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The Best Proximity Points for Weak $\mathcal{MT}$-cyclic Reich Type Contractions

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Abstract. In this paper, we introduce a weak $\mathcal{MT}$-cyclic Reich type contractions and obtain the existence theorems for best proximity point for self-mappings defined on the complete metric spaces. Our results improve and generalize some results in literature. Also, we give some applications of our results to solving some classes of non-linear integral and differential equations.

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1 Introduction

The theory of metric fixed point yields crucial tools to solve several differential and integral equations. One of the interesting topics of met-

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ric fixed point theory is the finding "best proximity point" in case of not reach the fixed point. On the other hand, instead of the considering a self-mappings, investigating the fixed point of cyclic mapping is very interesting and useful in nonlinear analysis (for example, see [20]). The concept of cyclic contraction mappings in uniformly convex Banach spaces is defined in [10] by Eldred and Veeramani. In the paper [10] they proved a best proximity point theorem for cyclic contraction.

The notion of cyclic Meir-Keeler contractions are introduced by Bari et al, see [4]. Bari et al proved the existence of a best proximity point for cyclic contractions in metric spaces in the case of two sets. For a cyclic map $f : A \cup B \to A \cup B$, Du and Lakzian [8] introduced a new class of maps called $\mathcal{MT}$-cyclic contraction with respect to function ϕ on $A \cup B$ which contains the cyclic contractions maps as a subclass (see Example A in [8]). Also, Du and Lakzian obtain some new existence and convergence theorems of best proximity points for cyclic contractions. Many authors have been investigated the existence, uniqueness and convergence of iterates to the best proximity point under weaker assumptions for a map f (see e. g. [2, 10, 16, 30, 31, 32, 41, 42]).

Afterward, Lakzian and Lin in [33] defined the concept of weak $\mathcal{MT}$-cyclic Kannan contractions with respect to function ϕ on $A \cup B$. Also, they established some new convergent and existence theorems of best proximity point theorems for cyclic contractions in uniformly Banach spaces, this results generalized a theorem by Petrić in [37]. See, also, e. g. [3, 9, 34, 35] for more details in this field.

Theorem 1.1. *[33] Let A and B be nonempty closed subsets of a complete metric space (X, d) such that $A \cap B \neq \emptyset$ and $T : A \cup B \to A \cup B$ be a weak $\mathcal{MT}$-cyclic Kannan contraction with respect to φ such that*

$$d(Tx, Ty) \leq \frac{1}{2}\varphi(d(x, y))[d(x, Tx) + d(y, Ty)] \text{ for any } x \in A \text{ and } y \in B.$$

Then T has a unique fixed point in $A \cap B$.

In this paper, first, we introduce a notion of weak $\mathcal{MT}$-cyclic Reich type contractions with respect to an $\mathcal{MT}$-function ϕ. Also, we shall give some new convergent and existence theorems for best proximity point theorems for self-mappings defined on a complete metric space. Our results improve and generalize some previous theorems in this field (see [16, 38, 39]). Also, we give some applications on non-linear integral and differential equations.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and $\mathbb{R}$ for the real numbers. Suppose that A and B are nonempty subsets of a nonempty set E. A map $f : A \cup B \to A \cup B$ is called a cyclic map if $f(A) \subset B$ and $f(B) \subset A$. Let (X, d) be a metric space and $f : A \cup B \to A \cup B$ a cyclic map. For any nonempty subsets A and B of X, let

$$dist(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

A point $x \in A \cup B$ is called a best proximity point for f if $d(x, fx) = dist(A, B)$. If $A \cap B$ is non-empty, then $dist(A, B) = 0$ and the best proximity point of f is no thing else than the fixed point of f.

For $c \in \mathbb{R}$, we recall that

$$\limsup_{x\to c} f(x) = \inf_{\varepsilon>0} \sup_{0<|x-c|<\varepsilon} f(x) \text{ and } \limsup_{x\to c^+} f(x) = \inf_{\varepsilon>0} \sup_{0<x-c<\varepsilon} f(x).$$

Definition 2.1. [6] A function $\phi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$-function if it satisfies Mizoguchi-Takahashi's condition (That is $\limsup_{s\to t+0} \phi(s) < 1$ for all $t \in [0, \infty)$).

Obviously, if $\phi : [0, \infty) \to [0, 1)$ is a nondecreasing or nonincreasing function, then ϕ is an $\mathcal{MT}$-function. So, in particular, if $\phi : [0, \infty) \to [0, 1)$ is defined by $\phi(t) = c$, where $c \in [0, 1)$, then ϕ is an $\mathcal{MT}$-function.

