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$\mathbb{D}-$Topological Duals of Bicomplex $\mathbb{BC}-$Modules $l_p^{\mathbb{k}}(\mathbb{BC})$

N. Değirmen
Ondokuz Mayıs University

B. Sağır*
Ondokuz Mayıs University

Abstract. In this paper, we present the concept of $\mathbb{D}-$topological dual of a $\mathbb{BC}-$module X with a $\mathbb{D}-$norm $\|.\|_{\mathbb{D},X}$ and we find $\mathbb{D}-$topological duals of $\mathbb{D}-$normed bicomplex $\mathbb{BC}-$modules $l_p^{\mathbb{k}}(\mathbb{BC})$ for $1 \leq p < \infty$. We also investigate some fundamental topological properties of bicomplex $\mathbb{BC}-$modules $l_p^{\mathbb{k}}(\mathbb{BC})$ by defining some topological concepts such as solid space, monotone space, $BK-$space, symmetric space in the bicomplex setting by using the $\mathbb{D}-$norm $|.|_{\mathbb{k}}$.

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Keywords and Phrases: $\mathbb{D}-$topological dual, bicomplex $\mathbb{BC}-$module, $\mathbb{D}-$solid space, $\mathbb{D}-$monotone space, $\mathbb{D}-$normed $BK-$space, $\mathbb{D}-$symmetric space

1 Introduction and Preliminaries

Continuous linear functionals from a Banach space to $\mathbb{C}$, the set of complex numbers, are a power tool in functional analysis and in the modern

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*Corresponding Author

theory of partial differential equations. This gives rise to the notion of the topological dual space. Dual spaces are studied by many mathematicians in a long period of time not only in many branches of mathematics that use linear spaces but also in mathematical physics with wide range of applications to many interesting problems in various directions. So, the studying of such spaces gains importance both in theoretical studies and in application areas.

Dual spaces are used to describe measures, distributions, Hilbert spaces, multipliers, composition operators, weak topologies, seminorms and compact operators.

On the other hand, in many models of physical systems such as quantum mechanic, statistical mechanics, quantum field theory, and so forth, the possible states of the system in question can be associated with linear functionals on appropriate Banach spaces.

In this regard, the two main purposes of researchers about topological duals are to determine the dual spaces of particular Banach spaces and to prove general theorems that give relationship between Banach spaces and their topological duals.

Duals of function spaces have been studied for years and have been used to find their multipliers. For example, one of the studies is [29] in which the multipliers of weighted Lebesgue spaces. In addition, sequence spaces play a key role in various fields of mathematics. The most popular sequence spaces are the spaces l_p which consist of absolutely $p-$summable complex sequences and they have a lot of useful applications. After then, in the light of all, the necessity of knowing the duals of these spaces in order to work on multipliers of sequence spaces has started similar studies in sequence spaces. For some of these works we refer the reader to [15, 34, 4, 3].

On the other hand, since sequence spaces have rich topological and geometric properties, researchers are motivated to use them and to obtain a new view and applications in different sequence spaces. In the literature, there are many results [10, 16, 9, 17, 35, 22, 12, 14], where discussed the topological properties of various sequence spaces.

In search for development of special algebras, Corrado Segre [30] introduced the concept of bicomplex numbers in 1892. In 1991, Price [24] laid the foundations for properties of bicomplex numbers, bicomplex function theory and bicomplex holomorphic functions. After that time, Alpay et al. [1] gave an extensive survey of functional analysis with bicomplex scalars and also discussed many new ideas. Besides, the references [5], [20], [18] and [19] contain bicomplex versions of some known function spaces, operators and related theorems. For other significant studies on bicomplex analysis, see [25, 21]. Our first work on bicomplex analysis is [26] in which we studied bicomplex sequence spaces with Euclidean norm in the set of bicomplex numbers.

So far, there have been great many attempts by topologists to extend topological spaces. For several related works, see [2, 33, 8, 23, 6, 31, 32].

Grossman and Katz [13] introduced non-Newtonian calculus as an alternative to classical calculus. A generator is a one-to-one function whose domain is $\mathbb{R}$, the set of all real numbers, and whose range is a subset of $\mathbb{R}$. The range of generator α is denoted by $\mathbb{R}(N)$ and is called non-Newtonian real line. If we take the identity function I instead of the generator α , then non-Newtonian real numbers turn into the real numbers. Therefore, the results obtained with respect to non-Newtonian calculus are stronger than those of classical calculus. So, numerous researchers have made some generalizations and extensions in various aspects on non-Newtonian calculus. As one of main articles on this topic, Duyar et al. [11] carried some well-known notions and topological results for real line to non-Newtonian real line.

In order to obtain more general results, Sager and Sağır [27] defined the quasi-Banach algebra $\mathbb{BC}(N)$ by defining non-Newtonian bicomplex numbers as a generalization of both bicomplex numbers and non-Newtonian complex numbers and also, examined the validity of the non-Newtonian bicomplex version of the well-known Hölder's and Minkowski's inequalities for sums. Hereupon, Sager and Sağır [28] constructed vector spaces $l_p(\mathbb{BC}(N))$ of absolutely $p-$summable $*-$bicomplex sequences over the field $\mathbb{C}(N)$ and showed that these vector spaces are Banach spaces with the $*-$norm $\ddot{\|}\,.\,\ddot{\|}_{2,l_p(\mathbb{BC}(N))}$ by using Minkowski's inequality in $\mathbb{BC}(N)$ with respect to $\ddot{\|}\,.\,\ddot{\|}_2$ obtained in [27].

As the foundation of this paper, Değirmen and Sağır [7] established bicomplex $\mathbb{BC}-$modules $l_p^{\Bbbk}(\mathbb{BC})$ and examined some geometric properties in bicomplex setting with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D},\cdot}$. We hope that the reference [7] will become a starting point for deeper developments of it and many applications.

This paper is organized as follows: First, we summarize a number of known results on the algebra of bicomplex numbers, which will be needed in other sections. In the next section, we give some inclusion relations related to bicomplex $\mathbb{BC}-$modules $l_p^{\Bbbk}(\mathbb{BC})$. Then, we define the concept of $\mathbb{D}-$topological dual of a $\mathbb{BC}-$module X with a hyperbolic valued norm $\|.\|_{\mathbb{D},X}$ and investigate $\mathbb{D}-$topological duals of bicomplex $\mathbb{BC}-$modules $l_p^{\Bbbk}(\mathbb{BC})$ for $1 \leq p < \infty$. Finally, we show that bicomplex $\mathbb{BC}-$modules $l_p^{\Bbbk}(\mathbb{BC})$ have some properties such as being $\mathbb{D}-$solid space, $\mathbb{D}-$monotone space, $\mathbb{D}-$normed $BK-$space, $\mathbb{D}-$symmetric space which are also defined in this work. Comparing with works with respect to some real valued norms in the literature, we believe that our results in the paper creates a new and different field of study in functional analysis with bicomplex scalars.

Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ forms a Banach algebra and a $\mathbb{BC}-$module with respect to the addition, scalar multiplication, multiplication and Euclidean norm defined as

$$\begin{aligned} z + w &= (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2), \\ \lambda.z &= \lambda.(z_1 + jz_2) = \lambda z_1 + j\lambda z_2, \\ z \times w &= zw = (z_1 + jz_2)(w_1 + jw_2) = (z_1w_1 - z_2w_2) + j(z_1w_2 + z_2w_1), \\ |.| &: \mathbb{BC} \to \mathbb{R}, z \to |z| = \sqrt{|z_1|^2 + |z_2|^2} \end{aligned}$$

for all $z = z_1 + jz_2, w = w_1 + jw_2 \in \mathbb{BC}$, $\lambda \in \mathbb{R}$.

The set of hyperbolic numbers $\mathbb{D}$ is defined by $\mathbb{D} = \{x + \Bbbk y : x, y \in \mathbb{R}, \Bbbk = ij\}$.

Let $z = z_1 + jz_2$ be any bicomplex number in $\mathbb{BC}$. There are three types of conjugates in $\mathbb{BC}$ as follows:

$$z^{\dagger_1} = \overline{z_1} + j\overline{z_2}, \quad z^{\dagger_2} = z_1 - jz_2, \quad z^{\dagger_3} = \overline{z_1} - j\overline{z_2},$$

where $\overline{z_1}, \overline{z_2}$ are the complex conjugates of $z_1, z_2 \in \mathbb{C}(i)$. Also, we know three types moduli as follows:

$$\begin{aligned} |z|_i^2 &= zz^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(i), \\ |z|_j^2 &= zz^{\dagger_1} = \left(|z_1|^2 - |z_2|^2\right) + j\left(2\Re\left(z_1.\overline{z_2}\right)\right) \in \mathbb{C}(j), \\ |z|_{\mathbb{k}}^2 &= zz^{\dagger_3} = \left(|z_1|^2 + |z_2|^2\right) + \mathbb{k}\left(-\Im\left(z_1.\overline{z_2}\right)\right) \in \mathbb{D}. \end{aligned}$$

We say that z is invertible if $|z|_i \neq 0$, that is, $z_1^2 + z_2^2 \neq 0$ and its inverse is given by $z^{-1} = \frac{z^{\dagger_2}}{|z|_i^2}$.

