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Module Bounded Approximate Amenability of Banach Algebras

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Abstract. In this study we continue an investigation of the notion of module approximate amenability of a Banach algebra $\mathcal{A}$ which is a module over another Banach algebra $\mathfrak{A}$. In fact we introduce the class of module boundedly approximately amenable Banach algebras (*m.b.app.am.*) . It is shown that the class of module boundedly approximately amenable Banach algebra is different from the class of amenable Banach algebras. Also, we show that for an inverse semigroup S with the set of idempotent E, $l^1(S)$ is module boundedly approximately amenable as $l^1(E)$-module if and only if S is amenable. Further examples are given of l^1-semigroup Banach algebras which are module boundedly approximately amenable but are not amenable.

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1 Introduction

The concept of approximate amenability was introduced by Ghahramani and Loy in 2004 [6]. They showed that the class of approximately amenable Banach algebras is larger than the class of amenable Banach algebras. Also, they proved that the group algebra $L^1(G)$ is approximately amenable if and only if G is amenable, but this fails to be true for any discrete semigroup S. In fact for any semigroup S just approximately amenability of $l^1(S)$ implies the amenability of S [7]. Also, they introduced the class of boundedly approximately amenable Banach algebras. Ghahramani and Read built a boundedly approximately amenable Banach algebra which has no right bounded approximate identity [8, Corollary 3.2], and so it is not amenable.

Amini considered a Banach algebra $\mathcal{A}$ over another Banach algebra $\mathfrak{A}$ as an $\mathfrak{A}$-module and introduced the concept of module amenability of Banach algebras [1]. He showed that under some natural conditions, for an inverse semigroup S with the set of idempotent E, $l^1(S)$ is $l^1(E)$-module amenable if and only if S is amenable. Amini defined a bounded virtual diagonal for $\mathcal{A}$ and proved that existing this diagonal implies the module amenability of $\mathcal{A}$. Yazdanpanah and Najafi defined the module approximate amenability of Banach algebras [13]. Pourmahmood and Bodaghi investigated the notions of module approximate amenability and module approximate contractibility for Banach algebras [12]. They showed that the classes of module approximately amenable and module approximately contractible Banach algebras are the same. They defined the unital Banach algebra $\mathcal{B} = \mathcal{A} \oplus \mathfrak{A}^{\#}$ as $\mathfrak{A}^{\#}$-module unitization of $\mathcal{A}$ which also is a $\mathfrak{A}^{\#}$-module with compatible actions and proved that the module approximate amenability (contractibility) of $\mathcal{A}$ and $\mathcal{B}$ is equivalent. Similar to module amenability, in approximate version for an inverse semigroup S with the set of idempotent E they concluded that $l^1(S)$ is $l^1(E)$-module approximately amenable if and only if S is amenable. As amenability, module version of another cohomological notion of Banach algebras such as module approximate biprojectivity and module approximate biflatness are verified recently in [3].

In this paper we consider $\mathcal{A}$ as an $\mathfrak{A}$-module Banach algebra and introduce the bounded version of $\mathfrak{A}$-module approximate amenability of $\mathcal{A}$. Here we show that the module bounded approximate amenability of

$\mathcal{A}$ and $\mathcal{B}$ are equivalent. Also, we prove that the existence of a net in $(\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^{\#}}\mathcal{B})^{**}$ is equivalent to module bounded approximate amenability of $\mathcal{B}$.

Also, we get that, for an inverse semigroup S with the set of idempotent E, the equivalence relation between amenability of S and module approximate amenability of $l^1(S)$ (as an $l^1(E)$-module) is true in boundedly version.

Throughout the paper, we shall use the abbreviation *m.b.app.am.* for module boundedly approximately amenable, *b.a.i.* for bounded approximate identity, *m.b.r.a.i.* for multiplier-bounded right approximate identity and *m.b.l.a.i.* for multiplier-bounded left approximate identity.

2 Notations and preliminaries

We first recall some definitions . Let $\mathcal{A}$ be a Banach algebra, and X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to X$ is called a derivation if

$$D(a \cdot b) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

For each $x \in X$, we define the map $ad_x : \mathcal{A} \to X$ by

$$ad_x(a) = a \cdot x - x \cdot a \qquad (a \in X). \tag{1}$$

It is easy to see that ad_x is a derivation. Derivations of this form are called *inner derivations*.

