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Original Research Paper

A Semi-analytical Solutions of Fractional Riccati Differential Equation via Singular and Non-singular Operators

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Abstract. In this paper, we investigate the solution of the Riccati differential equations of fractional order (FRDEs) involving Caputo derivative (CD), Caputo-Fabrizio derivative (CFD) or Atangana-Baleanu derivative (ABD) using a semi-analytical iterative approach. Temimi and Ansari introduced this method and called it TAM. The linearization of the method involves splitting the problem into a linear and nonlinear part. While this approach is a semi-analytical iterative technique, there is a significant amount of analytical work where the computational times of integrals are needed to be carried out numerically. The novelty and the motivation behind this paper is the comparison of the time used in minutes which is given for three derivatives: CD, CFD and ABD. Meanwhile, the comparison of the approximate solutions with CD, CFD and ABD are presented. With the help of the software *Mathematica*, the computational results have been obtained.

AMS Subject Classification: 26A33; 43A08; 35R11.
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1 Introduction

A problem of the non-linear differential equation is the Riccati differential equations (RDEs) which is written as follows [1]

$$D^{\theta}z(t) - \sum_{i=0}^{2} p_i(t)\, z^i(t) = 0, \tag{1}$$

in which $p_i(t)$ are constant functions, $z(0) = z_0$, $t \in \mathbb{R}$, $0 < t \leq T$ and $0 < \theta \leq 1$. The operator D^{θ} can denotes CD, CFD or ABD of order θ.

Scholars are requested to check papers that have been introduced to get a better grasp of thoroughgoing introduction about RDEs and its history in [39]. About the motivation and significance for RDEs, it has a basic role in some of the applications such as river flows, in pattern formation in dynamic games, econometric models, linear systems with Markovian jumps, stochastic systems, control theory, diffusion problems, random processes, robust stabilization, bending of beams, network synthesis, chemical reactions, the motion of rotating mass around body, pendulum and financial mathematics [34, 39, 45, 48, 50].

The other two applied instances include the time-independent static Schrodinger equation in one dimension and the non-autonomous and in-homogeneous diffusion-type equation (for detales see [17, 20, 43]).

In real world, we can see nowadays some utilization of differential equations featuring real order derivative (or fractional differential equations (FDEs)) in variant fields of sciences and some areas of engineering. Some of these topics and applications may be involved in sciences such as biology, physics, environmental science and fields of engineering such as image processing, control, signal processing, dynamic systems and mechanics, which were stated by *Sun* and his associates in [44].

To learn more about the differential and integral with non integer order and its properties, interested scholars and researchers can be referred to useful books and many articles written in this field in [11–15, 21, 25, 27, 29–33, 38, 40].

We will refer in the following section to the most widely used approximate methods for RDEs and FRDEs, including homotopic perturbation method [36], differential transform method [8], finite difference method [26], optimal homotopy asymptotic method [23], wavelet method [49], Adomian decomposition method [18], variational iteration technique [37], differential quadrature method [24], Lie-group shooting method, reproducing kernel Hilbert space method [41], collocation method [22], Runge-Kutta method [7] and etc [1, 10, 35, 42].

The novelty and motivation of this paper is the comparison of the time used in minutes given for three derivatives CD, CFD and ABD, using TAM's approximate method for the fractional Riccati's differential equation. Meanwhile, the comparison of the approximate solutions with CD, CFD and ABD are presented. After calculations, the results have showed that relations between total errors are $CD < CFD < ABD$. In addition, the results have showed the order of CPU speed in calculation in Mathematica are $CFD < ABD < CD$.

Our article is organized as: In Section 2, basic definitions and notations are offered. In Section 3, we elaborate on the methodology of TAM. Convergence of this method and error analysis are verified in Section 4. We provide the applications and results in Section 5.

2 Basic Definitions And Notations

In this part, we define and offer some known fractional derivatives, including some fractional integrals and important properties which are necessary for next sections.