The numbers $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ form idempotent basis of bicomplex numbers and hence any bicomplex number $z = z_1 + jz_2$ is uniquely written as $z = e_1\beta_1 + e_2\beta_2$ where $\beta_1 = z_1 - iz_2$, $\beta_2 = z_1 + iz_2 \in \mathbb{C}(i)$. This formula is called the idempotent representation of z.

Let $z_n = z_{1n}e_1 + z_{2n}e_2 \in \mathbb{BC}$ for all $n \in \mathbb{N}$. Then, (z_n) converges if and only if (z_{1n}) and (z_{2n}) converge and also, $\lim\limits_{n\to\infty} z_n = \left(\lim\limits_{n\to\infty} z_{1n}\right)e_1 + \left(\lim\limits_{n\to\infty} z_{2n}\right)e_2$.

Let $z_n = z_{1n}e_1 + z_{2n}e_2 \in \mathbb{BC}$ for all $n \in \mathbb{N}$. Then, $\sum\limits_{n=1}^{\infty} z_n$ converges if and only if $\sum\limits_{n=1}^{\infty} z_{1n}$ and $\sum\limits_{n=1}^{\infty} z_{2n}$ converge and also, $\sum\limits_{n=1}^{\infty} z_n = \left(\sum\limits_{n=1}^{\infty} z_{1n}\right)e_1 + \left(\sum\limits_{n=1}^{\infty} z_{2n}\right)e_2$ [24].

Let $\alpha = x + \mathbb{k}y$ be any hyperbolic number. Then, we have the equality $\alpha = e_1\alpha_1 + e_2\alpha_2$,where $\alpha_1 = x + y, \alpha_2 = x - y \in \mathbb{R}$. If $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, then α is called a positive hyperbolic number. Therefore, the set of positive hyperbolic numbers is denoted by $\mathbb{D}^+$, that is, $\mathbb{D}^+ = \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}$. For two hyperbolic numbers α and β; if their difference $\beta - \alpha \in \mathbb{D}^+$ (or $\beta - \alpha \in \mathbb{D}^+ - \{0\}$), then we write $\alpha \precsim \beta$ (or $\alpha \precnsim \beta$). For $\alpha = e_1\alpha_1 + e_2\alpha_2$, $\beta = \beta_1 e_1 + \beta_2 e_2 \in \mathbb{D}$ with real numbers $\alpha_1, \alpha_2, \beta_1$ and β_2, we have that

$$\begin{aligned} \alpha &\precsim \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2, \\ \alpha &\precnsim \beta \text{ if and only if } \alpha \neq \beta,\ \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2, \\ \alpha &\prec \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2. \end{aligned}$$

This relation $\precsim$ is reflexive, anti - symmetric, transitive and so defines a partial order on $\mathbb{D}$.

The following statements are true for any $\alpha, \beta, \gamma, \delta \in \mathbb{D}, z, w \in \mathbb{BC}$:

(i) $|z+w|_{\Bbbk} \precsim |z|_{\Bbbk} + |w|_{\Bbbk}$, $|zw|_{\Bbbk} = |z|_{\Bbbk}\, |w|_{\Bbbk}$ and $\left|\frac{z}{w}\right|_{\Bbbk} = \frac{|z|_{\Bbbk}}{|w|_{\Bbbk}}$ where w is invertible.

(ii) If $\alpha \in \mathbb{D}^{+}$, then $|\alpha|_{\Bbbk} = \alpha$ and $|\alpha z|_{\Bbbk} = \alpha\, |z|_{\Bbbk}$

(iii) $|z|_{\Bbbk} = |\beta_1|\, e_1 + |\beta_2|\, e_2$ for $z = \beta_1 e_1 +\ \beta_2 e_2$.

(iv) If $\alpha \precsim \beta$ and $\gamma \precsim \delta$, then $\alpha + \gamma \precsim \beta + \delta$.

(v) If $\alpha \precsim \beta$ and $0 \precsim \gamma$, then $\alpha\gamma \precsim \beta\gamma$.

(vi) If $\alpha \precsim \beta$ and $\beta \precsim \gamma$, then $\alpha \precsim \gamma$.

Let X be a $\mathbb{BC}-$module. A map $\|.\|_{\mathbb{D}} : X \to \mathbb{D}^{+}$ is said to be a hyperbolic valued norm or $\mathbb{D}-$norm on X if it satisfies the following properties:

(i) $\|x\|_{\mathbb{D}} = 0$ if and only if $x = 0$.

(ii) $\|\mu x\|_{\mathbb{D}} = |\mu|_{\Bbbk} . \|x\|_{\mathbb{D}}$ for all $x \in X, \mu \in \mathbb{BC}$.

(iii) $\|x+y\|_{\mathbb{D}} \precsim \|x\|_{\mathbb{D}} + \|y\|_{\mathbb{D}}$ for all $x, y \in X$.

A sequence (x_n) in X converges to $x_0 \in X$ with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$ if for every $0 \precnsim \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_0\|_{\mathbb{D}} \precnsim \varepsilon$ for all $n \geq n_0$.

A sequence (x_n) in X is Cauchy sequence with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$ if for every $0 \precnsim \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\|_{\mathbb{D}} \precnsim \varepsilon$ for any $n, m \geq n_0$.

If every Cauchy sequence in X with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$ converges to $x_0 \in X$ with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$, then we say that X is complete with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$.

$|.|_{\Bbbk} : \mathbb{BC} \to \mathbb{D}^{+}$ is the hyperbolic valued norm on the $\mathbb{BC}-$module $\mathbb{BC}$ and also, $\mathbb{BC}$ is complete with respect to the hyperbolic valued norm $|.|_{\Bbbk}$.

Let $A \subset \mathbb{D}$. If A has a $\mathbb{D}-$upper or a $\mathbb{D}-$lower bound α, then this means that for any $a \in A$ there holds that a is comparable with α and $a \precsim \alpha$ or $\alpha \precsim a$. If A is a set $\mathbb{D}-$bounded from above, we define the notion of its $\mathbb{D}-$supremum, denoted by $\sup_{\mathbb{D}} A$, to be the least $\mathbb{D}-$upper bound for A, and its $\mathbb{D}-$infimum $\inf_{\mathbb{D}} A$ to be the greatest $\mathbb{D}-$lower bound for A. The least $\mathbb{D}-$upper bound here means that $\sup_{\mathbb{D}} A \precsim \alpha$ for any $\mathbb{D}-$upper bound α even if not all of the $\mathbb{D}-$upper bounds are comparable. The least $\mathbb{D}-$lower bound here means that

$\alpha \precsim \inf_{\mathbb{D}} A$ for any $\mathbb{D}-$lower bound α even if not all of the $\mathbb{D}-$lower bounds are comparable. Consider the sets $A_1 = \{a_1 : a_1e_1 + a_2e_2 \in A\}$ and
$A_2 = \{a_2 : a_1e_1 + a_2e_2 \in A\}$. If A is $\mathbb{D}-$bounded from above, then $\sup_{\mathbb{D}} A = \sup A_1e_1 + \sup A_2e_2$. If A is $\mathbb{D}-$bounded from below, then $\inf_{\mathbb{D}} A = \inf A_1e_1 + \inf A_2e_2$. Also, the followings hold:

(i) If A and B are $\mathbb{D}-$bounded from below, then so is $A+B$ and $\inf_{\mathbb{D}}(A+B) = \inf_{\mathbb{D}} A + \inf_{\mathbb{D}} B$.

(ii) If A and B are $\mathbb{D}-$bounded from above, then so is $A+B$ and $\sup_{\mathbb{D}}(A+B) = \sup_{\mathbb{D}} A + \sup_{\mathbb{D}} B$.

(iii) If $A, B \subset \mathbb{D}^+$ and A and B are $\mathbb{D}-$bounded from below, then so is $A.B$ and $\inf_{\mathbb{D}}(A.B) = \inf_{\mathbb{D}} A . \inf_{\mathbb{D}} B$.

(iv) If $A, B \subset \mathbb{D}^+$ and A and B are $\mathbb{D}-$bounded from above, then so is $A.B$ and $\sup_{\mathbb{D}}(A.B) = \sup_{\mathbb{D}} A . \sup_{\mathbb{D}} B$ [20].

