\ 1 & t \in (0,1) \cap Q^c. \end{cases}$$

$b(t) = \frac{1}{c(t)}$ and

$$\mathcal{U}(t, w_1, w_2, w_3, w_4) = b(t)(\Sigma_{i=1}^4 \frac{|w_i|}{1+|w_i|}) + d(t).$$

Consider the pointwise defined equation

$$^c\mathcal{D}^{\sqrt{11}} w(t) + \mathcal{U}(t, w(t), w'(t), ^c\mathcal{D}^{\frac{2}{3}} w(t), \int_0^t w(s)ds) = 0 \tag{3}$$

with boundary condition $w(0) = w''(0) = 0$ and $w(\eta) + \int_0^1 w(s)ds = 0$, in which $\eta \in (0,1)$ is fixed. Then, for all $(w_1, w_2, w_3, w_4), (z_1, z_2, z_3, z_4) \in X^4$ and almost $t \in [0,1]$ we have

$$\begin{aligned} &\Big|\mathcal{U}(t, w_1, w_2, w_3, w_4) - \mathcal{U}(t, z_1, z_2, z_3, z_4)\Big| \\ &= b(t)\Big|\Sigma_{i=1}^4 (\frac{|w_i|}{1+|w_i|} - \frac{|z_i|}{1+|z_i|})\Big| \leq b(t)\Sigma_{i=1}^4 \frac{|w_i - z_i|}{1+|w_i - z_i|} \\ &= b(t)\Lambda(w_1 - z_1, w_2 - z_2, w_3 - z_3, w_4 - z_4), \end{aligned}$$

where

$$\Lambda(z_1, z_2, z_3, z_4) = \Sigma_{i=1}^4 \frac{|z_i|}{1+|z_i|}.$$

Simply speaking, $\lim_{z \to 0^+} \frac{\Lambda(z,z,z,z)}{z} = 4 < \infty$, $\lim_{z \to \infty} \frac{\Lambda(z,z,z,z)}{z} = 0$ and $b(t) \in L^1[0,1]$. Note that if $\phi(w(t)) = \int_0^t w(s)ds$, then

$$|\phi(w(t)) - \phi(z(t))| \leq \int_0^t |w(s) - z(s)|ds \leq \|w - z\|t,$$

for all $t \in [0,1]$, so $\|\phi(w) - \phi(z)\| \leq \|w - z\|$. Therefore all the conditions of Theorem (3.3) are held, so by therem (3.3), the pointwisedefined equation (3) has a solution.

Now, we want to consider two pointwise defined differential equaions

$$ {}^{c}\mathcal{D}^{\sigma} w(t) + \mathcal{U}(t, w(t), w'(t), {}^{c}\mathcal{D}^{\beta} w(t), \phi(w(t))) = 0 \tag{4} $$

and

$$ {}^{c}\mathcal{D}^{\sigma} z(t) + \mathcal{V}(t, z(t), z'(t), {}^{c}\mathcal{D}^{\gamma} z(t), \phi(z(t))) = 0, \tag{5} $$

when $\sigma \geq 2$, $\gamma, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that for all $w_1, w_2 \in X$, $\|\phi(w_1) - \phi(w_2)\| \leq a_0\|w_1 - w_2\| + a_1\|w_1' - w_2'\|$, for some $a_0, a_1, \in [0,\infty)$ and $\mathcal{U}, \mathcal{V} : [0,1] \times X^4 \to \mathbb{R}$ are two fuctions that are singular at some set with measure zero, under boundary conditions $w(0) = z(0) = 0$ for $\sigma \in [2,3)$ and

$$ w(0) = w''(0) = w^{(n_0)}(0) = z(0) = z''(0) = z^{(n_0)}(0) = 0 $$

where $n_0 = [\sigma] + 1$ for $\sigma \in [3,\infty)$ and also $w(\eta) + \int_0^1 w(s)ds = z(\eta) + \int_0^1 z(s)ds = 0$. We will show that under some conditions, these two equations have the same solution.