Definition 2.1. The CD [27] of order θ $(n-1 < \theta \leq n)$, $t > a$ and $n \in \mathbb{N}$, is stated as

$$^{CD}D^{\theta}z(t) = \frac{1}{\Gamma(n-\theta)}\int_a^t (t-s)^{\theta-1}z^{(n)}(s)ds. \tag{2}$$

Definition 2.2. The Reimann-Liouville's integral (RLI) [27] of order θ, is stated as

$$^{RLI}I^{\theta}\left(z(t)\right) = \frac{1}{\Gamma(\theta)}\int_a^t (t-s)^{\theta-1}z(s)ds, \quad \theta > 0, \quad t > a. \tag{3}$$

Definition 2.3. The ABD [9] of order θ $(n < \theta \leq n+1)$, $t > a$ and $n \in \mathbb{Z}^{+}$, is stated as

$$^{ABD}D^{n+\theta}z(t) = \frac{T(\theta)}{1-\theta}\int_a^t E_{\theta}\Big(-\theta\frac{(t-s)^{\theta}}{1-\theta}\Big)z^{(n+1)}(s)ds, \tag{4}$$

in which, E_θ is one-parameter Mittag-Leffler function, and $T(\theta)$, is called the normalization function featuring $T(0) = T(1) = 1$.

Definition 2.4. The Atangana-Baleanu's integral (ABI) [9] of order $0 < \theta \leq 1$, is defined by

$$ {}^{ABI}I^{\theta}\left(z(t)\right) = \frac{1-\theta}{T(\theta)} z(t) + \frac{\theta}{T(\theta)} \ {}^{RLI}I^{\theta}\left(z(t)\right), \tag{5}$$

in which, $t > a$ and $T(\theta)$ is called, the normalization function featuring $T(0) = T(1) = 1$.

Remark 2.5. The Atangana-Baleanu's integral of order $\theta(n < \theta \leq n+1)$, $t > a$ and $n \in \mathbb{Z}^+$, is stated as

$$ {}^{ABI}I^{\theta}\left(z(t)\right) = \frac{1-\theta+n}{M(\theta-n)} I^{n} z(t) + \frac{\theta-n}{M(\theta-n)} I^{\theta} z(t). \tag{6}$$

Definition 2.6. The CFD [16] of order θ $(n < \theta \leq n+1)$, $t > a$ and $n \in \mathbb{Z}^+$, is stated as

$$ {}^{CFD}D^{n+\theta} z(t) = \frac{T(\theta)}{1-\theta} \int_a^t \exp\Big(-\theta \frac{t-s}{1-\theta}\Big) z^{(n+1)}(s) ds, \tag{7}$$

in which, $T(\theta)$ is called the normalization function featuring $T(0) = T(1) = 1$.

Definition 2.7. The Caputo-Fabrizio's integral (CFI) [28] of order $0 < \theta \leq 1$, is stated as

$$ {}^{CFI}I^{\theta}\left(z(t)\right) = \frac{1-\theta}{T(\theta)} z(t) + \frac{\theta}{T(\theta)} I\left(z(t)\right), \tag{8}$$

in which, $t > a$ and $T(\theta)$ is called the normalization function featuring $T(0) = T(1) = 1$.

Remark 2.8. The Caputo-Fabrizio's integral of order $\theta(n < \theta \leq n+1)$, $t > a$ and $n \in \mathbb{Z}^+$, is stated as

$$ {}^{CFI}I^{\theta}\left(z(t)\right) = \frac{1-\theta+n}{M(\theta-n)} I^{n}\left(z(t)\right) + \frac{\theta-n}{M(\theta-n)} I^{n+1}\left(z(t)\right). \tag{9}$$

3 The Methodology Of TAM

To explain the TAM [2, 4–6, 46, 47], assume that the nonlinear differential equation is below featuring boundary assumptions

$$\begin{cases} L\left[z(t)\right] + N\left[z(t)\right] + \nu(t) = 0, \\ B\left(z, \frac{dz}{dt}\right) = 0, \end{cases} \tag{10}$$

in which t represents the independent variable, $z(t)$ is the unfamiliar function, $B(*)$ is a boundary operator, $\nu(t)$ is a given familiar function, $L(*)$ is the linear operator and $N(*)$ is the nonlinear operator. For Eq. (1), we consider $L\left[z(t)\right] = D^{\theta} z(t)$, $N\left[z(t)\right] = -\sum_{i=0}^{2} p_i(t)\, z^i(t)$ and $\nu(t) = 0$.

The TAM will start with an initial guess $z_0(t)$. To gain function $z(t)$ as a solution, we solve the following system of boundary value problems:

$$\begin{cases} L\left[z_0(t)\right] + \nu(t) = 0,\ B\left(z_0, \frac{dz_0}{dt}\right) = 0, \\ L\left[z_1(t)\right] + N\left[z_0(t)\right] + \nu(t) = 0,\ B\left(z_1, \frac{dz_1}{dt}\right) = 0, \\ L\left[z_2(t)\right] + N\left[z_1(t)\right] + \nu(t) = 0,\ B\left(z_2, \frac{dz_2}{dt}\right) = 0, \\ \vdots \\ L\left[z_n(t)\right] + N\left[z_n(t)\right] + \nu(t) = 0,\ B\left(z_n, \frac{dz_n}{dt}\right) = 0. \end{cases} \tag{11}$$

Then, by $z = \lim_{n\to\infty} z_n$, the solution is given.