Remark 2.2. Note that if ϕ is an $\mathcal{MT}$-function then clearly $\psi := \frac{2\phi}{3-\phi}$ is an $\mathcal{MT}$-function.

Example 2.3. [6] Let $\phi : [0, \infty) \to [0, 1)$ be defined by

$$\phi(t) := \begin{cases} \frac{\sin t}{t} & , \text{if } t \in (0, \frac{\pi}{2}] \\ 0 & , \text{otherwise.} \end{cases}$$

Since $\limsup_{s\to 0^+} \phi(s) = 1$, ϕ is not an $\mathcal{MT}$-function.

For some characterizations of $\mathcal{MT}$-functions see e. g. [7].

3 Best proximity point results for weak $\mathcal{MT}$-cyclic Reich type contractions

In this section, we present our main results. We, first, introduce the weak $\mathcal{MT}$-cyclic Reich type contraction with respect to auxiliary $\mathcal{MT}$-function ϕ.

Definition 3.1. Let A and B be nonempty subsets of a metric space (X, d). Suppose that a map $f : A \cup B \to A \cup B$ satisfies

(MTR1) f is a cyclic map, i.e. $f(A) \subset B$ and $f(B) \subset A$;

(MTR2) there exists an $\mathcal{MT}$-function $\phi : [0, \infty) \to [0, 1)$ such that

$$\begin{aligned} d(fx, fy) \leq & \frac{1}{3}\phi(d(x, y))[d(x, y) + d(x, fx) + d(y, fy)] \\ & + (1 - \phi(d(x, y)))dist(A, B), \end{aligned}$$

for any $x \in A$ and $y \in B$, then f is called a weak $\mathcal{MT}$-cyclic Reich type contraction with respect to ϕ on $A \cup B$.
This contraction is said to be cyclic Reich type contraction, if $\phi \equiv \alpha$ for some $\alpha \in [0, 1)$.

Note that in the above definition, if $A \cap B \neq \emptyset$, then $dist(A, B) = 0$. and f becomes the mapping from $A \cap B$ into $A \cap B$ and $(MTR2)$ changes as follows:

$$d(fx, fy) \leq \frac{1}{3}\phi(d(x, y))[d(x, y) + d(x, fx) + d(y, fy)].$$

Du and Lakzian in [8] gave an example of a map f which is an $\mathcal{MT}$-cyclic contraction but not a cyclic contraction. It is easy to see that the same example is also an $\mathcal{MT}$-cyclic Reich type contraction but it is not a cyclic Reich type contraction; so the class of $\mathcal{MT}$-cyclic Reich type contractions are bigger than their cyclic Reich type contractions.

For the main results of this section, we need the following lemmas.

Lemma 3.2. *Let A and B be nonempty closed subsets of a metric space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying:*

(i) $d(x_n, y_n) \to dist(A, B)$.
(ii) $d(z_n, y_n) \to dist(A, B)$.
Then $d(x_n, z_n) \to 0$.

Proof. The proof is similar to [10]. $\square$

Lemma 3.3. *Let A and B be nonempty subsets of a metric space X and $f : A \cup B \to A \cup B$ be a map satisfying*

$$\begin{aligned} d(fx, f^2x) \leq & \frac{1}{3}\phi(d(x, fx))[2d(x, fx) + d(fx, f^2x)] \\ & + (1 - \phi(d(x, fx)))dist(A, B), \end{aligned}$$

for all $x \in A \cup B$ and an $\mathcal{MT}$-function ϕ. Then there is an $\mathcal{MT}$-function ψ such that

$$d(f^2x, fx) \leq \psi(d(fx, x))d(fx, x) + (1 - \psi(d(fx, x)))dist(A, B).$$

for each $x \in A \cup B$.

Proof. It sufficient to put $\psi = \frac{2\phi}{3-\phi}$. $\square$

Theorem 3.4. *Let A and B be nonempty closed subsets of a complete metric space X and $f : A \cup B \to A \cup B$ be a cyclic map satisfying*

$$\begin{aligned} d(fx, f^2x) \leq & \frac{1}{3}\phi(d(x, fx))[2d(x, fx) + d(fx, f^2x)] \\ & + (1 - \phi(d(x, fx)))dist(A, B), \end{aligned}$$

for all $x \in A \cup B$ and an $\mathcal{MT}$-function ϕ. Then

(i) $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = dist(A, B)$ *for all* $x \in A \cup B$.

(ii) $\lim_{n\to\infty} d(f^{2n}x, f^{2n+2}x) = 0 = \lim_{n\to\infty} d(f^{2n-1}x, f^{2n+1}x)$ *for all* $x \in A \cup B$.