A bicomplex algebra A is a module over $\mathbb{BC}$ with a multiplication defined on it which satisfy the following properties:

(i) $x(y+z) = xy + xz$

(ii) $(x+y)z = xz + yz$

(iii) $x(yz) = (xy)z$

(iv) $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for all $x, y, z \in A$ and all scalars $\lambda \in \mathbb{BC}$. Also, a bicomplex algebra A with a $\mathbb{D}-$valued norm $\|.\|_{\mathbb{D}}$ relative to which A is a Banach module and such that $\|xy\|_{\mathbb{D}} \precsim \|x\|_{\mathbb{D}} \|y\|_{\mathbb{D}}$ for every x, y in A is called a $\mathbb{D}-$normed bicomplex Banach algebra [19]. For example, $\mathbb{BC}$ is a $\mathbb{D}-$normed bicomplex Banach algebra with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$.

Let X and Y be two $\mathbb{BC}-$modules. A map $T : X \to Y$ is said to be a linear operator if for any $x, y \in X$ and for any $\lambda \in \mathbb{BC}$ it is $T(\lambda x + y) = \lambda T(x) + T(y)$.

Let $T : X \to Y$ be a $\mathbb{BC}-$linear operator. Assume that X and Y have $\mathbb{D}-$norms $\|.\|_{\mathbb{D},X}$ and $\|.\|_{\mathbb{D},Y}$ respectively. The operator $T : X \to Y$ is called $\mathbb{D}-$bounded if there exists $\Lambda \in \mathbb{D}^+$ such that for any $x \in X$ one has $\|Tx\|_{\mathbb{D},Y} \precsim \Lambda \|x\|_{\mathbb{D},X}$. The $\mathbb{D}-$infimum of these Λ is called the norm of the operator T. It is clear that the set of $\mathbb{D}-$bounded $\mathbb{BC}-$linear operators is a $\mathbb{BC}-$module. The map $T \to \inf_{\mathbb{D}} \left\{ \Lambda \mid \text{for any } x \in X,\ \|Tx\|_{\mathbb{D},Y} \precsim \Lambda \|x\|_{\mathbb{D},X} \right\} : \|T\|_{\mathbb{D}}$ defines a hyperbolic valued norm on the $\mathbb{BC}-$module of all bounded operators.

Also, $\|T\|_{\mathbb{D}} = \sup_{\mathbb{D}} \left\{ \frac{\|Tw\|_{\mathbb{D},Y}}{\|w\|_{\mathbb{D},X}} | \left| \|w\|_{\mathbb{D},X} \right|_i \neq 0 \right\}$ [1].

We will denote the set of all $\mathbb{D}-$bounded $\mathbb{BC}-$linear operators from X to Y by $B_{\mathbb{BC}}(X, Y)$.

The following definitions and results are given in [7].

Let $z = z_1 e_1 + z_2 e_2, \alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathbb{BC}$ and $z \neq 0$. Then, we have $z^{\alpha} = z_1^{\alpha_1} e_1 + z_2^{\alpha_2} e_2$.

Let p and q be real numbers with $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $s_m, t_m \in \mathbb{BC}$ for $m \in \{1, 2, ..., n\}$. Then, we have

$$\sum_{m=1}^{n} |s_m t_m|_{\mathbb{k}} \precsim \left(\sum_{m=1}^{n} |s_m|_{\mathbb{k}}^{p} \right)^{\frac{1}{p}} \left(\sum_{m=1}^{n} |t_m|_{\mathbb{k}}^{q} \right)^{\frac{1}{q}}.$$

Let A be a bicomplex algebra and M be a module over $\mathbb{BC}$. M is said to be a left (right) bicomplex $A-$module if a mapping $A \times M \to M, (a, m) \to am$ ($M \times A \to M, (m, a) \to ma$) is specified which satisfies the following axioms:

(i) For each fixed $a \in A$, the mapping $m \to am$ ($m \to ma$) is linear on M;

(ii) For each fixed $m \in M$, the mapping $a \to am$ ($a \to ma$) is linear on A;

(iii) $a_1 (a_2 m) = (a_1 a_2) m$ ($(m a_1) a_2 = m (a_1 a_2)$) for all $a_1, a_2 \in A$ and $m \in M$.

M is said to be a bicomplex $A-$module if it is both a left bicomplex $A-$module and a right bicomplex $A-$module. The specified mapping $(a, m) \to am$ is called bicomplex module multiplication. Let A be a $D-$normed bicomplex algebra and M be a hyperbolic valued normed space with respect to hyperbolic valued norm $\|.\|_{\mathbb{D},M}$ on M. Then, M is said to be a $D-$normed left (right) bicomplex $A-$module if M is a left (right) bicomplex $A-$module and there exists a nonzero positive hyperbolic constant K such that $\|am\|_{\mathbb{D},M} \precsim K \|a\|_{\mathbb{D},A} \|m\|_{\mathbb{D},M}$ $\left(\|ma\|_{\mathbb{D},M} \precsim K \|m\|_{\mathbb{D},M} \|a\|_{\mathbb{D},A} \right)$ for all $a \in A, m \in M$. If M is both $\mathbb{D}-$normed left bicomplex $A-$module and $\mathbb{D}-$normed right bicomplex $A-$module, then M is called a $\mathbb{D}-$normed bicomplex $A-$module. A $\mathbb{D}-$normed left (right) bicomplex $A-$module is called a $\mathbb{D}-$normed Banach left (right) bicomplex $A-$module if it is complete as a $\mathbb{D}-$normed linear space. If M is both $\mathbb{D}-$normed Banach left bicomplex $A-$module and $\mathbb{D}-$normed Banach right bicomplex $A-$module, then M is called a $\mathbb{D}-$normed Banach bicomplex $A-$module.

Let X be a $\mathbb{BC}-$module. A map $|||.|||_{\mathbb{D}} : X \to \mathbb{D}^{+}$ is said to be a hyperbolic valued $p-$norm or $p_{\mathbb{D}}-$norm for $0 < p \leq 1$ if it satisfies the following properties:

(i) $|||x|||_{\mathbb{D}} = 0$ if and only if $x = 0$.

(ii) $|||\mu x|||_{\mathbb{D}} = |\mu|_{\mathbb{k}}^{p} \, |||x|||_{\mathbb{D}}$ for all $x \in X, \mu \in \mathbb{BC}$.

(iii) $|||x + y|||_{\mathbb{D}} \precsim |||x|||_{\mathbb{D}} + |||y|||_{\mathbb{D}}$ for all $x, y \in X$.

Then, X is said to be a hyperbolic valued $p-$normed space or $p_{\mathbb{D}}-$normed space.

A sequence (x_n) in X converges to $x_0 \in X$ with respect to the $p_{\mathbb{D}}-$norm $|||.|||_{\mathbb{D}}$ if for every $0 \precnsim \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that $|||x_n - x_0|||_{\mathbb{D}} \precnsim \varepsilon$ for all $n \geq n_0$.

A sequence (x_n) in X is Cauchy sequence with respect to the $p_{\mathbb{D}}-$norm $|||.|||_{\mathbb{D}}$ if for every $0 \precnsim \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that $|||x_n - x_m|||_{\mathbb{D}} \precnsim \varepsilon$ for any $n, m \geq n_0$.

If every Cauchy sequence in X with respect to the $p_{\mathbb{D}}-$norm $|||.|||_{\mathbb{D}}$ converges to $x_0 \in X$ with respect to the $p_{\mathbb{D}}-$norm $|||.|||_{\mathbb{D}}$, then we say that X is complete with respect to the $p_{\mathbb{D}}-$norm $|||.|||_{\mathbb{D}}$ or $p_{\mathbb{D}}-$normed Banach space.

Let A be a $\mathbb{D}-$normed bicomplex algebra and M be a $p_{\mathbb{D}}-$normed space with respect to $p_{\mathbb{D}}-$norm $|||.|||_{\mathbb{D},M}$ on M. Then, M is said to be a $p_{\mathbb{D}}-$normed left (right) bicomplex $A-$module if M is a left (right) bicomplex $A-$module and there exists a nonzero positive hyperbolic constant K such that $|||am|||_{\mathbb{D},M} \precsim K \, \|a\|_{\mathbb{D},A}^{p} \, |||m|||_{\mathbb{D},M}$ $\left(|||ma|||_{\mathbb{D},M} \precsim K \, |||m|||_{\mathbb{D},M} \, \|a\|_{\mathbb{D},A}^{p}\right)$ for all $a \in A, m \in M$. If M is both $p_{\mathbb{D}}-$normed left bicomplex $A-$module and $p_{\mathbb{D}}-$normed right bicomplex $A-$module, then M is called a $p_{\mathbb{D}}-$normed bicomplex $A-$module. A $p_{\mathbb{D}}-$normed left (right) bicomplex $A-$module is called a $p_{\mathbb{D}}-$normed Banach left (right) bicomplex $A-$module if it is complete as a $p_{\mathbb{D}}-$normed space. If M is both $p_{\mathbb{D}}-$normed Banach left bicomplex $A-$module and $p_{\mathbb{D}}-$normed Banach right bicomplex $A-$module, then M is called a $p_{\mathbb{D}}-$normed Banach bicomplex $A-$module.