4 Error And Convergence Of TAM

In this section, we consider the error and convergence of TAM.

4.1 Convergence of TAM

Theorem 4.1. *Suppose the solution components $z_0, z_1, z_2, \ldots,$ are defined as given in Eq.(11). If there exists $\theta \in (0, 1)$ such that $\|z_{i+1}\| \leq \theta \|z_i\|$ for all $i \geq i_0$ for some $i_0 \in \mathbb{N}$, then the series solution $\sum_{i=0}^{\infty} z_i$ converges to a solution.*

Proof. Suppose the sequences $\{V_p\}_{p=0}^{\infty}$ are specified with

$$\begin{aligned} V_0 =& z_0 \\ V_1 =& z_0 + z_1, \\ V_2 =& z_0 + z_1 + z_2, \\ &\dots \\ V_p =& z_0 + z_1 + z_2 + \ldots + z_p. \end{aligned} \tag{12}$$

It is enough to show that in the Hilbert space $\mathbb{R}$ the sequence $\{V_p\}_{p=0}^{\infty}$ is a Cauchy sequence. For this target, suppose

$$\begin{aligned} \|V_{p+1} - V_p\| &= \|z_{p+1}\| \\ &\leq \theta \|z_p\| \\ &\leq \theta^2 \|z_{p-1}\| \\ &\vdots \\ &\leq \theta^{p-i_0+1} \|z_{i_0}\|. \end{aligned}$$

Supposing that $p \geq q > i_0$ and for every $p, q \in \mathbb{N}$, we have

$$\begin{aligned} \|V_p - V_q\| &= \|(V_p - V_{p-1}) + (V_{p-1} - V_{p-2}) + \ldots + (V_q - V_{q-1})\| \\ &\leq \|(V_p - V_{p-1})\| + \|(V_{p-1} - V_{p-2})\| + \ldots + \|(V_q - V_{q-1})\| \\ &\leq \theta^{p-i_0} \|z_{i_0}\| + \theta^{p-i_0-1} \|z_{i_0}\| + \ldots + \theta^{q-i_0+1} \|z_{i_0}\| \\ &= \theta^{q-i_0+1} \left(\frac{1 - \theta^{p-q}}{1 - \theta} \right) \|z_{i_0}\|. \end{aligned}$$

It arrives at $\lim\limits_{\substack{p \to \infty \\ q \to \infty}} \|V_p - V_q\| = 0$, with regard to the $\theta \in (0, 1)$. So, in the Hilbert space $\mathbb{R}$, sequence $\{V_p\}_{p=0}^{\infty}$ is a Cauchy sequence and this implies that the series solution converges to series $\sum\limits_{i=0}^{\infty} z_i(t)$. $\square$

4.2 Error

In this subsection, to provide an error and the convergence criteria, we first recall the definition of L^2-norm on a certain domain ω for any continuous function h:

$$\|h\| = \sqrt{\int_{\omega} h^2 \, d\omega}.$$

Consider the following boundary value problems:

$$\begin{cases} L\left[z_0(t)\right] + \nu(t) = 0, \ B\left(z_0, \frac{dz_0}{dt}\right) = 0, \\ L\left[z_1(t)\right] + N\left[z_0(t)\right] + \nu(t) = 0, \ B\left(z_1, \frac{dz_1}{dt}\right) = 0, \\ L\left[z_2(t)\right] + N\left[z_1(t)\right] + \nu(t) = 0, \ B\left(z_2, \frac{dz_2}{dt}\right) = 0, \\ \vdots \\ L\left[z_{n+1}(t)\right] + N\left[z_n(t)\right] + \nu(t) = 0, \ B\left(z_{n+1}, \frac{dz_{n+1}}{dt}\right) = 0. \end{cases} \tag{13}$$

Then the error remainder is [3, 19]

$$ER_n(t) = L\left[z_n(t)\right] + N\left[z_n(t)\right] + \nu(t), \tag{14}$$

and the maximal error remainder is [3, 19]

$$MER_n(t) = \max_{0 \le t \le 1} \left|ER_n(t)\right|. \tag{15}$$

5 Applications And Results

Three examples in this part are now made for the purpose of making the readers understand the TAM for FRDEs more easily. The software Mathematica in these examples has been utilized for plots and computations.