A derivation $D : \mathcal{A} \to X$ is said to be boundedly approximately inner if there exists a net $(\xi_i) \subset X$ such that

$$D(a) = \lim_i ad_{\xi_i}(a) \qquad (a \in \mathcal{A})$$

and

$$\exists L > 0 : \sup \|ad_{\xi_i}(a)\| \leq L\|a\| \quad (a \in \mathcal{A}).$$

A Banach algebra $\mathcal{A}$ is boundedly approximately amenable if every bounded derivation $D : \mathcal{A} \to X^*$ is boundedly approximately inner, for each Banach $\mathcal{A}$-bimodule X, where X^* denotes the first dual of X which is a Banach $\mathcal{A}$-bimodule in the canonical way.

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions as follows:

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \qquad (ab) \cdot \alpha = a(b \cdot \alpha) \qquad (a, b \in \mathcal{A},\ \alpha \in \mathfrak{A}).$$

Let X be a left Banach $\mathcal{A}$-module and a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \quad (a \cdot x) \cdot \alpha = a \cdot (x \cdot \alpha), \quad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x,$$

for all $x \in X$, $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$ then X is called a left Banach $\mathcal{A}$-$\mathfrak{A}$-module, right and $\mathcal{A}$-$\mathfrak{A}$-bimodule are defined similarly. Moreover, if $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then X is called a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. Some examples of commutative and non-commutative $\mathcal{A}$-$\mathfrak{A}$-modules are given in [11]. If X is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then X^* is too, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as usual:

$$\begin{array}{ccc} < F \cdot \alpha, x > = < F, \alpha \cdot x > & , & < \alpha \cdot F, x > = < F, x \cdot \alpha > \\ < F \cdot a, x > = < F, a \cdot x > & , & < a \cdot F, x > = < F, x \cdot a > \end{array}$$

for all $\alpha \in \mathfrak{A}$, $a \in \mathcal{A}$, $x \in X$ and $F \in X^*$.

Note that, in general, $\mathcal{A}$ is not an $\mathcal{A}$-$\mathfrak{A}$-module because $\mathcal{A}$ does not satisfy in the compatibility condition $a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b$ for all $\alpha \in \mathfrak{A}$ and $a, b \in \mathcal{A}$. But when A is a commutative Banach $\mathfrak{A}$-module and acts on itself by multiplication, it is an $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A ($\mathfrak{A}-$)module derivation is a bounded map $D : \mathcal{A} \to X$ such that

$$\begin{aligned} D(a \pm b) &= D(a) \pm D(b) \\ D(a \cdot b) &= a \cdot D(b) + D(a) \cdot b \end{aligned}$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \qquad D(a \cdot \alpha) = D(a) \cdot \alpha \qquad (\alpha \in \mathfrak{A},\ a \in \mathcal{A})$$

Although D is not necessarily $\mathbb{C}$-linear, but still its boundedness implies its norm continuity. When X is a commutative $\mathfrak{A}$-bimodule, each $x \in X$ defines an *inner* module derivation as follows

$$ad_x(a) = a \cdot x - x \cdot a \quad (a \in A). \tag{2}$$

Remark that if $\mathcal{A}$ is a left (right) essential $\mathfrak{A}$-module, then every $\mathfrak{A}$-module derivation is also a derivation [12], in fact, it is $\mathbb{C}$-linear. If for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module derivation $D : \mathcal{A} \to X^*$ is inner, then $\mathfrak{A}$ is called module amenable (as an $\mathfrak{A}$-module).

Definition 2.1. Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras and $\mathcal{A}$ be an $\mathfrak{A}$-bimodule with compatible actions. Then $\mathcal{A}$ is module boundedly approximately amenable (*m.b.app.am.*) as an $\mathfrak{A}$-module if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module derivation $D : \mathcal{A} \to X^*$ is boundedly approximately inner;

Note that a left Banach $\mathfrak{A}$-module X is called left $\mathfrak{A}-$essential if the linear span of $\mathfrak{A} \cdot X = \{\alpha \cdot x \ : \ \alpha \in \mathfrak{A}, \ x \in X\}$ is dense in X. Right essential $\mathfrak{A}$-modules and two-sided essential $\mathfrak{A}$-bimodules are defined similarly.

Proposition 2.2. *Let $\mathcal{A}$ be b.app.am. that is essential as one-sided Banach $\mathfrak{A}$-module. Then $\mathcal{A}$ is m.b.app.am..*

Proof. According to descriptions above Definition 2.1 and our assumptions any module derivation is a derivation, so we conclude our proof. □

We will give Example 4.3 -(i) to show that the converse is not true in general.