(iii) *z is a best proximity point if and only if z is a fixed point of f^2.*

Proof. Let $x \in A \cup B$. Then for each $n \in \mathbb{N}$, $dist(A, B) \leq d(f^n x, f^{n+1}x)$. If there exists $j \in \mathbb{N}$ such that $f^j x = f^{j+1}x \in A \cap B$, then we have $\lim_{n\to\infty} d(f^n x, f^{n+1}x)| = 0$ and $dist(A, B) = 0$; therefore (i) is true. So it suffices to consider the case $f^{n+1}x \neq f^n x$, for each $n \in \mathbb{N}$.

The sequence $\{d(f^n x, f^{n+1}x)\}$ is bounded below and nondecreasing in $(0,\infty)$ and so it is convergent. Since for each $x \in A \cup B$,

$$\begin{aligned}
d(f^n x, f^{n+1}x) &\leq \frac{1}{3}\phi(d(f^{n-1}x, f^n x))[2d(f^{n-1}x, f^n x) + d(f^n x, f^{n+1}x)] \\
&+ (1 - \phi(d(f^{n-1}x, f^n x)))dist(A, B) \\
&\leq \frac{1}{3}\phi(d(f^{n-1}x, f^n x))[2d(f^{n-1}x, f^n x) + d(f^n x, f^{n+1}x)] \\
&+ \frac{1}{3}(1 - \phi(d(f^{n-1}x, f^n x)))[d(f^{n-1}x, f^n x) \\
&+ 2d(f^n x, f^{n+1}x)],
\end{aligned}$$

also,

$$\begin{aligned}
\phi(d(f^{n-1}x, f^n x)) + 1)d(f^n x, f^{n+1}x) \leq \\
(\phi(d(f^{n-1}x, f^n x)) + 1)d(f^{n-1}x, f^n x)
\end{aligned}$$

and therefore $d(f^n x, f^{n+1}x) \leq d(f^{n-1}x, f^n x)$.
Set

$$t := \lim_{n\to\infty} d(f^n x, f^{n+1}x). \tag{1}$$

Since $\psi = \frac{2\phi}{3-\phi}$ is an $\mathcal{MT}$-function, by [5, Remark 2.5 (iii)], there exist $r_t \in [0, 1)$ and $\varepsilon_t > 0$ such that $\psi(s) \leq r_t$ for all $s \in [t, t + \varepsilon_t)$. By (1), there exists $\ell \in \mathbb{N}$, such that

$$t \leq d(f^n x, f^{n+1}x) < t + \varepsilon_t$$

for all $n \in \mathbb{N}$ with $n \geq \ell$. Hence $\psi(d(f^n x, f^{n+1}x)) \leq r_t$ for all $n \geq \ell$. Let

$$\lambda := \max\{\psi(d(f^1 x, f^2 x)), \psi(d(f^2 x, f^3 x)), \cdots, \psi(d(f^{\ell-1}x, f^\ell x)), r_t\}.$$

Then $0 \leq \psi(d(f^n x, f^{n+1}x)) \leq \lambda < 1$ for all $n \in \mathbb{N}$. Note that by Lemma 3.3

$$\begin{aligned}
d(f^n x, f^{n+1}x) &\leq \frac{1}{3}\phi(d(f^{n-1}x, f^n x))[2d(f^{n-1}x, T^n x) + d(f^n x, f^{n+1}x)] \\
&+ (1 - \phi(d(f^{n-1}x, f^n x)))dist(A, B),
\end{aligned}$$

implies that

$$\begin{aligned} d(f^n x, f^{n+1} x) &\leq \psi(d(f^{n-1} x, f^n x)) d(f^{n-1} x, f^n x) \\ &\quad + (1 - \psi(d(f^{n-1} x, f^n x))) dist(A, B) \\ &\leq \psi(d(f^{n-1} x, f^n x)) d(f^{n-1} x, f^n x) + dist(A, B) \\ &\leq \lambda d(f^{n-1} x, f^n x) + dist(A, B). \end{aligned}$$

Inductively we conclude that

$$dist(A, B) \leq d(f^n x, f^{n+1} x) \leq \lambda^n d(x, fx) + dist(A, B). \tag{2}$$

Since $\lambda \in (0,1)$, we have $\lim_{n\to\infty} \lambda^n = 0$. Therefore (2) implies that $\lim_{n\to\infty} d(f^n x, f^{n+1} x) = dist(A, B)$. So (i) is proved.