Example 5.1. For the first example, we offer the FRDEs:

$$D^{\theta} z(t) = t^4 \Gamma(3-\theta)^2 - 2t^{2-\theta} - z^2(t), \quad t \in (0,1), \quad \theta \in (0,1], \tag{16}$$

including the incipient conditions

$$z_0 = z(0) = 0, \tag{17}$$

via the exact solution $z(t) = t^2 \Gamma(3-\theta)$.

Table 1 represents the present method for $\theta = 0.9$ and $n = 3$. $TAM(z_3)$ is approximate solution with three iterations.

Table 1: Comparison between the approximate solutions of the sample 5.1 with $\theta = 0.9$.

t	$TAM(z_3)$			Exact
	CD	CFD	ABD	
0	0	0	0	0
0.2	0.0418594	0.0418462	0.0418453	0.0418594
0.4	0.167437	0.167297	0.167294	0.167438
0.6	0.376734	0.376601	0.376583	0.376735
0.8	0.669749	0.6696	0.66959	0.669751
Total Errors	4×10^{-6}	4.32×10^{-4}	4.711×10^{-4}	0.0

Example 5.2. For the second example, we offer the FRDEs [1]:

$$D^{\theta} z(t) = 1 + 2z(t) - z^2(t), \quad t \in (0,1)\,, \quad \theta \in (0,1]\,, \tag{18}$$

including the incipient conditions

$$z_0 = z(0) = 0. \tag{19}$$

Toward $\theta = 1$, the solution that we have gained is in accordance with the precise solution $z(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right)$.

Table 2, represents the present method for $\theta = 1$, $n = 3$ and the achieved results of fractional variational iteration method (FVI), modified homotopic perturbation method (MHPM), trigonometric transform method (TTM) and Padé-variational iteration method (PVI) [1].

Table 2: Comparison between the approximate solutions of the sample 5.2, with $\theta = 1$.

t	MHPM	FVI	PVI	TTM	$TAM(z_3)$			Exact
					CD	CFD	ABD	
0.6	1.370240	1.873658	1.331462	0.953566	0.945156	0.945156	0.945156	0.953653
0.7	1.367499	2.112944	1.497600	1.152949	1.14483	1.14483	1.14483	1.15308
0.8	1.794879	2.260134	1.630234	1.346364	1.34155	1.34155	1.34155	1.34655
0.9	1.962239	2.339134	1.724439	1.526911	1.52769	1.52769	1.52769	1.52715
1.0	2.087384	2.379356	1.776542	1.689498	1.69524	1.69524	1.69524	1.68976

Due to increasing the amount n in this method (Table 3), a much better approximate solution can be achieved.

The time of CPU used in minutes for Example 5.2, with $\theta = 1$ featuring CD, CFD and ABD, is shown in Table 4, for $n = 3$ and $n = 8$.

Table 3: Comparison between the approximate solutions of 5.2 accompanied by different iterations.

t	*CD* $n=3$	*CD* $n=8$	*CFD* $n=3$	*CFD* $n=8$	*ABD* $n=3$	*ABD* $n=8$	Exact
0	0	0	0	0	0	0	0
0.2	0.241586	0.241977	0.241586	0.241977	0.241586		0.241977
0.4	0.564013	0.567812	0.564013	0.567812	0.564013	0.567812	0.567812
0.6	0.945156	0.953566	0.945156	0.953566	0.945156	0.953566	0.953566
0.8	1.34155	1.34636	1.34155	1.34636	1.34155	1.34636	1.34636
1.0	1.69524	1.6895	1.69524	1.6895	1.69524	1.6895	1.6895

Table 4: Duration used in minutes for Example 5.2.

	CD $n=3$	*CD* $n=8$	*CFD* $n=3$	*CFD* $n=8$	*ABD* $n=3$	*ABD* $n=8$
Total Times	0.000520833	0.343229	0.000260417	0.0294271	0.00510745	0.560677

Example 5.3. For the third example, we offer the FRDEs:

$$D_t^{\theta} z(t) - z(t) - z(t)^2 = 0, \quad t \in (0,1)\,, \quad \theta \in (0,1]\,, \tag{20}$$

including the incipient conditions

$$z_0 = z(0) = 0.5. \tag{21}$$

For $\theta = 1$ exact solution is $z(t) = \frac{\exp(-t)}{\exp(-t)+1}$.

The exact and fifth approximate answers for Example 5.3, with $\theta = 0.8$ through applying TAM can be seen with the CD, CFD and ABD for Eq. (20) in Figure 1.