Let $\mathcal{A}\widehat{\otimes}\mathcal{A}$ be the projective tensor product of $\mathcal{A}$ which is a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule. Now consider the module projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ which is the quotient space $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}}$ where $I_{\mathcal{A}}$ is the closed linear span of $\big\{a\cdot\alpha\otimes b - a\otimes\alpha\cdot b : \ \alpha\in\mathfrak{A}, \, a, b \in \mathcal{A}\big\}$. Also, consider the closed ideal $J_{\mathcal{A}}$ of $\mathcal{A}$ generated by the elements $(a\cdot\alpha)b - a(\alpha\cdot b)$ for $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$.

It follows that $I_{\mathcal{A}}$ and $J_{\mathcal{A}}$ are both $\mathcal{A}$-submodules and $\mathfrak{A}$-submodules of $(\mathcal{A}\widehat{\otimes}\mathcal{A})$ and $\mathcal{A}$, respectively. Both of the quotients $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ and $\mathcal{A}/J_{\mathcal{A}}$ are $\mathcal{A}$-modules and $\mathfrak{A}$-modules. Also, $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})$ is a $\mathcal{A}$-$\mathfrak{A}$-module if $\mathcal{A}$ is a $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, when $\mathcal{A}$ acts on $A/J_{\mathcal{A}}$ canonically, then $\mathcal{A}/J_{\mathcal{A}}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module

Consider $\omega_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ defined by $\omega_{\mathcal{A}}(a\otimes b) = ab$, $(a, b \in \mathcal{A})$ and extended by linearity. Then both ω and its second conjugate ω^{**} are

$\mathcal{A}$-module homomorphisms. We define $\tilde{\omega}_{\mathcal{A}}: (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}) = (\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}} \longrightarrow \mathcal{A}/J_{\mathcal{A}}$ by

$$\tilde{\omega}_{\mathcal{A}}(a\widehat{\otimes}b + I_{\mathcal{A}}) = ab + J_{\mathcal{A}}. \quad , \quad (a, b \in \mathcal{A}).$$

We denote by $\Box$ the first Arens product on $\mathcal{A}^{**}$, the second dual of $\mathcal{A}$. We assume that $\mathcal{A}^{**}$ is equipped with the first Arens product.

For a Banach algebra $\mathfrak{A}$, its unitization, denoted by $\mathfrak{A}^{\#}$, is the Banach algebra $\mathfrak{A} \oplus \mathbb{C}$ with the multiplication

$$(u, \alpha)(v, \beta) = (uv + \beta u + \alpha v, \alpha\beta) \qquad (u, v \in \mathfrak{A},\ \alpha, \beta \in \mathbb{C}).$$

Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions and let $\mathcal{B} = (\mathcal{A} \oplus \mathfrak{A}^{\#}, \bullet)$, where the multiplication $\bullet$ is defined through

$$(a, u) \bullet (b, v) = (ab + a \cdot v + u \cdot b, uv) \qquad (\ a, b \in \mathcal{A},\ \ u, v \in \mathfrak{A}^{\#}).$$

$\mathcal{B}$ is called the module unitization of $\mathcal{A}$. Consider the module actions of $\mathfrak{A}^{\#}$ on $\mathcal{B}$ as follows:

$$u \cdot (a, v) = (u \cdot a, uv), \ \ (a, v) \cdot u = (a \cdot u, vu) \qquad (\ a \in \mathcal{A},\ \ u, v \in \mathfrak{A}^{\#}).$$

Then $\mathcal{B}$ is a unital Banach algebra and a Banach $\mathfrak{A}^{\#}$-bimodule with compatible actions.

Proposition 2.3. *Let $\mathcal{A}$ be a Banach algebra and an $\mathfrak{A}$-bimodule with compatible actions. Then the following are equivalent:*

(i) *$\mathcal{A}$ is $\mathfrak{A}^{\#}$-module boundedly approximately amenable;*

(ii) *$\mathcal{B}$ is $\mathfrak{A}^{\#}$-module boundedly approximately amenable;*

If, in addition $\mathcal{A}$ is a left or right essential $\mathfrak{A}$-module, then (i) and (ii) are equivalent to

(iii) *$\mathcal{A}$ is $\mathfrak{A}$-module boundedly approximately amenable.*

Proof. Since every $\mathfrak{A}^{\#}$-module derivation on $\mathcal{B}$ reduces to a $\mathfrak{A}^{\#}$-module derivation from $\mathcal{A}$, by vanishing on $\mathfrak{A}^{\#}$, the proposition can be proved in essentially the same way as [12, Theorem 3.1]. $\Box$

The following lemma which is analogous to [12, Lemma 3.2], will be used in the proof of Theorem 3.1 implicitly.