For (ii), let $x \in A \cup B$. Then by (i), $\lim_{n\to\infty} d(f^{2n} x, f^{2n+1} x) = dist(A, B)$ and $\lim_{n\to\infty} d(f^{2n+2} x, f^{2n+1} x) = dist(A, B)$. Lemma 2.2 concludes that

$$\lim_{n\to\infty} d(f^{2n} x, f^{2n+2} x) = 0,$$

for any $x \in A \cup B$. Similarly, since

$$\lim_{n\to\infty} d(f^{2n} x, f^{2n+1} x) = dist(A, B) = \lim_{n\to\infty} d(f^{2n} x, f^{2n-1} x),$$

we conclude that $\lim_{n\to\infty} d(f^{2n-1} x, f^{2n+1} x) = 0$, for any $x \in A \cup B$.

Now we prove (iii). Let z be a fixed point of f^2 but it is not a best proximity point of f, i.e. $dist(A, B) < d(z, fz)$. Then by Lemma 3.3 we have

$$\begin{aligned} d(z, fz) &= d(f^2 z, fz) \\ &\leq \psi(d(z, fz)) d(z, fz) + (1 - \psi(d(z, fz))) dist(A, B) \\ &< \psi(d(z, fz)) d(z, fz) + (1 - \psi(d(z, fz))) d(z, fz) \\ &= d(z, fz), \end{aligned}$$

a contradiction.

Now, if z is a best proximity point of f, i.e. $d(z, fz) = dist(A, B)$, then by a similar method as above we get $d(f^2 z, fz) = dist(A, B)$. So by Lemma 3.2, $f^2 z = z$ which show that (iii) is true. $\square$

Theorem 3.5. *Let A and B be nonempty closed subsets of a complete metric space (X,d) and $f : A\cup B \to A\cup B$ be an* weak $\mathcal{MT}$-cyclic Reich type contraction *with respect to an $\mathcal{MT}$-function ϕ. Then f has a best proximity point z in A. In this case z is a fixed point of f^2 and fz is a best proximity point of f in B.*

Proof. First of all, we show that $\{f^{2n}x\}$ is a Cauchy sequence, for each $x \in A\cup B$. For this, by Lemma 3.2, it sufficient to show that

$$\lim_{n\to\infty}\lim_{m\to\infty} d(f^{2m}x, f^{2n+1}x) = dist(A,B).$$

Now since $d(f^{2m}x, f^{2n+1}x) \le \frac{1}{3}\phi(d(f^{2m-1}x, f^{2n}x))[d(f^{2m-1}x, f^{2n}x) + d(f^{2m-1}x, f^{2m}x)+d(f^{2n+1}x, f^{2n}x)]+(1-\phi(d(f^{2m-1}x, f^{2n}x)))dist(A,B)$, the part (i) of Lemma 3.4 implies that

$$\begin{aligned}\lim_{n,m} d(f^{2m}x, f^{2n+1}x) &\le \lim_{n,m}\frac{1}{3}\phi(d(f^{2m-1}x, f^{2n}x))d(f^{2m-1}x, f^{2n}x)\\ &\quad + (1-\frac{1}{3}\phi(d(f^{2m-1}x, f^{2n}x)))dist(A,B).\end{aligned}$$

Then by an inductive method we conclude that

$$\begin{aligned}dist(A,B) &\le \lim_{n,m} d(f^{2m}x, f^{2n+1}x)\\ &\le \lim_{n,m}\frac{1}{3^{2n+1}}\phi_1\cdots\phi_{2n+1}d(f^{2m-2n}x, x)\\ &+ \lim_{n,m}(1-\frac{1}{3^{2n+1}}\phi_1\cdots\phi_{2n+1})dist(A,B)\\ &= dist(A,B).\end{aligned}$$

Where $\phi_i = \phi(d(f^{2m-i}x, f^{2n+1-i}x))$, for each positive integer i. Therefore $\lim_{n,m} d(f^{2m}x, f^{2n+1}x) = dist(A,B)$.

Now if we consider $x \in A$, then since A is closed, it is complete and so by the Cauchyness of $\{f^{2n}x\}$, there is $z \in A$ such that $\{f^{2n}x\}$ is convergent to z (We have a similar proof when $x \in B$). Now we have by Lemma 3.4-(i)

$$dist(A,B) \le d(f^{2n-1}x, z) \le d(f^{2n-1}x, f^{2n}x)+d(f^{(2n)}x, z) \to dist(A,B). \tag{3}$$

Therefore $\lim_n d(f^{2n-1}x, z) = dist(A, B)$.
On the other hand

$$\begin{aligned} d(z, fz) &\leq d(z, f^{2n}x) + d(f^{2n}x, fz) \\ &\leq d(z, f^{2n}x) \\ &+ \frac{1}{3}\phi(d(f^{2n-1}x, z))[d(f^{2n-1}x, z) + d(f^{2n-1}x, f^{2n}x) + d(z, fz)] \\ &+ (1 - \phi(d(f^{2n-1}x, z)))dist(A, B). \end{aligned}$$