The sets $l_p^{\mathbb{k}}(\mathbb{BC})$ for $0 < p < \infty$ and $l_\infty^{\mathbb{k}}(\mathbb{BC})$ are defined by using hyperbolic valued norm $|.|_{\Bbbk}$ as follows:

$$\begin{aligned} l_{\infty}^{\Bbbk}(\mathbb{BC}) &:= \left\{ \zeta = (\zeta_n) \in w(\mathbb{BC}) : \sup_{\mathbb{D}} \{ |\zeta_n|_{\Bbbk} : n \in \mathbb{N} \} \text{ is finite} \right\}, \\ l_{p}^{\Bbbk}(\mathbb{BC}) &:= \left\{ \zeta = (\zeta_n) \in w(\mathbb{BC}) : \sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^{p} \text{ converges} \right\} \text{for } 0 < p < \infty \end{aligned}$$

which are new sequence spaces as bicomplex sequence space with respect to the hyperbolic valued norm $|.|_{\Bbbk}$. Note that $\zeta = (\zeta_n) = (\zeta_{n1}e_1 + \zeta_{n2}e_2) \in l_{\infty}^{\Bbbk}(\mathbb{BC})$ means that $\sup\{|\zeta_{n1}| : n \in \mathbb{N}\} < \infty$ and $\sup\{|\zeta_{n2}| : n \in \mathbb{N}\} < \infty$. Also, $\zeta = (\zeta_n) = (\zeta_{n1}e_1 + \zeta_{n2}e_2) \in l_{p}^{\Bbbk}(\mathbb{BC})$ means that the series $\sum\limits_{n=1}^{\infty} |\zeta_{n1}|^p$ and $\sum\limits_{n=1}^{\infty} |\zeta_{n2}|^p$ converge.

$l_{\infty}^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D},l_{\infty}^{\Bbbk}(\mathbb{BC})}$ defined by $\|s\|_{\mathbb{D},l_{\infty}^{\Bbbk}(\mathbb{BC})} = \sup_{\mathbb{D}}\{|s_n|_{\Bbbk} : n \in \mathbb{N}\}$ for all $s = (s_n) \in l_{\infty}^{\Bbbk}(\mathbb{BC})$.

For $0 < p < 1$, $l_{p}^{\Bbbk}(\mathbb{BC})$ is a $p_{\mathbb{D}}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the $p_{\mathbb{D}}-$norm $\|.\|_{\mathbb{D},l_{p}^{\Bbbk}(\mathbb{BC})}$ defined by $\|s\|_{\mathbb{D},l_{p}^{\Bbbk}(\mathbb{BC})} = \sum\limits_{n=1}^{\infty} |s_n|_{\Bbbk}^{p}$ for all $s = (s_n) \in l_{p}^{\Bbbk}(\mathbb{BC})$.

For $1 \leq p < \infty$, $l_{p}^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D},l_{p}^{\Bbbk}(\mathbb{BC})}$ defined by $\|s\|_{\mathbb{D},l_{p}^{\Bbbk}(\mathbb{BC})} = \left(\sum\limits_{n=1}^{\infty} |s_n|_{\Bbbk}^{p}\right)^{\frac{1}{p}}$ for all $s = (s_n) \in l_{p}^{\Bbbk}(\mathbb{BC})$.

2 Inclusion Reations For Bicomplex $\mathbb{BC}-$ Modules $l_{p}^{\Bbbk}(\mathbb{BC})$

In this part, we deal with some inclusion relations related to bicomplex $\mathbb{BC}-$modules $l_{p}^{\Bbbk}(\mathbb{BC})$.

Theorem 2.1. *For $0 < p < q < \infty$, we have the inclusion $l_{p}^{\Bbbk}(\mathbb{BC}) \subset l_{q}^{\Bbbk}(\mathbb{BC})$. Also, this inclusion strictly holds, where $1 \leq p < q < \infty$.*

Proof. Assume that $\zeta = (\zeta_n) \in l_p^{\Bbbk}(\mathbb{BC})$. Then, the series $\sum\limits_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^p$ converges. Therefore, we conclude that there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|\zeta_n|_{\Bbbk} \precnsim 1$ for all $n \geq n_0$ and so $|\zeta_n|_{\Bbbk}^{q-p} \precnsim 1$ for all $n \geq n_0$ since $0 < q - p$.

Let $M = \sup_{\mathbb{D}} \left\{ |\zeta_1|_{\Bbbk}^{q-p}, |\zeta_2|_{\Bbbk}^{q-p}, ..., |\zeta_{n_0}|_{\Bbbk}^{q-p}, 1 \right\}$. Thus, since

$$\sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^q = \sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^{q-p} |\zeta_n|_{\Bbbk}^p \precsim M \sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^p,$$

we obtain that $l_p^{\Bbbk}(\mathbb{BC}) \subset l_q^{\Bbbk}(\mathbb{BC})$.

We now want to show that the inclusion is strict for $1 \leq p < q < \infty$. Consider the sequence $\zeta = (\zeta_n)$ defined by $\zeta_n = \frac{1}{n^{\frac{1}{p}}} e_1 + \frac{i}{n^{\frac{1}{p}}} e_2$ for all $n \in \mathbb{N}$. Then, we get

$$\begin{aligned} \sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^q &= \sum_{n=1}^{\infty} \left| \frac{1}{n^{\frac{1}{p}}} e_1 + \frac{i}{n^{\frac{1}{p}}} e_2 \right|_{\Bbbk}^q \\ &= \sum_{n=1}^{\infty} \left(\left| \frac{1}{n^{\frac{1}{p}}} \right| e_1 + \left| \frac{i}{n^{\frac{1}{p}}} \right| e_2 \right)^q \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{q}{p}}} \right) e_1 + \left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{q}{p}}} \right) e_2. \end{aligned}$$

Also, since $\frac{q}{p} > 1$, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{q}{p}}}$ converges which implies that $\sum\limits_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^q$ converges and hence, $\zeta = (\zeta_n) \in l_p^{\Bbbk}(\mathbb{BC})$.

On the other hand, observe that

$$\sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^p = \sum_{n=1}^{\infty} \left| \frac{1}{n^{\frac{1}{p}}} e_1 + \frac{i}{n^{\frac{1}{p}}} e_2 \right|_{\Bbbk}^p = \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) e_1 + \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) e_2.$$

Hence, since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ doesn't converge, $\sum\limits_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^p$ doesn't converge. This shows that $\zeta = (\zeta_n) \notin l_p^{\Bbbk}(\mathbb{BC})$. Then, $l_p^{\Bbbk}(\mathbb{BC}) \subset l_q^{\Bbbk}(\mathbb{BC})$ is a strict inclusion for $1 \leq p < q < \infty$. This completes the proof. $\square$

Theorem 2.2. *For $0 < p < \infty$, we have the inclusion $l_p^{\Bbbk}(\mathbb{BC}) \subset l_\infty^{\Bbbk}(\mathbb{BC})$. Also, this inclusion strictly holds, where $1 \leq p < \infty$.*

Proof. Assume that $\zeta = (\zeta_n) \in l_p^{\Bbbk}(\mathbb{BC})$. Then, the series $\sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^p$ converges. Therefore, we conclude that there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $|\zeta_n|_{\Bbbk} \precnsim 1$ for all $n \geq n_0$.

Let $M = \sup_{\mathbb{D}} \{|\zeta_1|_{\Bbbk}, |\zeta_2|_{\Bbbk}, ..., |\zeta_{n_0}|_{\Bbbk}, 1\}$. Thus, since

$$\sup_{\mathbb{D}} \{|\zeta_n|_{\Bbbk} : n \in \mathbb{N}\} \precsim \sup_{\mathbb{D}} \{|\zeta_1|_{\Bbbk}, |\zeta_2|_{\Bbbk}, ..., |\zeta_{n_0}|_{\Bbbk}, 1\} = M$$

we obtain that $l_p^{\Bbbk}(\mathbb{BC}) \subset l_\infty^{\Bbbk}(\mathbb{BC})$.

Now, our goal is to show that there exists a sequence such that it is in $l_\infty^{\Bbbk}(\mathbb{BC})$ but not in $l_p^{\Bbbk}(\mathbb{BC})$ for $1 \leq p < \infty$. For this purpose, consider the sequence $\zeta = (\zeta_n)$ defined by $\zeta_n = \frac{i}{n^{\frac{1}{p}}} e_1 + \frac{j}{n^{\frac{1}{p}}} e_2$ for all $n \in \mathbb{N}$. Then, we get

$$\begin{aligned}
\sup_{\mathbb{D}} \{|\zeta_n|_{\Bbbk} : n \in \mathbb{N}\} &= \sup_{\mathbb{D}} \left\{ \left| \frac{i}{n^{\frac{1}{p}}} e_1 + \frac{j}{n^{\frac{1}{p}}} e_2 \right|_{\Bbbk} : n \in \mathbb{N} \right\} \\
&= \sup_{\mathbb{D}} \left\{ \left| \frac{i}{n^{\frac{1}{p}}} \right| e_1 + \left| \frac{j}{n^{\frac{1}{p}}} \right| e_2 : n \in \mathbb{N} \right\} \\
&= \sup_{\mathbb{D}} \left\{ \frac{1}{n^{\frac{1}{p}}} e_1 + \frac{1}{n^{\frac{1}{p}}} e_2 : n \in \mathbb{N} \right\} \\
&= \sup \left\{ \frac{1}{n^{\frac{1}{p}}} : n \in \mathbb{N} \right\} e_1 + \sup \left\{ \frac{1}{n^{\frac{1}{p}}} : n \in \mathbb{N} \right\} e_2.
\end{aligned}$$

Also, we know that $\sup \left\{ \frac{1}{n^{\frac{1}{p}}} : n \in \mathbb{N} \right\} < \infty$. So, $\zeta = (\zeta_n) \in l_\infty^{\Bbbk}(\mathbb{BC})$.