For this, we define $\mathcal{F}, \mathcal{S} : X \to X$ as

$$ \mathcal{F}w(t) = \int_0^1 \kappa(t,s)\mathcal{U}(s, w(s), w'(s), {}^{c}\mathcal{D}^{\beta} w(s), \phi(w(s)))ds $$

and

$$ \mathcal{S}z(t) = \int_0^1 \kappa(t,s)\mathcal{V}(s, z(s), z'(s), {}^{c}\mathcal{D}^{\gamma} z(s), \phi(z(s)))ds $$

where $\kappa(t,s)$ is the Green function that defined by lemma (3.1). We will prove that $\mathcal{F}$ and $\mathcal{S}$ has a common fixed point, so two equations (4) and (5) have a same solution.

Theorem 3.5. *Let $\mathcal{U}, \mathcal{V} : [0,1] \times X^4 \to \mathbb{R}$ are continuous on $E \subset X$ with $m(E^c) = 0$ and there exist $b, \theta \in L^1[0,1]$, nondecreasing mapping $\Lambda : X^4 \to \mathbb{R}$ such that*

$$ \lim_{\|z_i\|\to 0} \frac{|\mathcal{V}(t, z_1, z_2, z_3, z_4)|}{\|z_i\|} \leq \theta(t) $$

and $|\mathcal{U}(t, w_1, w_2, w_3, xw_4)| \leq b(t)\Lambda(w_1, w_2, w_3, w_4)$ for all $(w_1, w_2, w_3, w_4) \in X^4$, $1 \leq i \leq 4$ and almost all $t \in [0,1]$. Also let

$$ \lim_{z\to 0^+} \frac{\Lambda(z, z, z, z)}{z} = q_0, $$

$m_1 := \int_0^1 b(s)ds < \frac{1}{A_{\sigma,\eta}}$ and $m_2 := \int_0^1 \theta(s)ds < \frac{1}{l_2 A_{\sigma,\eta}}$, where $l_1 = \max\{1, \frac{1}{\Gamma(2-\beta)}, a_0+a_1\}$, $l_2 = \max\{1, \frac{1}{\Gamma(2-\gamma)}, a_0+a_1\}$ and $q_0 \in [0, \frac{1}{l_1})$. If for all $(w_1, w_2, w_3, w_4), (z_1, z_2, z_3, z_4) \in X^4$ that $(w_1, w_2, w_3, w_4) \neq (z_1, z_2, z_3, z_4)$, almost all $t \in [0,1]$ and all $1 \leq i \leq 4$

$$\lim_{(\|w_i\|,\|z_i\|)\to(0^+,0^+)} \frac{\mathcal{U}(t, w_1, w_2, w_3, w_4) - \mathcal{V}(t, z_1, z_2, z_3, z_4)}{max\|w_i - z_i\|} = 0,$$

then the pointwise defined equations (4) and (5) have a common solution.

Proof. Since

$$\lim_{z\to 0^+} \frac{\Lambda(z,z,z,z)}{z} = q_0,$$

so for each $\epsilon > 0$, there exists $0 < \delta(\epsilon) \leq \epsilon$ such that $z \in (0, \delta(\epsilon)]$ implies that

$$\frac{\Lambda(z,z,z,z)}{z} < q_0 + \epsilon,$$

therefore

$$\Lambda(z,z,z,z) < (q_0 + \epsilon)z.$$

Let $\epsilon_1 > 0$ be such that $q_0 + \epsilon_1 < \frac{1}{l_1}$, then for all $z \in (0, \delta_1 := \delta(\epsilon_1)]$ it is concluded that

$$\Lambda(z,z,z,z) < (q_0 + \epsilon_1)z,$$

consequently

$$\Lambda(l_1 z, l_1 z, l_1 z, l_1 z) < (q_0 + \epsilon_1) l_1 z < z,$$

for all $z \in (0, \frac{\delta_1}{l_1}]$. On the other hand for all $w \in X$ and $t \in [0,1]$, we have