Then by taking a limit and applying Lemma 3.4-(i) and 3, we conclude that

$$(1 - \frac{1}{3}\phi(d(f^{2n-1}x, z)))d(z, fz) \leq (1 - \frac{1}{3}\phi(d(f^{2n-1}x, z)))dist(A, B).$$

it follows that $dist(A, B) \leq d(z, fz) \leq dist(A, B)$. Therefore z is a best proximity point of f in A. Now Lemma 3.4-(iii) says that z is a fixed point of f^2 and so $d(fz, f^2z) = d(fz, z) = dist(A, B)$. That is fz is a best proximity point of f in B. □

The following example shows that the best proximity point in the last theorem may be not unique.

Example 3.6. Let $X = \mathbb{R}^2$, $A = [0, 1] \times [0, 1]$ and $B = [2, 3] \times [0, 1]$. For each $x, y, x', y' \in \mathbb{R}$, define

$$d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\};$$

and $f : A \cup B \to A \cup B$ by

$$f((x, y)) = \begin{cases} (2, y) & \text{if } (x, y) \in A \\ (1, y) & \text{if } (x, y) \in B \end{cases}$$

Then it is easy to see that $dist(A, B) = 1$ and for each $\mathcal{MT}$-function ϕ and $(x, y), (x', y') \in A \cup B)$ we have

$$\begin{aligned} d((f((x, y)), f((x', y')) \leq &\frac{1}{3}\phi(d((x, y), (x', y')))[d((x, y), (x', y')) \\ &+ d((x, y), f((x, y))) + d((x', y'), f((x', y')))] \\ &+ (1 - \phi(d((x, y), (x', y'))))dist(A, B). \end{aligned}$$

Note that $(1, y)$ is a best proximity point of f in A and $(2, y) = f((1, y))$ is a best proximity point of f in B, for each $y \in [0, 1]$.

The following theorem can be obtain immediately from Lemma 3.4 and Theorem 3.5.

Theorem 3.7. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Let $f : A \cup B \to A \cup B$ be a a cyclic map. Suppose that there exists a decreasing function $\tau : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned} d(fx, fy) \leq \frac{1}{3}\tau(d(x, y))[d(x, y) + d(x, fx) + d(y, fy)] \\ + (1 - \tau(d(x, y)))dist(A, B), \quad (x, y \in A \cup B). \end{aligned}$$

Then

(i) f has a unique best proximity point z in A .

(ii) The sequence $\{f^{2n}x\}$ converges to z for any starting point $x \in A$.

(iii) z is the unique fixed point of f^2 .

(iv) fz is a best proximity point of f in B .

Remark 3.8. If in Theorems 3.5 and 3.7 we put $\phi(t) = \tau(t) = k$, for all $t \in [0, \infty)$, where $k \in [0, 1)$, then ϕ and τ are $\mathcal{MT}$-function and decreasing function, respectively and so we can obtain Theorem 10 in [16] as the special case.

In the case that $dist(A, B) = 0$, we can obtain the following corollary that generalize Reich theorem [38].

Corollary 3.9. *Suppose that A and B are nonempty closed subsets of a complete metric space (X, d) such that $A \cap B \neq \emptyset$ and $f : A \cup B \to A \cup B$ is a weak $\mathcal{MT}$-cyclic Reich type contraction with respect to ϕ. Then f has a unique fixed point z in $A \cap B$.*

Proof. By Theorem 3.5, f has a best proximity point z in A. That is $d(z, fz) = dist(A, B) = 0$ and so $fz = z$. This says that z is a fixed point of f and also belongs to B.
Therefore it sufficient to show that z is unique. Suppose that v is another point, i.e. $d(v, z) \neq 0$, such that $fv = v$. Then we have

$$\begin{aligned} d(v,z) &= d(fv,fz) \\ &\leq \frac{1}{3}\phi(d(v,z))[d(v,z)+d(fv,v)+d(z,fz)] \\ &= \frac{1}{3}\phi(d(v,z))[d(v,z)]. \end{aligned}$$

Thus, $\phi(d(v,z)) \geq 3$, a contradiction. Thus, $v = z$. $\square$

Remark 3.10. If in Corollary 3.9, we put $\phi(t) = k$, for all $t \in [0,\infty)$, where $k \in [0,1)$, then one can obtain Reich's theorem in [38] as the special case. Regarding the analogy, we omit corollary and remark for Reich's theorem in [39].