Lemma 2.4. *If $\mathcal{A}$ has a bounded approximate identity, then it is module boundedly approximately amenable iff every $\mathfrak{A}$-module derivation $D : \mathcal{A} \to X^*$ is boundedly approximately inner for each commutative $\mathcal{A}$-pseudo-unital Banach $\mathcal{A}$-$\mathfrak{A}$-module X.*

3 Bounded approximate module amenability of Banach algebras

In this section we provide some equivalent conditions for the module bounded approximate amenability in terms of diagonal for $\mathcal{B}$ with results related to the existence of bounded approximate identity for $\mathcal{A}$. It is shown that if $\mathcal{A}^{**}$ is *m.b.app.am.*, so is $\mathcal{A}$ when $\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module and $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})$ is commutative as $\mathcal{A}$-$\mathfrak{A}$-module. Finally, the $l^1(E)$-module bounded approximate amenability of $l^1(S)$ and $l^1(S)^{**}$ are characterized where S is an inverse semigroups with the set of idempotent elements E.

Note that Example 6.1 in [6] is a non-amenable Banach algebra that is boundedly approximately amenable [7, Remark 5.2]. So two notions 'bounded approximate amenability' and 'amenability' do not coincide. Since these are the special cases of module bounded approximate amenability and module amenability with $\mathfrak{A}$=$\mathbb{C}$, respectively, then module bounded approximate amenability *and* module amenability are different notions.

Now we prove a proposition for *m.b.app.am.* Banach algebras, as follows:

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions. Let also $\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B}$ be commutative as a $\mathfrak{A}^\#$-module. Then the following are equivalent:*

(i) *$\mathcal{B}$ is m.b.app.am. as a $\mathfrak{A}^\#$-module;*

(ii) *There exist a net $(M_i) \subset (\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B})^{**}$ and $L > 0$ such that for all $b \in B$, $b \cdot M_i - M_i \cdot b \longrightarrow 0$, $\|b \cdot M_i - M_i \cdot b\| \leq L\|b\|$, $\tilde{\omega}_{\mathcal{B}}^{**}(M_i) \to 1_{\mathcal{B}}$ and $\tilde{\omega}_{\mathcal{B}}^{**}(M_i)$ is bounded;*

(iii) *There exist a net $(M_i) \subset (\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B})^{**}$ and $L > 0$ such that for* *all* $b \in \mathcal{B}$, $b \cdot M_i - M_i \cdot b \longrightarrow 0$, $\|b \cdot M_i - M_i \cdot b\| \leq L\|b\|$ *and* $\tilde{\omega}_{\mathcal{B}}^{**}(M_i) = 1_{\mathcal{B}}$.

Proof. (**i**) $\Longrightarrow$ (**iii**): Let $F = 1 \otimes_{\mathfrak{A}^\#} 1$. It is straightforward to check that the inner derivation $D_F : \mathcal{B} \to (\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B})^{**}$ satisfies $D_F(\mathcal{B}) \subset \ker \tilde{\omega}_{\mathcal{B}}^{**} = (\ker \tilde{\omega}_{\mathcal{B}})^{**}$, and so there exist a net $(N_i) \subset (\ker \tilde{\omega}_{\mathcal{B}}^{**})$ and a constant $k > 0$ such that $D_{N_i}(b) \longrightarrow D_F(b)$ and $\|D_{N_i}(b)\| \leq k\|b\|$ for all $b \in \mathcal{B}$. Letting $M_i = F - N_i$ for all i, we have

$$\tilde{\omega}_{\mathcal{B}}^{**}(M_i) = \tilde{\omega}_{\mathcal{B}}^{**}(F) - \tilde{\omega}_{\mathcal{B}}^{**}(N_i) = 1_{\mathcal{B}} - 0 = 1_{\mathcal{B}},$$

$$b \cdot M_i - M_i \cdot b = D_F(b) - D_{N_i}(b) \longrightarrow 0$$

and

$$\begin{aligned} \|b \cdot M_i - M_i \cdot b\| &\leq \parallel D_F(b) \parallel + \parallel D_{N_i}(b) \parallel \\ &\leq (\parallel D_F \parallel + k) \parallel b \parallel . \end{aligned}$$