The exact and third and fifth approximate answers in Table 5, featuring different values θ through applying TAM can be seen with the CD, CFD and ABD for Eq. (20).

Fig.2 shows the absolute error for $n = 5$ with TAM for various values of $0 \leq t \leq 1$ and $\theta = 0.8$.

The time of CPU used in minutes for Example 5.3, with different θ featuring CD, CFD and ABD, is shown in Table 6, for $n = 5$.

6 Conclusion

We have efficiently utilized TAM as a semi-analytical iterative technique to acquire approximate solution of the fractional Riccati differen-

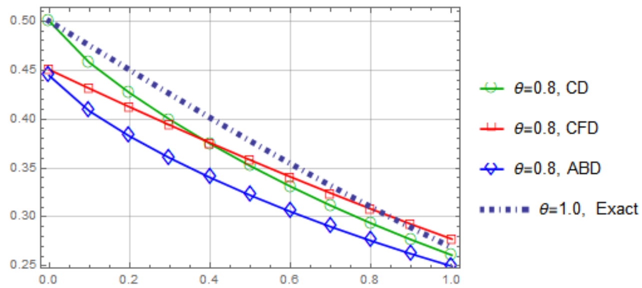


Figure 1: Agreement the exact solution and TAM with the CD, CFD and ABD featuring $\theta = 0.8$ and $n = 5$ for Eq.(20).

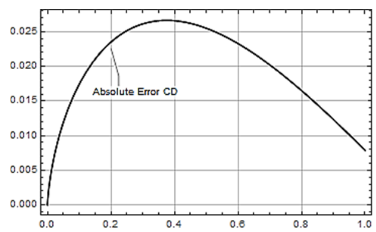
Table 5: Comparison between the approximate solutions of the sample 5.3 accompanied by different θ and different iterations.

	CD		*CFD*		*ABD*		
t	$\theta = 0.8, n = 5$	$\theta = 1, n = 3$	$\theta = 0.8, n = 5$	$\theta = 1, n = 3$	$\theta = 0.8, n = 5$	$\theta = 1, n = 3$	Exact
0	0.5	0.5	0.45049	0.5	0.444338	0.5	0.5
0.2	0.426648	0.450166	0.412313	0.450166	0.382389	0.450166	0.450166
0.4	0.374723	0.401312	0.375686	0.401312	0.340215	0.401312	0.401312
0.6	0.331046	0.35434	0.340894	0.35434	0.305384	0.35434	0.35434
0.8	0.293511	0.309997	0.308139	0.309997	0.275623	0.309997	0.309997
1.0	0.261056	0.268812	0.277548	0.268812	0.249752	0.268812	0.268812

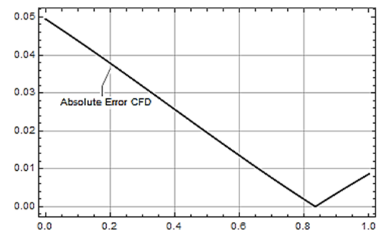
Table 6: Duration used in minutes for Example 5.3.

θ	CD	CFD	ABD
0.01	0.179167	0.00286458	0.211198
0.5	0.141146	0.003125	0.405729
0.8	0.186458	0.00338542	0.609375
1.0	0.0015625	0.0015625	0.00182292
Total Times	0.508334	0.0109375	1.22812

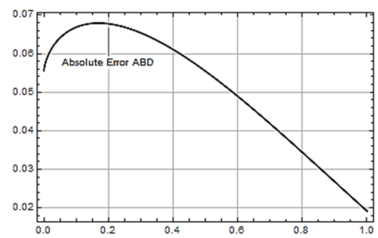
tial equations (FRDEs). While this approach is a semi-analytical iterative technique, there is a significant amount of analytical work where the computational times of integrals are needed to be carried out numerically. There is one concern that readers may have and that is the analytical part of the solution. In essence, the differentiation and integration involved, depending on the problem can get challenging at times. The results demonstrate that via few iterations of TAM, we can achieve



(a) Absolute error for Caputo derivative.



(b) Absolute error for Caputo-Fabrizio derivative.



(c) Absolute error for Atangana-Baleanu derivative.

Figure 2: Absolute errors for Example 5.3.

useful approximate solutions. We utilized TAM and presented a comparison between approximate solution with CD, CFD and ABD for the FRDEs. The results showed that the relations between total errors are $CD < CFD < ABD$. In addition, the results show the order of CPU speed in calculation in Mathematica are $CFD < ABD < CD$.

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