$$\begin{aligned}
|\mathcal{F}w(t)| &\leq \int_0^1 |\kappa(t,s)| \Big| \mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s))) \Big| ds \\
&\leq \int_0^1 A_{\sigma,\eta} t(1-t)^{\sigma-1} b(s) \Lambda(w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s))) ds \\
&\leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \int_0^1 b(s) \Lambda(\|w\|, \|w'\|, \|{}^c\mathcal{D}^\beta w\|, \|\phi(w)\|) ds \\
&\leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \Lambda(\|w\|, \|w'\|, \frac{\|w'\|}{\Gamma(2-\beta)}, a_0\|w\| + a_1\|w'\|) \int_0^1 b(s) ds \\
&\leq A_{\sigma,\eta} t(1-t)^{\sigma-1} \Lambda(l_1\|w\|_*, l_1\|w\|_*, l_1\|w\|_*, l_1\|w\|_*) m_1.
\end{aligned}$$

So, if $\|w\|_* \in (0, \frac{\delta_1}{l_1}]$, then

$$|\mathcal{F}w(t)| \leq A_{\sigma,\eta} t(1-t)^{\sigma-1}\|w\|_* m_1 \leq \|w\|_* t(1-t)^{\sigma-1}$$

thus, it is resulted that $\|\mathcal{F}w\| \leq \|w\|_*$. Also we have

$$\begin{aligned}
&|\mathcal{F}'w(t)| \leq \int_0^1 |\frac{\partial \kappa(t,s)}{\partial t}||\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s)))|ds \\
&\leq \int_0^1 A_{\sigma,\eta}(1-t)^{\sigma-1} b(s)\Lambda(w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s)))ds \\
&\leq A_{\sigma,\eta}(1-t)^{\sigma-1} \int_0^1 b(s)\Lambda(\|w\|, \|w'\|, \|{}^c\mathcal{D}^\beta w\|, \|\phi(w)\|)ds \\
&\leq A_{\sigma,\eta}(1-t)^{\sigma-1}\Lambda(\|w\|, \|w'\|, \frac{\|w'\|}{\Gamma(2-\beta)}, a_0\|w\| + a_1\|w'\|) \int_0^1 b(s)ds \\
&\leq A_{\sigma,\eta}(1-t)^{\sigma-1}\Lambda(l_1\|w\|_*, l_1\|w\|_*, l_1\|w\|_*, l_1\|w\|_*)m_1.
\end{aligned}$$

Therefore, if $\|w\|_* \in (0, \frac{\delta_1}{l_1}]$, then

$$|\mathcal{F}'w(t)| \leq A_{\sigma,\eta}(1-t)^{\sigma-1}\|w\|_* m_1 \leq \|w\|_*(1-t)^{\sigma-1},$$

so, we conclude that $\|\mathcal{F}'w\| \leq \|w\|_*$. Hence if $\|w\|_* \in (0, \frac{\delta_1}{l_1}]$ then

$$\|\mathcal{F}w\|_* = \max\{\|\mathcal{F}w\|, \|\mathcal{F}'w\|\} \leq \|w\|_*. \tag{6}$$

By the assumptions, for all $1 \leq i \leq 4$ and almost all $t \in [0, 1]$,

$$\lim_{\|z_i\| \to 0} \frac{|\mathcal{V}(t, z_1, z_2, z_3, z_4)|}{\|z_i\|} \leq \theta(t),$$

so, for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, such that $\|z_i\| \in (0, \delta(\epsilon)]$ implies

$$|\mathcal{V}(t, z_1, z_2, z_3, z_4)| \leq (\theta(t) + \epsilon)\|z_i\|.$$

Thus, for $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $l_2\|z\| \in (0, \delta(\epsilon)]$, it follows

$$\begin{aligned}
|\mathcal{V}(t, z, z', {}^c\mathcal{D}^\gamma z, \phi(z))| &\leq (\theta(t) + \epsilon)\max\{\|z\|, \|z'\|, \|{}^c\mathcal{D}^\gamma z\|, \|\phi(z)\|\|\} \\
&\leq (\theta(t) + \epsilon)l_2\|z\|_*.
\end{aligned}$$