Therefore (**iii**) holds for $L = \parallel D_F \parallel + k$.
(**iii**) $\Longrightarrow$ (**ii**): is obvious.
(**ii**) $\Longrightarrow$ (**i**): It is similar to [12, Theorem 3.3], with this additional notion that $\sup_i \|\tilde{\omega}_{\mathcal{B}}^{**}(M_i)\| < \infty$. So we have

$$\begin{aligned} \|ad_{f_i}(b)\| &\leq \|F\|\|b \cdot M_i - M_i \cdot b\| + \|D(b)\|\|\tilde{\omega}_{\mathcal{B}}^{**}(M_i)\| \\ &\leq \|D\|\|b\|L + \|D\|\|b\| \sup_i \|\tilde{\omega}_{\mathcal{B}}^{**}(M_i)\|, \end{aligned}$$

for all i and $b \in \mathcal{B}$. So $\|ad_{f_i}(b)\| \leq K\|b\|$ for all $b \in \mathcal{B}$, where $K = \|D\|(L + \sup_i \|\tilde{\omega}_{\mathcal{B}}^{**}(M_i)\|)$. $\square$

Remark that by using [4, Lemma 3.1] we can conclude that when $\mathcal{A}$ is *m.b.app.am.* as a commutative Banach $\mathfrak{A}$-module, it has left and right approximate identity.

Theorem 3.2. *Suppose that $\mathcal{A}$ is a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions which is m.b.app.am.. Also, let $\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B}$ be commutative as Banach $\mathfrak{A}^\#$-module. Then exist a constant $L > 0$, nets $(m_i) \subset \left(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A}\right)^{**}$ and $(a_i), (b_i) \subset \mathcal{A}^{**}$ such that for all $a \in \mathcal{A}$ we have*

(i) $\tilde{\omega}_{\mathcal{A}}^{**}(m_i) = a_i + b_i$;

(ii) $b_i \cdot a \longrightarrow a,\ \|b_i \cdot a\| \leq L\|a\|$ *for all* i*;*

(iii) $a \cdot a_i \longrightarrow a,\ \|a \cdot a_i\| \leq L\|a\|$ *for all* i*;*

(iv) $a \cdot m_i - m_i \cdot a + a_i \otimes a - a \otimes b_i \longrightarrow 0,$
$\|a \cdot m_i - m_i \cdot a + a_i \otimes a - a \otimes b_i\| \leq L\|a\|$ *for all* i*.*

Proof. By Theorem 3.1 there is a net $(M_i) \subset (\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B})^{**}$ and a constant $L > 0$ satisfying $b \cdot M_i - M_i \cdot b \longrightarrow 0$, $\|b \cdot M_i - M_i \cdot b\| \leq L\|b\|$ and $\tilde{\omega}_{\mathcal{B}}^{**}(M_i) = 1_{\mathcal{B}}$ for all $b \in \mathcal{B}$. Following

$$\begin{aligned}(\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B})^{**} &= ((\mathcal{A}\oplus\mathfrak{A}^\#)\widehat{\otimes}_{\mathfrak{A}^\#}(\mathcal{A}\oplus\mathfrak{A}^\#))^{**}\\ &= (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A})^{**}\oplus(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathfrak{A}^\#)^{**}\\ &\oplus (\mathfrak{A}^\#\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A})^{**}\oplus(\mathfrak{A}^\#\widehat{\otimes}_{\mathfrak{A}^\#}\mathfrak{A}^\#)^{**},\end{aligned}$$

we can write

$$M_i' = m_i - (a_i \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#}) - (1_{\mathfrak{A}^\#} \otimes_{\mathfrak{A}^\#} b_i) + (t_i \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#})\ ,$$

for some $(m_i) \subset \big(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A}\big)^{**}$, $(a_i), (b_i) \subset \mathcal{A}^{**}$, and $(t_i) \subset (\mathfrak{A}^\#)^{**}$. Applying $\tilde{\omega}_{\mathcal{B}}^{**}(M_i) = 1_{\mathcal{B}}$ yields

$$\tilde{\omega}_{\mathcal{A}}^{**}(m_i) - a_i - b_i + t_i = 1_{\mathcal{B}} = (0,1) \in (\mathcal{A}\oplus\mathfrak{A}^\#).$$

This follows that $\tilde{\omega}_{\mathcal{A}}^{**}(m_i) - a_i - b_i = 0$, and $t_i = 1$, for all i. Also, we have

$$\begin{aligned} a\cdot M_i' - M_i'\cdot a &= \big((a\cdot m_i - m_i\cdot a) + (a_i \otimes_{\mathfrak{A}^\#} a) - (a \otimes_{\mathfrak{A}^\#} b_i)\big)\\ &+ (1_{\mathfrak{A}^\#} \otimes_{\mathfrak{A}^\#} b_i a - 1_{\mathfrak{A}^\#} \otimes_{\mathfrak{A}^\#} a)\\ &+ (-a a_i \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#} + a \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#}) \longrightarrow 0, \end{aligned} \tag{3}$$