We give an examples illustrating Corollary 3.9.

Example 3.11. Let Γ be a locally compact group and $X = L^p(\Gamma)$. Consider the compact subsets K and K' in Γ such that $K \cap K' = \emptyset$ and $|K| = |K'|$, where $|K| = \lambda(K)$ and λ is the Haar measure of Γ. Suppose $A = \{\alpha\chi_K; \alpha \in \mathbb{C}\}$ and $B = \{\alpha\chi_{K'}; \alpha \in \mathbb{C}\}$, where χ_K is the characteristic function on the set K. Then $A \cap B = \{0\}$. Define $f : A \cup B \to A \cup B$ by

$$fx = \begin{cases} \frac{\alpha}{2}\chi_{K'} & \text{if } x \in A \\ \frac{\alpha}{2}\chi_K & \text{if } x \in B \end{cases}$$

Then it is easy to see that for each $\mathcal{MT}$-function ϕ with $\frac{3}{5} \leq \phi < 1$, we have

$$\|fx - fy\|_p \leq \frac{1}{3}\phi(\|x-y\|_p)[\|x-y\|_p + \|x-fx\|_p + \|y-fy\|_p],$$

for all $x, y \in A \cup B$. Also, $0 \in A \cap B$ is the unique fixed point of f.

Corollary 3.12. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Suppose that $f : A \cup B \to A \cup B$ satisfies*

$$\begin{aligned} d(fx,fy) \leq \phi(d(x,y)) \max\{&d(x,y), \frac{1}{2}(d(x,fx)+d(y,fy)), \frac{1}{3}(d(x,y) \\ &+ d(x,fx)+d(y,fy))\} \\ &+ (1-\phi(d(x,y)))dist(A,B), \quad (x,y \in A \cup B). \end{aligned}$$

Then f has a best proximity point z in A. In this case z is a fixed point of f^2 and fz is a best proximity point of f in B.

Proof. We have three cases:
1. if $\max\{d(x,y), \frac{1}{2}(d(x,fx)+d(y,fy)), \frac{1}{3}(d(x,y)+d(x,fx)+d(y,fy))\} = d(x,y)$, then the proof can imply from Theorem 2.1 in [8];
2. if $\max\{d(x,y), \frac{1}{2}(d(x,fx)+d(y,fy)), \frac{1}{3}(d(x,y)+d(x,fx)+d(y,fy))\} = \frac{1}{2}(d(x,fx)+d(y,fy))$, then the proof can imply from Theorem 3.8 in [33];
3. if $\max\{d(x,y), \frac{1}{2}(d(x,fx)+d(y,fy)), \frac{1}{3}(d(x,y)+d(x,fx)+d(y,fy))\} = \frac{1}{3}(d(x,y)+d(x,fx)+d(y,fy))$, then the proof can imply from Theorem 3.5.

$\square$

Corollary 3.13. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Let $f : A \cup B \to A \cup B$ be a a cyclic map. Suppose that there exists a decreasing function $\tau : [0,\infty) \to [0,1)$ such that*

$$\begin{aligned} d(fx,fy) \leq& \tau(d(x,y)) \max\{d(x,y), \frac{1}{2}(d(x,fx)+d(y,fy)), \frac{1}{3}(d(x,y) \\ &+ d(x,fx)+d(y,fy))\} \\ &+(1-\tau(d(x,y)))dist(A,B), \quad (x,y \in A \cup B). \end{aligned}$$

Then

(i) f has a unique best proximity point z in A .

(ii) The sequence $\{f^{2n}x\}$ converges to z for any starting point $x \in A$.

(iii) z is the unique fixed point of f^2 .

(iv) fz is a best proximity point of f in B .

Remark 3.14. In Corollaries 3.12 and 3.13 if $\phi = \tau = k$, where $k \in [0,1)$ then we can obtain Corollary 15 in [16] as special case.

4 Applications

Let Ω be a locally compact group. Consider $C_0(\Omega)$, of all continuous functions from Ω to $\mathbb{C}$ which vanishes at infinity, with the metric

$$d(f,g) = \|f-g\|_u = \sup\{|f(\xi)-g(\xi)|; \xi \in \omega\} \quad (f,g \in C_0(\Omega)).$$

Define $\|.\|_1 : C_0(\Omega) \to \mathbb{R}^+ \cup \{0\}$ by

$$\|f\|_1 = \int_\Omega |f(x)|dx \quad (f \in C_0(\Omega)).$$

Where dx is used for $d\lambda(x)$ and λ is the left Haar measure on Ω. See [12] for more studying.