On the other hand, we have

$$\sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk}^p = \sum_{n=1}^{\infty} \left| \frac{i}{n^{\frac{1}{p}}} e_1 + \frac{j}{n^{\frac{1}{p}}} e_2 \right|_{\Bbbk}^p = \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) e_1 + \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) e_2,$$

and this implies that $\zeta = (\zeta_n) \notin l_p^{\Bbbk}(\mathbb{BC})$. The proof is completed. □

3 Some Topological Properties of Bicomplex $\mathbb{BC}-$Modules $l_p^{\mathbb{k}}(\mathbb{BC})$

In this part, our aim is to evince that bicomplex $\mathbb{BC}-$modules $l_p^{\mathbb{k}}(\mathbb{BC})$ have some topological properties which are defined in this section.

Let's give bicomplex versions of solid space, monotone space, $BK-$space, symmetric space by using the hyperbolic valued norm $|.|_{\mathbb{k}}$.

Definition 3.1. Let X be a bicomplex sequence space with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$ and

$$\widetilde{X} := \left\{ \begin{array}{c} (s_n) \in w(\mathbb{BC}) : \text{there exists } (t_n) \in X \\ \text{such that } |s_n|_{\mathbb{k}} \precsim |t_n|_{\mathbb{k}} \text{ for all } n \in \mathbb{N} \end{array} \right\}.$$

If $\widetilde{X} \subset X$, then X is called a $\mathbb{D}-$solid space or $\mathbb{D}-$normal space.

Definition 3.2. Let X be a bicomplex sequence space with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$, $A := \{(s_n) \in w(\mathbb{BC}) : s_n \in \{0,1\} \text{ for all } n \in \mathbb{N}\}$ and $M_0 = sp\{A\}$. If $M_0 X \subset X$, then X is called a $\mathbb{D}-$monotone space.

Definition 3.3. Let X be a bicomplex sequence space with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$ and a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D},X}$. If $\zeta_l^{(n)} \to \zeta_l$ as $n \to \infty$ for all $l \in \mathbb{N}$ with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$ whenever $\zeta^{(n)} \to \zeta$ as $n \to \infty$ with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D},X}$, then X is called a $\mathbb{D}-$normed $BK-$space.

Definition 3.4. Let X be a bicomplex sequence space with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$ and $\pi := \{f : \mathbb{N} \to \mathbb{N} \text{ is one to one and onto}\}$. If $s_\sigma = \left(s_{\sigma(n)}\right) \in X$ whenever $(s_n) \in X$ and $\sigma \in \pi$, then X is called a $\mathbb{D}-$symmetric space.

The next result is the property of being $\mathbb{D}-$solid space of bicomplex $\mathbb{BC}-$module $l_\infty^{\mathbb{k}}(\mathbb{BC})$.

Theorem 3.5. $l_\infty^{\mathbb{k}}(\mathbb{BC})$ *is a* $\mathbb{D}-$*solid space.*

Proof. Let $(s_n) \in \overset{\sim}{l_\infty^{\Bbbk}(\mathbb{BC})}$ be arbitrary. Then, there exists $(t_n) \in l_\infty^{\Bbbk}(\mathbb{BC})$ such that $|s_n|_{\Bbbk} \precsim |t_n|_{\Bbbk}$ for all $n \in \mathbb{N}$. Therefore, $\sup_{\mathbb{D}}\{|t_n|_{\Bbbk} : n \in \mathbb{N}\}$ is finite, and so, $\sup_{\mathbb{D}}\{|s_n|_{\Bbbk} : n \in \mathbb{N}\}$ is finite. This shows that $(s_n) \in l_\infty^{\Bbbk}(\mathbb{BC})$. Then, we have the inclusion $\overset{\sim}{l_\infty^{\Bbbk}(\mathbb{BC})} \subset l_\infty^{\Bbbk}(\mathbb{BC})$ which means that $l_\infty^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$solid space. □

The following theorem gives the property of being $\mathbb{D}-$monotone space of bicomplex $\mathbb{BC}-$module $l_\infty^{\Bbbk}(\mathbb{BC})$.

Theorem 3.6. $l_\infty^{\Bbbk}(\mathbb{BC})$ *is a* $\mathbb{D}-$*monotone space.*

Proof. Let $(\zeta_n) \in M_0 l_\infty^{\Bbbk}(\mathbb{BC})$ be arbitrary. Then, there exist $(s_n) \in M_0$ and $(t_n) \in l_\infty^{\Bbbk}(\mathbb{BC})$ such that $(\zeta_n) = (s_n t_n)$. Therefore, $\sup_{\mathbb{D}}\{|t_n|_{\Bbbk} : n \in \mathbb{N}\}$ is finite and since $\{s_n : n \in \mathbb{N}\}$ is finite, we have $\sup_{\mathbb{D}}\{|s_n|_{\Bbbk} : n \in \mathbb{N}\}$ is finite. Then, since

$$\begin{aligned} \sup_{\mathbb{D}}\{|s_n t_n|_{\Bbbk} : n \in \mathbb{N}\} &= \sup_{\mathbb{D}}\{|s_n|_{\Bbbk}|t_n|_{\Bbbk} : n \in \mathbb{N}\} \\ &= \sup_{\mathbb{D}}\{|s_n|_{\Bbbk} : n \in \mathbb{N}\}\sup_{\mathbb{D}}\{|t_n|_{\Bbbk} : n \in \mathbb{N}\}, \end{aligned}$$

we say that $\sup_{\mathbb{D}}\{|s_n t_n|_{\Bbbk} : n \in \mathbb{N}\}$ is finite. This implies that $(\zeta_n) \in l_\infty^{\Bbbk}(\mathbb{BC})$. Then, we have the inclusion $M_0 l_\infty^{\Bbbk}(\mathbb{BC}) \subset l_\infty^{\Bbbk}(\mathbb{BC})$ which means that $l_\infty^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$monotone space. □

The next theorem shows that bicomplex $\mathbb{BC}-$module $l_\infty^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$normed $BK-$space.

Theorem 3.7. $l_\infty^{\Bbbk}(\mathbb{BC})$ *is a* $\mathbb{D}-$*normed* $BK-$*space.*

Proof. Let $\left(\zeta^{(n)}\right) \in l_\infty^{\Bbbk}(\mathbb{BC})$ such that $\zeta^{(n)} \to \zeta$ as $n \to \infty$ with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}, l_\infty^{\Bbbk}(\mathbb{BC})}$. Then, for every $0 \precnsim \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that $\left\|\zeta^{(n)} - \zeta\right\|_{\mathbb{D}, l_\infty^{\Bbbk}(\mathbb{BC})} \precnsim \varepsilon$ for all $n \geq n_0$. Thus, we have that $\sup_{\mathbb{D}}\left\{\left|\zeta_l^{(n)} - \zeta_l\right|_{\Bbbk} : l \in \mathbb{N}\right\} \precnsim \varepsilon$ for every $0 \precnsim \varepsilon$ and for all $n \geq n_0$. So, for any fixed $l \in \mathbb{N}$ we write $\left|\zeta_l^{(n)} - \zeta_l\right|_{\Bbbk} \precnsim \varepsilon$ for every $0 \precnsim \varepsilon$ and for all $n \geq n_0$ which means that $\left(\zeta_l^{(n)}\right)$ converges to the bicomplex number ζ_l with respect to the hyperbolic valued norm $|.|_{\Bbbk}$. Thus, the coordinates are continuous on $l_\infty^{\Bbbk}(\mathbb{BC})$ and we get the required result. □

The following result states the property of being $\mathbb{D}-$symmetric space of bicomplex $\mathbb{BC}-$module $l_{\infty}^{\Bbbk}(\mathbb{BC})$.