Since $m_2 < \frac{1}{l_2 A_{\sigma,\eta}}$, there exists $\epsilon_2 > 0$ such that $m_2 + \epsilon_2 < \frac{1}{l_2 A_{\sigma,\eta}}$. Put $\delta_2 := \delta(\epsilon_2)$, so if $\|z\| \in (0, \frac{\delta_2}{l_2}]$, then we have

$$|\mathcal{V}(t, z, z', {}^c\mathcal{D}^\gamma z, \phi(z))| \leq (\theta(t) + \epsilon_2) l_2 \|z\|_*.$$

Thus, for $z \in X$ in which $\|z\| \in (0, \frac{\delta_2}{l_2}]$, we conclude that

$$\begin{aligned}
|\mathcal{S}z(t)| &\leq \int_0^1 |\kappa(t,s)||\mathcal{V}(s, z(s), z'(s), {}^c\mathcal{D}^\gamma z(s), \phi(z(s)))|ds \\
&\leq \int_0^1 A_{\sigma,\eta} t(1-t)^{\alpha-1}(\theta(s) + \epsilon_2) l_2 \|z\|_* ds \\
&= t(1-t)^{\sigma-1} A_{\sigma,\eta}(\int_0^1 \theta(s)ds + \epsilon_2) l_2 \|z\|_* \\
&= t(1-t)^{\sigma-1} A_{\sigma,\eta}(m_2 + \epsilon_2) l_2 \|z\|_* \\
&\leq t(1-t)^{\sigma-1} \|z\|_*,
\end{aligned}$$

so $\|\mathcal{S}z\| \leq \|z\|_*$. Also for all $t \in [0,1]$ and $z \in X$ in which $\|z\| \in (0, \frac{\delta_2}{l_2}]$, we have

$$\begin{aligned}
|\mathcal{S}'z(t)| &\leq \int_0^1 |\frac{\partial \kappa(t,s)}{\partial t}||\mathcal{V}(s, z(s), z'(s), {}^c\mathcal{D}^\gamma z(s), \phi(z(s)))|ds \\
&\leq \int_0^1 A_{\sigma,\eta}(1-t)^{\alpha-1}(\theta(s) + \epsilon_2) l_2 \|z\|_* ds \\
&= (1-t)^{\sigma-1} A_{\sigma,\eta}(\int_0^1 \theta(s)ds + \epsilon_2) l_2 \|z\|_* \\
&= (1-t)^{\sigma-1} A_{\sigma,\eta}(m_2 + \epsilon_2) l_2 \|z\|_* \\
&\leq (1-t)^{\sigma-1} \|z\|_*,
\end{aligned}$$

so $\|\mathcal{S}'z\| \leq \|z\|_*$. Therefore,

$$\|\mathcal{S}z\|_* = \max\{\|\mathcal{S}z\|, \|\mathcal{S}'z\|\} \leq \|z\|_*. \tag{7}$$

Likewise, through the given assumptions for almost all $t \in [0,1]$, we have

$$\lim_{(\|w_i\|,\|z_i\|)\to(0^+,0^+)} \frac{\mathcal{U}(t, w_1, w_2, w_3, w_4) - \mathcal{V}(t, z_1, z_2, z_3, z_4)}{max\|w_i - z_i\|} = 0.$$

Put $\|w_k - z_k\| := max_{1\leq j\leq 4}\|w_i - z_i\|$ for some $1 \leq k \leq 4$, then for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|w_i\|, \|z_i\| \in (0, \delta]$ implies

$$|\mathcal{U}(t, w_1, w_2, w_3, w_4) - \mathcal{V}(t, z_1, z_2, z_3, z_4)| < \epsilon\|w_k - z_k\|.$$