Theorem 4.1. *Let A and B be nonempty closed subsets of $C_0(\Omega)$. Consider $\varphi, \psi \in C_0(\Omega)$, $\alpha \in \mathbb{C}$, a bounded continuous map $K : \Omega \times \Omega \to \mathbb{C}$ with a bound β and a cyclic map $T : A \cup B \to A \cup B$ which is defined by $Tf(\xi) = \varphi(\xi) + \alpha \int_\Omega K(\xi, \varrho)\psi(f(\varrho))d\varrho$ such that for some $\mathcal{MT}$-function ϕ we have*

$$3|\alpha|\beta\|\psi \circ f - \psi \circ g\|_1 \leq \phi(\|f - g\|)\|f - g\| \quad (f \in A, g \in B). \tag{4}$$

Then T has a best proximity point f_0 in A which is also a fixed point of T^2.

Proof. if $f \in M$ and $g \in N$, then 4 implies that $2|\alpha|\beta\|\psi \circ f - \psi \circ g\|_1 \leq \frac{2}{3}\phi(\|f - g\|)\|f - g\|$ and so $|\alpha|\beta\|\psi \circ f - \psi \circ g\|_1 \leq \frac{2}{3}\phi(\|f - g\|)\|f - g\| - |\alpha|\beta\|\psi \circ f - \psi \circ g\|_1$. Therefore since $\frac{1}{3}\phi \leq 1$ and for each $x, y \in \Omega$, $|K(x, y)| \leq \beta$, we have for each $\xi \in \Omega$;

$$\begin{aligned}
d(Tf, Tg) &= \|Tf - Tg\| \\
&= \sup_{\xi \in \Omega} |\alpha \int_\Omega K(\xi, \varrho)(\psi(f(\varrho)) - \psi(g(\varrho)))d\varrho| \\
&\leq |\alpha|\beta\|\psi \circ f - \psi \circ g\|_1 \\
&\leq \frac{2}{3}\phi(\|f - g\|)\|f - g\| - |\alpha|\beta\|\psi \circ f - \psi \circ g\|_1 \\
&\leq \frac{2}{3}\phi(\|f - g\|)\|f - g\| - \frac{1}{3}\phi(\|f - g\|)|\alpha|\beta\|\psi \circ f - \psi \circ g\|_1 \\
&\leq \frac{2}{3}\phi(\|f - g\|)\|f - g\| \\
&- \frac{1}{3}\phi(\|f - g\|)|\alpha| \int_\Omega |K(\xi, \varrho)||\psi(f(\varrho)) - \psi(g(\varrho))|d\varrho \\
&\leq \frac{1}{3}\phi(\|f - g\|)(2\|f - g\| \\
&- |\alpha| \int_\Omega |K(\xi, \varrho)||\psi(f(\varrho)) - \psi(g(\varrho))|d\varrho).
\end{aligned}$$

Hence

$$
\begin{aligned}
d(Tf,Tg) &\leq \frac{1}{3}\phi(\|f-g\|)(2\|f-g\| \\
&- |\alpha| \| \int_\Omega K(.,\varrho)|\psi(f(\varrho)) - \psi(g(\varrho))| d\varrho \|) \\
&\leq \frac{1}{3}\phi(\|f-g\|)(\|f-g\| \\
&+ \|f-g-|\alpha| \int_\Omega K(.,\varrho)|\psi(f(\varrho)) - \psi(g(\varrho))| d\varrho \|) \\
&\leq \frac{1}{3}\phi(\|f-g\|)(\|f-g\| \\
&+ \|f-g-|\alpha| \int_\Omega K(.,\varrho)\psi(f(\varrho)) - \psi(g(\varrho)) d\varrho \|) \\
&\leq \frac{1}{3}\phi(\|f-g\|)(\|f-g\| + \|f-\varphi-\alpha\int_\Omega K(.,\varrho)\psi(g(\varrho))d\varrho\| \\
&+ \|g-\varphi-\alpha\int_\Omega K(.,\varrho)\psi(f(\varrho))d\varrho\|) \\
&= \frac{1}{3}\phi(\|f-g\|)(\|f-g\| + \|f-Tg\| + \|g-Tf\|) \\
&\leq \frac{1}{3}\phi(\|f-g\|)(\|f-g\| + \|f-Tg\| + \|g-Tf\|) \\
&+ (1-\phi(\|f-g\|)dist(A,B).
\end{aligned}
$$

Therefore T is a *weak $\mathcal{MT}$-cyclic Reich type contraction* with respect to ϕ and the results follows from Theorem 3.7. □

Corollary 4.2. *(i) With conditions of Theorem 4.1, the integral equation*

$$
f(\xi) = \varphi(\xi) + \alpha \int_\Omega K(\xi,\varrho)\psi(\varphi(\varrho) + \alpha\int_\Omega K(\varrho,\eta)\psi(f(\eta))d\eta)d\varrho \tag{5}
$$

has a solution.