for all $a \in \mathcal{A}$. Hence

$$\big((a\cdot m_i - m_i\cdot a) + (a_i \otimes_{\mathfrak{A}^\#} a) - (a \otimes_{\mathfrak{A}^\#} b_i)\big) \longrightarrow 0,$$

$$(1_{\mathfrak{A}^\#} \otimes_{\mathfrak{A}^\#} b_i a - 1_{\mathfrak{A}^\#} \otimes_{\mathfrak{A}^\#} a) \longrightarrow 0,$$

$$(-a a_i \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#} + a \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#}) \longrightarrow 0,$$

we may conclude

$$1_{\mathfrak{A}^\#} \otimes_{\mathfrak{A}^\#} (b_i a - a) \longrightarrow 0 \qquad \Longrightarrow \qquad b_i a \longrightarrow a,$$
$$(aa_i - a) \otimes_{\mathfrak{A}^\#} 1_{\mathfrak{A}^\#} \longrightarrow 0 \qquad \Longrightarrow \qquad aa_i \longrightarrow a,$$

for all $a \in \mathcal{A}$. The left side of 3 is bounded by $L\|a\|$ for all i and $a \in \mathcal{A}$, then we get

$$\|a \cdot m_i - m_i \cdot a + a_i \otimes a - a \otimes b_i\| < L\|a\|,$$
$$\|b_i a\| \leq L\|a\|,$$
$$\|aa_i\| \leq L\|a\|.$$

□

Theorem 3.3. *Suppose that $\mathcal{A}$ is a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions which is m.b.app.am. and has both m.b.l.a.i. and m.b.r.a.i.. Also $\mathcal{B}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{B}$ is commutative as Banach $\mathfrak{A}^\#$-module. Then $\mathcal{A}$ has a b.a.i..*

Proof. Let (f_γ) and (e_β) be left and right multiplier-bounded approximate identities for $\mathcal{A}$, respectively. So there is $K > 0$ such that

$$\|a \cdot e_\beta\| \leq K\|a\|, \quad \|f_\gamma \cdot a\| \leq K\|a\| \tag{4}$$

for all $a \in \mathcal{A}$ and for all β, γ. From this relation and projective tensor norm we have

$$\|f_\gamma \cdot m\|_{\widehat{\otimes}} = \Big\| \sum_{n=1}^{\infty} f_\gamma \cdot a_n \otimes b_n \Big\|_{\widehat{\otimes}} \leq K \sum_{n=1}^{\infty} \|a_n\| \, \|b_n\| \qquad (m \in \mathcal{A}\widehat{\otimes}\mathcal{A})$$

for any representation $m = \sum_{n=1}^{\infty} a_n \otimes b_n$, and so $\|f_\gamma \cdot m\|_{\widehat{\otimes}} \leq K\|m\|_{\widehat{\otimes}}$. By passing to the quotient we have $\|f_\gamma \cdot m\|_{\widehat{\otimes}_{\mathfrak{A}^\#}} \leq K\|m\|_{\widehat{\otimes}_{\mathfrak{A}^\#}}$ for all $m \in \big(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A}\big)$ and all γ, where the index $\widehat{\otimes}_{\mathfrak{A}^\#}$ in the norm, denotes the norm on $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A}$ that from now on, it will be omitted.

According to Goldestine's Theorem for any $T \in (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A})^{**}$ there exists a net $(m_j) \subseteq \mathcal{A}\widehat{\otimes}_{\mathfrak{A}^\#}\mathcal{A}$ such that $m_j \xrightarrow{w^*} T$ and $\sup_j \|m_j\| \leq \|T\|$.

Using this and the ω^*-continuity of the left module action of $\mathcal{A}$ on $(A\widehat{\otimes}_{\mathfrak{A}\#}A)^{**}$ yield

$$f_\gamma \cdot m_j \xrightarrow{w^*} f_\gamma \cdot T, \quad \|f_\gamma \cdot m_j\| \leq K\|m_j\| \leq K\|T\|.$$

So $\|f_\gamma \cdot T\| \leq K\|T\|$. By the same argument we have

$$\|m \cdot e_\beta\| \leq K\|m\|, \quad \|T \cdot e_\beta\| \leq K\|T\|, \tag{5}$$

for all $m \in \mathcal{A}\widehat{\otimes}_{\mathfrak{A}\#}\mathcal{A}$ and $T \in (\mathcal{A}\widehat{\otimes}_{\mathfrak{A}\#}\mathcal{A})^{**}$.