Let $0 < \epsilon_3 < \frac{1}{A_{\sigma,\eta}}$ and $\delta_3 := \delta(\epsilon_3)$, then if $\|w\|, \|z\| \in (0, \frac{\delta_3}{l_3}]$, we have

$$\begin{aligned}
&|\mathcal{U}(t, w, w', {}^c\mathcal{D}^\beta w, \phi(w)) - \mathcal{V}(t, z, z', {}^c\mathcal{D}^\gamma z, \phi(z))| \\
&< \epsilon_3 \max\{\|w - z\|, \|w' - z'\|, \|{}^c\mathcal{D}^\beta w - {}^c\mathcal{D}^\gamma z\|, \|\phi(w) - \phi(z)\|\} \\
&\leq \epsilon_3 l_3 \|w - z\|_*,
\end{aligned}$$

where $l_3 = \max\{l_1, l_2, |\frac{1}{\Gamma(2-\beta)} - \frac{1}{\Gamma(2-\gamma)}|\} = \max\{l_1, l_2\}$. So if $\|w\|, \|z\| \in (0, \delta_3]$, then

$$|\mathcal{U}(t, w, w', {}^c\mathcal{D}^\beta w, \phi(w)) - \mathcal{V}(t, z, z', {}^c\mathcal{D}^\gamma z, \phi(z))| \leq \epsilon_3\|w - z\|_*. \tag{8}$$

Now, let $\delta_M = \min\{\frac{\delta_1}{l_1}, \frac{\delta_2}{l_2}, \delta_3\}$, define $\alpha : X^2 \to [0, \infty)$ as

$$\alpha(x, y) = \begin{cases} 1 & \|w\|_*, \|z\|_* \in (0, \delta_M] \\ 0 & other\ wise \end{cases}$$

and $\psi : \mathbb{R} \to \mathbb{R}$ as $\psi(t) = \epsilon_3 A_{\sigma,\eta} t$. So, $\psi \in \Psi$ is obviuos. If $\alpha(w, z) \geq 1$ then $\|w\|_*, \|z\|_* \in (0, \delta_M]$, so by (7), $\|\mathcal{S}w\|_* \leq \|x\|_* \leq \delta_M$. Likewise, via (6), $\|\mathcal{F}y\|_* \leq \|y\|_* \leq \delta_M$, so $\alpha(\mathcal{S}w, \mathcal{F}z) \geq 1$. If $w \in X$ be such that $\|w\|_* \leq \delta_M$, then $\|\mathcal{S}w\|_* \leq \delta_M$, so it is concluded that there exists $w_0 \in X$ such that $\alpha(w_0, \mathcal{S}w_0) \geq 1$. To check the continuity $\mathcal{F}$, let $E \subset [0, 1]$ be a set which $\mathcal{U}(t, ., ., ., .)$ is not continuous on that, then $m(E) = 0$ where m is the Lebesgue measure in $\mathbb{R}$, and let $w_n \to w$ as

$n \to \infty$. So for all $t \in [0,1]$ we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{F}w_n(t) &= \lim_{n\to\infty} \int_0^1 \kappa(t,s)\mathcal{U}(s, w_n(s), w_n'(s), {}^c\mathcal{D}^\beta w_n(s), \phi(w_n(s)))ds \\
&= \lim_{n\to\infty} \int_{E^c} \kappa(t,s)\mathcal{U}(s, w_n(s), w_n'(s), {}^c\mathcal{D}^\beta w_n(s), \phi(w_n(s)))ds \\
&+ \lim_{n\to\infty} \int_{E} \kappa(t,s)\mathcal{U}(s, w_n(s), w_n'(s), {}^c\mathcal{D}^\beta w_n(s), \phi(w_n(s)))ds \\
&= \int_{E^c} \kappa(t,s)\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s)))ds \\
&= \int_0^1 \kappa(t,s)\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s)))ds \\
&= \mathcal{F}w(t).
\end{aligned}$$

Similarly, $\lim_{n\to\infty} \mathcal{F}'w_n(t) = \mathcal{F}'w(t)$ is obtained for all $t \in [0,1]$, so it is concluded that $\mathcal{F}$ is a continuous mapping in $(X, \|.\|_*)$. On the other hand, for all $t \in [0,1]$ we deduce that

$$\begin{aligned}
|\mathcal{F}w(t) - \mathcal{S}z(t)| \le \int_0^1 |\kappa(t,s)| \Big| &\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s))) \\
&-\mathcal{V}(s, z(s), z'(s), {}^c\mathcal{D}^\beta z(s), \phi(z(s)))\Big| ds \\
\le \quad A_{\sigma,\eta} t(1-t)^{\sigma-1} \int_0^1 \Big| &\mathcal{U}(s, w(s), w'(s), {}^c\mathcal{D}^\beta w(s), \phi(w(s))) \\
&-\mathcal{V}(s, z(s), z'(s), {}^c\mathcal{D}^\beta z(s), \phi(z(s)))\Big| ds.
\end{aligned}$$