(ii) If in addition $A \cap B \neq \emptyset$, then the integral equation

$$
f(\xi) = \varphi(\xi) + \lambda \int_\Omega K(\xi,\varrho)\psi(f(\varrho))d\varrho \tag{6}
$$

has a solution.

Proof. (i) Obviously the fixed point of f^2 is the solution of 5.

(ii) In this case the best proximity of f is a fixed point of f and obviously it is the solution of 6. $\square$

Example 4.3. Let $\psi \in C_0(\Omega)$, $\alpha \geq 0$, $K : \Omega \times \Omega \to \mathbb{C}$ be a bounded continuous map with a bound β and with nonneqative range which satisfy in equation 4 and $\psi(\xi) \geq 0$ for each $\xi \leq 0$ and $\psi(\xi) \leq 0$ for each $\xi \geq 0$. Then the equation 5 has a solution. Since it suficient to consider $A = C_0^+(\Omega)$ and $B = C_0^-(\Omega)$ of positive and negative functions of $C_0(\Omega)$, respectively. Then Obviously f is a cyclic map. Therefore conditions of Corollary 4.2 (i) hold.

Example 4.4. Let $\varphi, \psi \in C_0^+(\Omega) \cup \{0\}$, $\alpha \geq 0$, $K : \Omega \times \Omega \to \mathbb{C}$ be a bounded continuous map with a bound β and with nonneqative range which satisfy in equation 4. Then the equations 5 and 6 have solution. Since if we put $A = C_0^+(\Omega)$ and $B = C_0(\Omega)$, then obviously f is a cyclic map and $A \cap B \neq \emptyset$.

Remark 4.5. When $A \cap B \neq \emptyset$, then with conditions of Theorem 4.1 for $S = T|_M : M \to N$, there exists an $f_0 \in M$ such that $\|f_0 - Sf_0\| = dist(A, B) = min\{\|f - Sf\|; f \in M\}$.

Theorem 4.6. *Let $(X, d) = (C(\mathbb{R}), \|.\|)$, with $\|f\| = \sup\{|f(x); x \in \mathbb{R}\}$, for each $f \in C(\mathbb{R})$. Consider the integrable map $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and $S \in C(\mathbb{R})$ such that $\lim_{x \to \infty} \int_0^x h(t, S(x))dt < \frac{1}{3\alpha}$ for some $\alpha > 0$. Then the differential equation $\frac{\partial^2 f}{\partial x^2} = -\alpha f(x) h^2(x, S(x))$ has a solution $f \in C(\mathbb{R})$.*

Proof. Put $A = C^+(\mathbb{R})$ and $B = C^-(\mathbb{R})$. Then $dist(A, B) = 0$ and since h is positive, $f : C(\mathbb{R}) \to C(\mathbb{R})$ with $ff(x) = -\alpha \int_0^x f(t)g(t, S(x))dt$ is cyclic and $\int_0^x h(t, S(x))dt$ is increasing. So

$$\sup\{|\int_0^x h(t, S(x))dt|; x \in \mathbb{R}\} = lim_{x \to \infty} \int_0^x h(t, S(x))dt < \frac{1}{3\alpha}.$$

Now $\phi = 3\alpha \lim_{x\to\infty} \int_0^x h(t, S(x))dt$ is an $\mathcal{MT}$-function and we have for each $f, g \in C(\mathbb{R})$;

$$\begin{aligned}
\|Tf - Tg\| &= \alpha \sup\{|\int_0^x (f(t) - g(t))h(t, S(x))dt|; x \in \mathbb{R}\} \\
&\le \alpha\|f - g\| \sup\{|\int_0^x h(t, S(x))dt|; x \in \mathbb{R}\} \\
&= \frac{1}{3}\phi(\|f - g\|)\|f - g\| \\
&\le \frac{1}{3}\phi(\|f - g\|)(\|f - g\| + \|f - Tg\| + \|g - Tf\|).
\end{aligned}$$

That is f is a weak $\mathcal{MT}$-cyclic Reich contraction with respect to ϕ on $A \cup B$. Now Theorem 3.7 implies that f^2 has a fixed point $f \in C(\mathbb{R})$ which obviously it is the solution of the above differential equation. □

Suggestion. Many studies have investigated in the fixed point theory on partial metric spaces. See e. g. [1, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 36, 40]. It can also be interesting to study the results of this paper and the results in [3, 33, 34, 35] on partial metric spaces and recommended for further work.

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