Let the nets (a_i) and (b_i) and the constant L satisfy in the previous theorem. Suppose, on the contrary, that the net (f_γ) is unbounded. According to Theorem 3.2-(iv) for every i and γ we have

$$\|f_\gamma \cdot m_i - m_i \cdot f_\gamma - f_\gamma \otimes b_i + a_i \otimes f_\gamma\| \leq L\|f_\gamma\|.$$

Applying (5) gives

$$\|\big(f_\gamma \cdot m_i - m_i \cdot f_\gamma - f_\gamma \otimes b_i + a_i \otimes f_\gamma\big) \cdot e_\beta\| \leq KL\|f_\gamma\|,$$

for all i, β and γ. Utilizing this relation, the triangle inequality and left-multiplier boundedness of the net (f_γ) we get

$$\begin{aligned}\|f_\gamma\|\|b_i \cdot e_\beta\| &\leq KL\|f_\gamma\| + \|f_\gamma \cdot (m_i \cdot e_\beta)\| + \|m_i \cdot (f_\gamma \cdot e_\beta)\| \\ &\quad + \|a_i \cdot (f_\gamma \cdot e_\beta)\| \\ &\leq KL\|f_\gamma\| + 2K\|m_i\|\|e_\beta\| + K\|a_i\|\|e_\beta\|,\end{aligned} \tag{6}$$

for all i, β and γ. So we have

$$\|b_i \cdot e_\beta\| \leq KL + \tfrac{1}{\|f_\gamma\|}\big(2K\|m_i\|\|e_\beta\| + K\|a_i\|\|e_\beta\|\big).$$

For fixed i and β, our assumption regarding unboundedness of (f_γ) implies:

$$\|b_i \cdot e_\beta\| \leq KL.$$

Taking limits with respect to i, according to Theorem 3.2, we obtain $\|e_\beta\| \leq KL$ for each β. Using (e_β) as a right approximate identity and (f_γ) as a $m.b.l.a.i$ and then, applying the latter inequality we find out

$$\|f_\gamma\| = \lim_\beta \|f_\gamma \cdot e_\beta\| \leq \lim_\beta K\|e_\beta\| \leq K^2L$$

for all γ. This contradicts our assumption that the net (f_γ) is unbounded.

A similar argument shows that the net (e_β) is also bounded. Therefore, $\mathcal{A}$ has a bounded approximate identity. $\square$

Remark 3.4. Ghahramani and Read made a Banach algebra $\mathcal{A}$ which was *b.app.am*, but they proved that $\mathcal{A} \oplus \mathcal{A}^{op}$ is not *app.am* [8, Theorem 4.1]. So the direct sum of two *m.b.app.am.* Banach algebras is not necessarily *m.b.app.am.*.

Proposition 3.5. *Suppose that $\mathcal{A}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module and $(\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A})$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. If $\mathcal{A}^{**}$ is m.b.app.am., so is $\mathcal{A}$.*

Proof. Consider Theorem 3.1 for $\mathcal{B}^{**}$, since the role of $\mathcal{B}^{**}$ for $\mathcal{A}^{**}$ is the same as the role of $\mathcal{B}$ for $\mathcal{A}$. We follow the notations of [12, Proposition 3.7], the proof is similar to the proof of this proposition with these additional assumptions:

(i) $\tilde{\omega}^{**}_{\mathcal{B}^{**}}(\theta_j)$ is bounded;

(ii) $\|b \cdot \theta_j - \theta_j \cdot b\| < L\|b\|$ for all $b \in \mathcal{B}^{**}$ and $L > 0$.

Since Ω_u is a bounded mapping, T is canonical embedding and Ω_u and T and their adjoints are $\mathcal{B}$-$\mathfrak{A}^{\#}$-module homomorphisms, then for $M_j = T^*\big(\Omega_u^{**}(\theta_j)\big)$ exists a $C > 0$ such that $\|b \cdot M_j - M_j \cdot b\| \leq C\|b\|$, for all $b \in \mathcal{B}$.

Actually λ and its adjoint are $\mathcal{B}$-$\mathfrak{A}^{\#}$-module homomorphisms, so according to the proof of [12, Proposition 3.7] we have $\tilde{\omega}^{**}_{\mathcal{B}}(M_j) = \lambda^{**}\big(\tilde{\omega}^{**}_{\mathcal{B}^{**}}(\theta_j)\big)$. Moreover λ and its adjoint are continuous, so $\tilde{\omega}^{**}_{\mathcal{B}}(M_j)$ is bounded. $\square$

We can get *m.b.app.am.* version of Johnson's Theorem for inverse semigroups. For an inverse semigroups S with the set of idempotent elements E, in fact E is a commutative subsemigroup of S, so $l^1(E)$ is a commutative subalgebra of $l^1(S)$. Suppose that $l^1(E)$ acts on $l^1(S)$ and its second dual with trivial left action $\delta_e \cdot \delta_s = \delta_s$ and the right action $\delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e$ for all $e \in E$ and $s \in S$. So $l^1(S)$ is a Banach $l^1(E)$-module with compatible actions [1]. Hence the closed ideal $J_{l^1(S)}$ of $\mathcal{A} = l^1(S)$ is the closed linear span of $\{\delta_{set} - \delta_{st} \quad : s,t \in S, e \in E\}$.