Theorem 3.8. *$l_{\infty}^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$symmetric space.*

Proof. Let $(s_n) \in l_{\infty}^{\Bbbk}(\mathbb{BC})$ and $\sigma \in \pi$. Then, since $\sigma : \mathbb{N} \to \mathbb{N}$ is a injective and surjective function, we have $\left\{ \left| s_{\sigma(n)} \right|_{\Bbbk} : n \in \mathbb{N} \right\} = \{ |s_n|_{\Bbbk} : n \in \mathbb{N} \}$. Thus, the equality $\sup_{\mathbb{D}} \left\{ \left| s_{\sigma(n)} \right|_{\Bbbk} : n \in \mathbb{N} \right\} = \sup_{\mathbb{D}} \{ |s_n|_{\Bbbk} : n \in \mathbb{N} \}$ holds. Since $\sup_{\mathbb{D}} \{ |s_n|_{\Bbbk} : n \in \mathbb{N} \}$ is finite, we have that $\sup_{\mathbb{D}} \left\{ \left| s_{\sigma(n)} \right|_{\Bbbk} : n \in \mathbb{N} \right\}$ is finite. This means that $\left(s_{\sigma(n)} \right) \in l_{\infty}^{\Bbbk}(\mathbb{BC})$, as required. □

The following theorems in the rest of this section of the paper are the analogues of the above Theorem 3.5, Theorem 3.6, Theorem 3.7 and Theorem 3.8 in bicomplex $\mathbb{BC}-$modules $l_{p}^{\Bbbk}(\mathbb{BC})$.

Theorem 3.9. *$l_{p}^{\Bbbk}(\mathbb{BC})$ for $0 < p < \infty$ is a $\mathbb{D}-$solid space.*

Proof. Let $(s_n) \in l_{p}^{\Bbbk}\widetilde{(\mathbb{BC})}$ be arbitrary. Then, there exists $(t_n) \in l_{p}^{\Bbbk}(\mathbb{BC})$ such that $|s_n|_{\Bbbk} \precsim |t_n|_{\Bbbk}$ for all $n \in \mathbb{N}$. So, we write $|s_n|_{\Bbbk}^{p} \precsim |t_n|_{\Bbbk}^{p}$ for all $n \in \mathbb{N}$. Therefore, the series $\sum\limits_{n=1}^{\infty} |t_n|_{\Bbbk}^{p}$ is convergent, the comparison test implies that $\sum\limits_{n=1}^{\infty} |t_n|_{\Bbbk}^{p}$ converges. This shows that $(s_n) \in l_{p}^{\Bbbk}(\mathbb{BC})$. Then, we have the inclusion $l_{p}^{\Bbbk}\widetilde{(\mathbb{BC})} \subset l_{p}^{\Bbbk}(\mathbb{BC})$ which means that $l_{p}^{\Bbbk}(\mathbb{BC})$ is a $\mathbb{D}-$solid space. □

Theorem 3.10. *$l_{p}^{\Bbbk}(\mathbb{BC})$ for $0 < p < \infty$ is a $\mathbb{D}-$monotone space.*

Proof. Let $(\zeta_n) \in M_0 l_{p}^{\Bbbk}(\mathbb{BC})$ be arbitrary. Then, there exist $(s_n) \in M_0$ and $(t_n) \in l_{p}^{\Bbbk}(\mathbb{BC})$ such that $(\zeta_n) = (s_n t_n)$. Therefore, the series $\sum\limits_{n=1}^{\infty} |t_n|_{\Bbbk}^{p}$ converges and since $\{s_n : n \in \mathbb{N}\}$ is finite, we have $\sup_{\mathbb{D}} \{ |s_n|_{\Bbbk} : n \in \mathbb{N} \}$ is finite, and so $\sup_{\mathbb{D}} \left\{ |s_n|_{\Bbbk}^{p} : n \in \mathbb{N} \right\}$ is finite. Then, since

$$
\begin{aligned}
\sum_{n=1}^{\infty} |s_n t_n|_{\Bbbk}^{p} &= \sum_{n=1}^{\infty} |s_n|_{\Bbbk}^{p} |t_n|_{\Bbbk}^{p} \\
&\precsim \sup_{\mathbb{D}} \left\{ |s_n|_{\Bbbk}^{p} : n \in \mathbb{N} \right\} \sum_{n=1}^{\infty} |t_n|_{\Bbbk}^{p},
\end{aligned}
$$

we say that $\sum_{n=1}^{\infty} |s_n t_n|_{\mathbb{k}}^p$ converges. This implies that $(\zeta_n) \in l_p^{\mathbb{k}}(\mathbb{BC})$. Then, we have the inclusion $M_0 l_p^{\mathbb{k}}(\mathbb{BC}) \subset l_p^{\mathbb{k}}(\mathbb{BC})$ which means that $l_p^{\mathbb{k}}(\mathbb{BC})$ is a $\mathbb{D}-$monotone space. □

Theorem 3.11. *$l_p^{\mathbb{k}}(\mathbb{BC})$ for $1 \leq p < \infty$ is a $\mathbb{D}-$normed $BK-$space.*

Proof. Let $\left(\zeta^{(n)}\right) \in l_p^{\mathbb{k}}(\mathbb{BC})$ such that $\zeta^{(n)} \to \zeta$ as $n \to \infty$ with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}, l_p^{\mathbb{k}}(\mathbb{BC})}$. Then, for every $0 \precnsim \varepsilon$ there exists $n_0 \in \mathbb{N}$ such that $\left\|\zeta^{(n)} - \zeta\right\|_{\mathbb{D}, l_p^{\mathbb{k}}(\mathbb{BC})} \precnsim \varepsilon$ for all $n \geq n_0$. Thus, we have that $\left(\sum_{l=1}^{\infty} \left|\zeta_l^{(n)} - \zeta_l\right|_{\mathbb{k}}^p\right)^{\frac{1}{p}} \precnsim \varepsilon$ for every $0 \precnsim \varepsilon$ and for all $n \geq n_0$. So, for any fixed $l \in \mathbb{N}$ we write $\left|\zeta_l^{(n)} - \zeta_l\right|_{\mathbb{k}}^p \precnsim \varepsilon^p$ and hence $\left|\zeta_l^{(n)} - \zeta_l\right|_{\mathbb{k}} \precnsim \varepsilon$ for every $0 \precnsim \varepsilon$ and for all $n \geq n_0$ which means that $\left(\zeta_l^{(n)}\right)$ converges to the bicomplex number ζ_l with respect to the hyperbolic valued norm $|.|_{\mathbb{k}}$. Thus, the coordinates are continuous on $l_p^{\mathbb{k}}(\mathbb{BC})$ for $1 \leq p < \infty$ and we get the required result. □

Theorem 3.12. *$l_p^{\mathbb{k}}(\mathbb{BC})$ for $0 < p < \infty$ is a $\mathbb{D}-$symmetric space.*

Proof. Let $(s_n) \in l_p^{\mathbb{k}}(\mathbb{BC})$ and $\sigma \in \pi$. Then, since $\sigma : \mathbb{N} \to \mathbb{N}$ is a injective and surjective function, we have the equalities $\left\{\left|s_{\sigma(n)}\right|_{\mathbb{k}} : n \in \mathbb{N}\right\} = \left\{|s_n|_{\mathbb{k}} : n \in \mathbb{N}\right\}$ and $\left\{\left|s_{\sigma(n)}\right|_{\mathbb{k}}^p : n \in \mathbb{N}\right\} = \left\{|s_n|_{\mathbb{k}}^p : n \in \mathbb{N}\right\}$. So, we can write $\sum_{n=1}^{\infty} \left|s_{\sigma(n)}\right|_{\mathbb{k}}^p = \sum_{n=1}^{\infty} |s_n|_{\mathbb{k}}^p$. Thus, since $\sum_{n=1}^{\infty} |s_n|_{\mathbb{k}}^p$ converges, we obtain that $\sum_{n=1}^{\infty} \left|s_{\sigma(n)}\right|_{\mathbb{k}}^p$ converges. This means that $\left(s_{\sigma(n)}\right) \in l_p^{\mathbb{k}}(\mathbb{BC})$, as required. □

4 $\mathbb{D}-$Topological Duals of $\mathbb{D}-$Normed Bicomplex $\mathbb{BC}-$Modules $l_p^{\mathbb{k}}(\mathbb{BC})$ for $1 \leq p < \infty$

We start this part with the following definitions.

Definition 4.1. A sequence (x_n) in a $\mathbb{D}-$normed Banach bicomplex $\mathbb{BC}-$module X is a called a $\mathbb{D}-$Schauder basis of X if for every $x \in X$ there exists a unique sequence $(\zeta_n) \subset \mathbb{BC}$ such that $x = \sum\limits_{n=1}^{\infty} \zeta_n x_n,$, that is, such that $\lim\limits_{N\to\infty} \left\| x - \sum\limits_{n=1}^{N} \zeta_n x_n \right\|_{\mathbb{D},X} = 0.$

Definition 4.2. Let X be a $\mathbb{BC}-$module. Assume that X has $\mathbb{D}-$norm $\|.\|_{\mathbb{D},X}$. Then, the $\mathbb{BC}-$module $B_{\mathbb{BC}}(X, \mathbb{BC})$ is called $\mathbb{D}-$topological dual of X and is denoted by $X^{*_{\mathbb{D}}}$.