Therefore, when $\|w\|_*, \|z\|_* \in (0, \delta_M]$, by (8), it implies that

$$|\mathcal{F}w(t) - \mathcal{S}z(t)| \quad \le \quad A_{\sigma,\eta} t(1-t)^{\sigma-1}\epsilon_3 \|w - z\|_*,$$

consequently

$$\|\mathcal{F}w - \mathcal{S}z\| \quad \le \quad A_{\sigma,\eta}\epsilon_3 \|x - y\|_* = \psi(\|w - z\|_*).$$

In a similar manner, we have

$$\|\mathcal{F}'w - \mathcal{S}'z\| \quad \le \quad A_{\sigma,\eta}\epsilon_3 \|w - z\|_* = \psi(\|w - z\|_*),$$

hence

$$\|\mathcal{F}w - \mathcal{S}z\|_* = \max\{\|\mathcal{F}'w - \mathcal{S}'z\|, \|\mathcal{F}w - \mathcal{S}z\|\} \leq \psi(\|w - z\|_*).$$

Therefore, regarding Lemma (2.3), both equations (4) and (5) have a common solution. □

Example 3.6. Consider the following pointwise defined equations

$$^c\mathcal{D}^{\frac{5}{2}}w(t) + \frac{0.5}{p(t)}(\|w(t)\|^2 + \|w'(t)\|^2 + \|^c\mathcal{D}^{\frac{1}{2}}w(t)\|^2 + \|\int_0^t w(s)ds\|^2) = 0$$

and

$$^c\mathcal{D}^{\frac{5}{2}}z(t) + \frac{0.3}{\sqrt{t}}(\|z(t)\| + \|z'(t)\| + \|^c\mathcal{D}^{\frac{1}{3}}z(t)\| + \|\int_0^t z(s)ds\|) = 0$$

with boundary conditions $w(0) = z(0) = 0$ and $w(\frac{1}{2}) + \int_0^1 w(s)ds = z(\frac{1}{2}) + \int_0^1 z(s)ds = 0$, where

$$p(t) = \begin{cases} 1 & t \in [0,1]|\{\delta_1, ..., \delta_k\} \\ 0 & t \in \{\delta_1, ..., \delta_k\}. \end{cases}$$

Put $\Lambda(w_1, w_2, w_3, w_4) = \Sigma_{i=1}^4 \|w_i\|^2$, $\phi(w(t)) = \int_0^t w(s)ds$, $b(t) = \frac{0.5}{p(t)}$,

$$\mathcal{U}(t, w_1, w_2, w_3, w_4) = \Lambda(w_1, w_2, w_3, w_4),$$

$\theta(t) = \frac{0.3}{\sqrt{t}}$ and

$$\mathcal{V}(t, z_1, z_2, z_3, z_4) = \theta(t)\Sigma_{i=1}^4 \|z_i\|,$$

then $\|\phi(w) - \phi(z)\| \leq \|w - z\|$, $l_1 = \max\{1, \frac{1}{\Gamma(2-\frac{1}{2})}\} = \frac{2}{\sqrt{\pi}}$, $l_2 = \max\{1, \frac{1}{\Gamma(2-\frac{1}{3})}\} = \frac{1}{\Gamma(\frac{5}{3})}$, $q_0 = \lim_{z\to 0^+} \frac{\Lambda(z,z,z,z)}{z} = 0 < \frac{1}{l_1}$,

$$A_{\sigma,\eta} = \frac{2(1 + \frac{5}{2})}{(1+1)\Gamma(\frac{5}{2} + 1)} = \frac{28}{15\sqrt{\pi}},$$