Now consider the equivalence relation $\approx$ on S as $s \approx t$ if and only if $\delta_s - \delta_t \in J_{l^1(S)}$, for all $s, t \in S$. We can bring our intended propositions.

The next proposition holds because the m.am version ([1, Theorem 3.1]) and m.app.am version ([12, Theorem 3.9]) hold.

Proposition 3.6. *Let S be an inverse semigroup with the idempotent elements set E. Then $l^1(S)$ is m.b.app.am. as $l^1(E)$-module iff S is amenable.*

Applying both results ([11, Theorem 2.11]) and ([12, Theorem 3.10]) yields the following proposition.

Proposition 3.7. *Suppose that S is an inverse semigroup with the set of idempotent elements E. Then $l^1(S)^{**}$ is m.b.app.am. as $l^1(E)$-module iff $\frac{S}{\approx}$ is finite.*

4 Examples

Example 4.1. Let $(\mathcal{A}_n)$ be a sequence of amenable Banach algebras. According to [7, Remark 5.2] the Banach algebra $\mathcal{C} = c_0 - \oplus_{n=1}^{\infty} \mathcal{A}_n^{\#}$ is *b.app.am*. Then $\mathcal{C}$ is *m.b.app.am.* as $\mathbb{C}$-module. If their amenability constant $M(\mathcal{A}_n)$ (the infimum of the norms of virtual diagonals of $\mathcal{A}_n$) tends to ∞, then $\mathcal{C}$ is not amenable.

Example 4.2. Suppose that $K(l^1)$ is the Banach algebra of all compact operators on $l^1 = \{(x_i) : \|x\|_1 = \sum |x_i| < \infty, x_i \in \mathbb{C}\}$. According to [8, Lemma 2.4] the Banach algebra $\mathcal{A}^{(n)} = (K(l^1), \|.\|_n)$ has a $l.b.a.i$ with the bound 1 but the smallest bound of any $r.b.a.i$ in $\mathcal{A}^{(n)}$ is $n+1$. Thus the Banach algebra $\mathcal{A} = c_0 - \oplus_{n=1}^{\infty} \mathcal{A}^{(n)}$ has a $l.b.a.i$ but has no $m.b.r.a.i$. We can consider $\mathcal{A} = c_0 - \oplus_{n=1}^{\infty} \mathcal{A}^{(n)}$ as a (commutative) Banach $\mathbb{C}$-module which is *m.b.app.am.* but has no *b.a.i.* (so according to [1, Proposition 2.2] is not *m.am*).

In the next example we see some Banach algebras that are *m.b.app.am.* but are not *b.app.am* in the classical case.

Example 4.3. **(i)** Suppose that $\mathcal{C}$ is the bicyclic semigroup in two generators, then by [2], $\frac{\mathcal{C}}{\approx} \simeq \mathbb{Z}$. So $\frac{\mathcal{C}}{\approx}$ is infinite. Applying Proposition 3.7, $l^1(\mathcal{C})^{**}$ is not *m.b.app.am.* as $l^1(E)$-module.

$\mathcal{C}$ is amenable semigroup [5, Examples]. So by Proposition 3.6, $l^1(\mathcal{C})$ is *m.b.app.am.* as $l^1(E)$-module. However according to [9, Theorem], $l^1(\mathcal{C})$ is not *b.app.am.*.

(ii) Suppose that G is a group and I is a non-empty set and $S = M(G, I)$ is the *Brandt inverse semigroup* corresponding to the group G and the index set I. It is shown in [11, Example 3.2] that $\frac{S}{\approx}$ is trivial group. According to Proposition 3.7 $l^1(S)^{**}$ is *m.b.app.am*. Therefore $l^1(S)$ is *m.b.app.am* as $l^1(E)$-module by Proposition 3.5. However we can get from [10, Theorem 4.5]that $l^1(S)$ is *b.app.am* iff $l^1(S)$ is amenable iff I is finite and G is amenable.

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