We are ready to investigate $\mathbb{D}-$topological duals of $\mathbb{D}-$normed bicomplex $\mathbb{BC}-$modules $l_p^{\mathbb{k}}(\mathbb{BC})$ that are bicomplex versions of topological duals of sequence spaces l_p from the literature using hyperbolic valued norm $\|.\|_{\mathbb{D},.}$.

Theorem 4.3. *The set $\mathbb{BC}^n$ forms a $\mathbb{BC}-$module and a hyperbolic valued normed space with respect to the addition, bicomplex scalar multiplication and hyperbolic valued norm defined as*

$$\begin{aligned} z +_n w &= (z_1 + w_1, ..., z_n + w_n), \\ \lambda \cdot_n z &= (\lambda z_1, ..., \lambda z_n), \\ \|.\|_{\mathbb{D},\mathbb{BC}^n} &: \mathbb{BC}^n \to \mathbb{D}^+, z \to \|z\|_{\mathbb{D},\mathbb{BC}^n} = \left(\sum_{l=1}^{n} |z_l|_{\mathbb{k}}^2 \right)^{\frac{1}{2}} \end{aligned}$$

for all $z = (z_1, z_2, ..., z_n), w = (w_1, w_2, ..., w_n) \in \mathbb{BC}^n$, $\lambda \in \mathbb{BC}$.

Proof. If we apply the definitions of $\mathbb{BC}-$module and hyperbolic valued normed space to the set $\mathbb{BC}^n$, we get the required results. $\square$

Theorem 4.4. *$\mathbb{D}-$Topological dual of $\mathbb{BC}^n$ is $\mathbb{BC}^n$, that is, $[\mathbb{BC}^n]^{*_{\mathbb{D}}} = \mathbb{BC}^n$.*

Proof. Let $e_1 = (1, 0, 0, ..., 0)$, $e_2 = (0, 1, 0, ..., 0)$, ..., $e_n = (0, 0, ..., 0, 1)$. Then, $\{e_1, e_2, ..., e_n\}$ is a $\mathbb{D}-$Schauder basis of $\mathbb{BC}^n$.

Define $T : [\mathbb{BC}^n]^{*_{\mathbb{D}}} \to \mathbb{BC}^n$, $f \to Tf = (f(e_1), f(e_2), ..., f(e_n))$. It is easy to show that $\|f\|_{\mathbb{D}} = \|Tf\|_{\mathbb{D},\mathbb{BC}^n}$ for every $f \in [\mathbb{BC}^n]^{*_{\mathbb{D}}}$. Also, T is a surjective $\mathbb{BC}-$linear operator. Thus, T is a surjective isometric $\mathbb{BC}-$module isomorphism with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$. Consequently, we get $[\mathbb{BC}^n]^{*_{\mathbb{D}}} = \mathbb{BC}^n$. $\square$

Theorem 4.5. *$\mathbb{D}-$Topological dual of $l_1^{\Bbbk}(\mathbb{BC})$ is $l_\infty^{\Bbbk}(\mathbb{BC})$, that is, $\left[l_1^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}} = l_\infty^{\Bbbk}(\mathbb{BC})$.*

Proof. Let $e_1 = (1,0,0,0,...)$, $e_2 = (0,1,0,0,...)$, ..., $e_n = (0,0,...,0,1,0,0,...)$, ... Then, (e_n) is a $\mathbb{D}-$Schauder basis of $l_1^{\Bbbk}(\mathbb{BC})$ and so every $\zeta = (\zeta_1,\zeta_2,...,\zeta_n,...) \in l_1^{\Bbbk}(\mathbb{BC})$ can be written uniquely as $\zeta = \sum_{n=1}^{\infty} \zeta_n e_n$. Define

$T : \left[l_1^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}} \to l_\infty^{\Bbbk}(\mathbb{BC})$, $f \to Tf = (f(e_1), f(e_2), ..., f(e_n), ...)$.

Let $f \in \left[l_1^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}}$ be arbitrary. Since $e_n \in l_1^{\Bbbk}(\mathbb{BC})$ and $\|e_n\|_{\mathbb{D},l_1^{\Bbbk}(\mathbb{BC})} = 1$ for all $n \in \mathbb{N}$, we get

$$|f(e_n)|_{\Bbbk} \precsim \|f\|_{\mathbb{D}} \|e_n\|_{\mathbb{D},l_1^{\Bbbk}(\mathbb{BC})} = \|f\|_{\mathbb{D}},$$

and so,

$$\|f(e_n)\|_{\mathbb{D},l_\infty^{\Bbbk}(\mathbb{BC})} = \sup_{\mathbb{D}} \{|f(e_n)|_{\Bbbk} : n \in \mathbb{N}\} \precsim \|f\|_{\mathbb{D}}.$$

This means that $(f(e_n)) \in l_\infty^{\Bbbk}(\mathbb{BC})$.

On the other hand, for $f \in \left[l_1^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}}$, we obtain that $f(\zeta) = \sum_{n=1}^{\infty} \zeta_n f(e_n)$ and so by $\mathbb{D}-$boundedness of f

$$\begin{aligned}
|f(\zeta)|_{\Bbbk} &= \left|\sum_{n=1}^{\infty} \zeta_n f(e_n)\right|_{\Bbbk} \precsim \sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk} |f(e_n)|_{\Bbbk} \\
&\precsim \sup_{\mathbb{D}} \{|f(e_n)|_{\Bbbk} : n \in \mathbb{N}\} \sum_{n=1}^{\infty} |\zeta_n|_{\Bbbk} \\
&= \|f(e_n)\|_{\mathbb{D},l_\infty^{\Bbbk}(\mathbb{BC})} \|\zeta\|_{\mathbb{D},l_1^{\Bbbk}(\mathbb{BC})}.
\end{aligned}$$

If the $\mathbb{D}-$supremum is taken over all elements $\zeta \in l_1^{\Bbbk}(\mathbb{BC})$ with $\|\zeta\|_{\mathbb{D},l_1^{\Bbbk}(\mathbb{BC})} = 1$, we deduce that $\|f\|_{\mathbb{D}} \precsim \|f(e_n)\|_{\mathbb{D},l_\infty^{\Bbbk}(\mathbb{BC})}$. Consequently, $\|f\|_{\mathbb{D}} = \|f(e_n)\|_{\mathbb{D},l_\infty^{\Bbbk}(\mathbb{BC})}$ and so T is isometry.

Now, let $\nu = (\nu_1,\nu_2,...,\nu_n,...) \in l_\infty^{\Bbbk}(\mathbb{BC})$ be arbitrary. Define $g : l_1^{\Bbbk}(\mathbb{BC}) \to \mathbb{BC}$, $\rho = (\rho_n) \to g(\rho) = \sum_{n=1}^{\infty} \rho_n \nu_n$. It is clear that

$g(\rho+\alpha\sigma)=g(\rho)+\alpha g(\sigma)$ for all $\rho=(\rho_n), \sigma=(\sigma_n)\in l_1^{\mathbb{k}}(\mathbb{BC}), \alpha\in\mathbb{BC}$. Moreover, we have

$$\begin{aligned} |g(\rho)|_{\mathbb{k}} &= \left|\sum_{n=1}^{\infty}\rho_n\nu_n\right|_{\mathbb{k}} \precsim \sum_{n=1}^{\infty}|\rho_n|_{\mathbb{k}}|\nu_n|_{\mathbb{k}} \\ &\precsim \sup_{\mathbb{D}}\{|\nu_n|_{\mathbb{k}} : n\in\mathbb{N}\}\sum_{n=1}^{\infty}|\rho_n|_{\mathbb{k}} \\ &= \|\nu\|_{\mathbb{D},l_\infty^{\mathbb{k}}(\mathbb{BC})}\|\rho\|_{\mathbb{D},l_1^{\mathbb{k}}(\mathbb{BC})}, \end{aligned}$$

that implies that $g\in\left[l_1^{\mathbb{k}}(\mathbb{BC})\right]^{*\mathbb{D}}$. Hence, T is surjective.