$b, \theta \in L^1[0,1]$, $m_1 = \int_0^1 b(s)ds = 0.5 < \frac{1}{A_{\sigma,\eta}}$, $m_2 = \int_0^1 \theta(s)ds = 0.6 < \frac{1}{l_2 A_{\sigma,\eta}}$ and for all $(w_1, w_2, w_3, w_4), (z_1, z_2, z_3, z_4) \in X^4$ that $(w_1, w_2, w_3, w_4) \neq (z_1, z_2, z_3, z_4)$, almost all $t \in [0,1]$ and all $1 \le i \le 4$

$$\begin{aligned}
&\lim_{(\|w_i\|,\|z_i\|)\to(0^+,0^+)} \frac{|\mathcal{U}(t, w_1, w_2, w_3, w_4) - \mathcal{V}(t, z_1, z_2, z_3, z_4)|}{max\|w_i - z_i\|} \\
\le\ & |b(t) - \theta(t)| \lim_{(\|w_i\|,\|z_i\|)\to(0^+,0^+)} \frac{\Sigma_{i=1}^4 |\|w_i\|^2 - \|z_i\||}{max\|x_i - z_i\|} \\
\le\ & |b(t) - \theta(t)| \lim_{(\|x_i\|,\|z_i\|)\to(0^+,0^+)} \frac{\Sigma_{i=1}^4 |\|w_i\|^2 - \|w_i\|\|z_i\||}{max\|w_i - z_i\|} \\
=\ & |b(t) - \theta(t)| \lim_{(\|w_i\|,\|z_i\|)\to(0^+,0^+)} \frac{\Sigma_{i=1}^4 |\|w_i\|(\|w_i\| - \|z_i\|)|}{max\|x_i - y_i\|} \\
\le\ & |b(t) - \theta(t)| \lim_{(\|w_i\|,\|z_i\|)\to(0^+,0^+)} \frac{\Sigma_{i=1}^4 \|w_i\|\|w_i - z_i\|}{max\|w_i - z_i\|} \\
\le\ & |b(t) - \theta(t)| \lim_{\|w_i\|\to 0^+} \Sigma_{i=1}^4 \|w_i\| = 0.
\end{aligned}$$

Hence, based on Theorem (3.5) there is a common solution for both mentioned equations.

Corollary 3.7. *Let $\mathcal{U} : [0,1] \times X^4 \to \mathbb{R}$ be continuous on set $E \in X$ with $m(E^c) = 0$, there exists $b \in L^1[0,1]$ and nondecreasing mapping $\Lambda : X^4 \to \mathbb{R}$ such that $|\mathcal{U}(t, w_1, w_2, w_3, w_4)| \le b(t)\Lambda(w_1, w_2, w_3, w_4)$ for all $(w_1, w_2, w_3, w_4) \in X^4$ and almost all $t \in [0,1]$, also let*

$$\lim_{z\to 0^+} \frac{\Lambda(z, z, z, z)}{z} = q_0,$$

$m_1 := \int_0^1 b(s)ds < \frac{1}{A_{\sigma,\eta}}$, where $l_1 = \max\{1, \frac{1}{\Gamma(2-\beta)}, a_0 + a_1\}$ and $q_0 \in [0, \frac{1}{l_1})$. Then, the pointwise defined equation (4) has a solution.

Proof. In theorem (3.5), let for all $t \in [0,1]$ and $(w_1, w_2, w_3, w_4) \in X^4$,

$$\mathcal{V}(t, w_1, w_2, w_3, w_4) = \mathcal{U}(t, w_1, w_2, w_3, w_4).$$

Indicating all conditions of Theorem (3.5) is feasible. Therefore, the pointwise defined equation (4) has a solution.

□

4 Conclusion

Investigating of a solution for fractional differential equations has a specific importance, among which the singular ones have a significant role. In this paper, we consider a solution for a singular differential equation, then allocate some conditions to prove the existence of a common solution for two singular differential equations. Used new methods in this article, can help to examine other fractional differential equations.

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