Besides, for all $f,g\in\left[l_1^{\mathbb{k}}(\mathbb{BC})\right]^{*\mathbb{D}}, \alpha\in\mathbb{BC}$ we get

$$\begin{aligned} T(f+\alpha g) &= ((f+\alpha g)(e_1),(f+\alpha g)(e_2),...,(f+\alpha g)(e_n),...) \\ &= (f(e_1),...,f(e_n),...)+\alpha(g(e_1),...,g(e_n),...) \\ &= Tf+\alpha Tg, \end{aligned}$$

which shows that T is a $\mathbb{BC}-$linear operator. Thus, T is a surjective isometric $\mathbb{BC}-$module isomorphism with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$ and so, $\left[l_1^{\mathbb{k}}(\mathbb{BC})\right]^{*\mathbb{D}}=l_\infty^{\mathbb{k}}(\mathbb{BC})$. □

Theorem 4.6. *Let p and q be real numbers with $1<p<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$. $\mathbb{D}-$Topological dual of $l_p^{\mathbb{k}}(\mathbb{BC})$ is $l_q^{\mathbb{k}}(\mathbb{BC})$, that is, $\left[l_p^{\mathbb{k}}(\mathbb{BC})\right]^{*\mathbb{D}}=l_q^{\mathbb{k}}(\mathbb{BC})$.*

Proof. Let $e_1=(1,0,0,0,...)$, $e_2=(0,1,0,0,...)$, ..., $e_n=(0,0,...,0,1,0,0,...)$, ... Then, (e_n) is a $\mathbb{D}-$Schauder basis of $l_p^{\mathbb{k}}(\mathbb{BC})$ and so every $\zeta=(\zeta_1,\zeta_2,...,\zeta_n,...)\in l_p^{\mathbb{k}}(\mathbb{BC})$ can be written uniquely as $\zeta=\sum_{n=1}^{\infty}\zeta_n e_n$. Define

$T:\left[l_p^{\mathbb{k}}(\mathbb{BC})\right]^{*\mathbb{D}}\to l_q^{\mathbb{k}}(\mathbb{BC}),\ f\to Tf=(f(e_1),f(e_2),...,f(e_n),...)$.

Let $f\in\left[l_1^{\mathbb{k}}(\mathbb{BC})\right]^{*\mathbb{D}}$ be arbitrary. Define $(\zeta_n)=\left(\zeta_1^{(n)},\zeta_2^{(n)},...,\zeta_l^{(n)},...\right)\in l_p^{\mathbb{k}}(\mathbb{BC})$ by

$$\zeta_l^{(n)}=\begin{cases}\frac{|f(e_l)|_{\mathbb{k}}^q}{f(e_l)}, & l\leq n \text{ and } |f(e_l)|_i\neq 0\\ 0, & l>n \text{ and } |f(e_l)|_i=0\end{cases}.$$

Then, we get

$$f(\zeta_n)=\sum_{l=1}^{\infty}\zeta_l^{(n)}f(e_l)=\sum_{l=1}^{n}\frac{|f(e_l)|_{\Bbbk}^{q}}{f(e_l)}f(e_l)=\sum_{l=1}^{n}|f(e_l)|_{\Bbbk}^{q}.$$

Thus, by $\mathbb{D}-$boundedness of f we obtain

$$\begin{aligned}\sum_{l=1}^{n}|f(e_l)|_{\Bbbk}^{q} &= \left|\sum_{l=1}^{n}|f(e_l)|_{\Bbbk}^{q}\right|_{\Bbbk}=|f(\zeta_n)|_{\Bbbk}\precsim\|f\|_{\mathbb{D}}\|\zeta_n\|_{\mathbb{D},l_p^{\Bbbk}(\mathbb{BC})}\\ &= \|f\|_{\mathbb{D}}\left(\sum_{l=1}^{\infty}\left|\zeta_l^{(n)}\right|_{\Bbbk}^{p}\right)^{\frac{1}{p}}=\|f\|_{\mathbb{D}}\left(\sum_{l=1}^{n}\left|\frac{|f(e_l)|_{\Bbbk}^{q}}{f(e_l)}\right|_{\Bbbk}^{p}\right)^{\frac{1}{p}}\\ &= \|f\|_{\mathbb{D}}\left(\sum_{l=1}^{n}|f(e_l)|_{\Bbbk}^{(q-1)p}\right)^{\frac{1}{p}}=\|f\|_{\mathbb{D}}\left(\sum_{l=1}^{n}|f(e_l)|_{\Bbbk}^{q}\right)^{\frac{1}{p}}.\end{aligned}$$

Then, by the following the fact that

$$\left(\sum_{l=1}^{n}|f(e_l)|_{\Bbbk}^{q}\right)^{\frac{1}{q}}\precsim\|f\|_{\mathbb{D}},$$

we have

$$\|f(e_n)\|_{\mathbb{D},l_q^{\Bbbk}(\mathbb{BC})}=\left(\sum_{l=1}^{\infty}|f(e_l)|_{\Bbbk}^{q}\right)^{\frac{1}{q}}\precsim\|f\|_{\mathbb{D}}$$

as $n\to\infty$. Therefore, we conclude that $(f(e_l))\in l_q^{\Bbbk}(\mathbb{BC})$.

On the other hand, for $f\in\left[l_p^{\Bbbk}(\mathbb{BC})\right]^{*_{\mathbb{D}}}$, we obtain that $f(\zeta)=\sum\limits_{n=1}^{\infty}\zeta_n f(e_n)$ and so

$$\begin{aligned}|f(\zeta)|_{\Bbbk} &= \left|\sum_{n=1}^{\infty}\zeta_n f(e_n)\right|_{\Bbbk}\precsim\sum_{n=1}^{\infty}|\zeta_n f(e_n)|_{\Bbbk}\\ &\precsim \left(\sum_{n=1}^{\infty}|\zeta_n|_{\Bbbk}^{p}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}|f(e_n)|_{\Bbbk}^{q}\right)^{\frac{1}{q}}\\ &= \|\zeta\|_{\mathbb{D},l_p^{\Bbbk}(\mathbb{BC})}\|f(e_n)\|_{\mathbb{D},l_q^{\Bbbk}(\mathbb{BC})}.\end{aligned}$$

If the $\mathbb{D}-$supremum is taken over all elements $\zeta \in l_p^{\Bbbk}(\mathbb{BC})$ with $\|\zeta\|_{\mathbb{D},l_p^{\Bbbk}(\mathbb{BC})} = 1$, we deduce that $\|f\|_{\mathbb{D}} \precsim \|f(e_n)\|_{\mathbb{D},l_q^{\Bbbk}(\mathbb{BC})}$. Consequently, $\|f\|_{\mathbb{D}} = \|f(e_n)\|_{\mathbb{D},l_q^{\Bbbk}(\mathbb{BC})}$ and so T is isometry.

Now, let $\nu = (\nu_1, \nu_2, ..., \nu_n, ...) \in l_q^{\Bbbk}(\mathbb{BC})$ be arbitrary. Define $g : l_p^{\Bbbk}(\mathbb{BC}) \to \mathbb{BC}$, $\rho = (\rho_n) \to g(\rho) = \sum\limits_{n=1}^{\infty} \rho_n \nu_n$. It is clear that $g(\rho + \alpha\sigma) = g(\rho) + \alpha g(\sigma)$ for all $\rho = (\rho_n), \sigma = (\sigma_n) \in l_p^{\Bbbk}(\mathbb{BC}), \alpha \in \mathbb{BC}$. Moreover, we have

$$\begin{aligned}
|g(\rho)|_{\Bbbk} &= \left|\sum_{n=1}^{\infty} \rho_n \nu_n\right|_{\Bbbk} \precsim \sum_{n=1}^{\infty} |\rho_n \nu_n|_{\Bbbk} \\
&\precsim \left(\sum_{n=1}^{\infty} |\rho_n|_{\Bbbk}^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |\nu_n|_{\Bbbk}^q\right)^{\frac{1}{q}} \\
&= \|\rho\|_{\mathbb{D},l_p^{\Bbbk}(\mathbb{BC})} \|\nu\|_{\mathbb{D},l_q^{\Bbbk}(\mathbb{BC})},
\end{aligned}$$

that implies that $g \in \left[l_p^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}}$. Hence, T is surjective.

Besides, for all $f, g \in \left[l_p^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}}, \alpha \in \mathbb{BC}$ it is simple to show that $T(f + \alpha g) = Tf + \alpha Tg$ which proves that T is a $\mathbb{BC}-$linear operator. Thus, T is a surjective isometric $\mathbb{BC}-$module isomorphism with respect to the hyperbolic valued norm $\|.\|_{\mathbb{D}}$ and so, $\left[l_p^{\Bbbk}(\mathbb{BC})\right]^{*\mathbb{D}} = l_q^{\Bbbk}(\mathbb{BC})$. □

5 Conclusion and Future Works

This article introduces some notions such as $\mathbb{D}-$topological dual, $\mathbb{D}-$solid space, $\mathbb{D}-$monotone space, $\mathbb{D}-$normed $BK-$space, $\mathbb{D}-$symmetric space in bicomplex setting. Since they are crucial topics related to the topological properties of sequence spaces, our results will be hammering away at studying on other topological aspects of such spaces.

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Nilay Değirmen
Department of Mathematics
Graduated Ph.D. of Mathematics
Ondokuz Mayıs University
Samsun, Turkey
E-mail: nilay.sager@omu.edu.tr

Birsen Sağır
Department of Mathematics
Professor of Mathematics
Ondokuz Mayıs University
Samsun, Turkey
E-mail: bduyar@omu.edu.tr