INVERSE EIGENVALUE PROBLEM OF BISYMMETRIC NONNEGATIVE MATRICES

A.M. NAZARI[1], P. ASLAMI, A. NEZAMI

ABSTRACT. This paper considers an inverse eigenvalue problem for bisymmetric nonnegative matrices. We first discuss the specified structure of the bisymmetric matrices. Then for a given set of real numbers of order maximum five with special conditions, we construct a nonnegative bisymmetric matrix such that the given set is its spectrum. Finally, we solve the problem for arbitrary order n in the special case of the spectrum.

1. Introduction

Bisymmetric matrices have been widely discussed since 1939, and are very useful in communication theory, engineering and statistics [1]. In fact, symmetric Toeplitz matrices and persymmetric Hankel matrices are two useful examples of bisymmetric matrices. The bisymmetric nonnegative inverse eigenvalue problem is the problem of finding necessary and sufficient conditions for a list of n real numbers to be the spectrum of an $n \times n$ bisymmetric nonnegative matrix. If there exists an $n \times n$ bisymmetric nonnegative matrix A with spectrum σ, we say that is realizable and that A realizes σ.

The nonnegative inverse eigenvalue problem is very difficult and it is solved only for $n = 3$ by Loewy and London and for matrices with trace 0 of order $n = 4$ by Reams and for $n = 5$ in some special cases by Nazari and Sherafat in 2012 [5]. Recently Nazari et.al solve symmetric nonnegative inverse eigenvalue problem (SNIEP) with one positive eigenvalue and nonnegative summation [6].

Through this paper the following notation is used. The spectral radius of nonnegative matrix A denoted by $\rho(A)$. For nonnegative matrices the largest eigenvalue is called Perron eigenvalue and denoted by λ_1 and we have $\lambda_1 = \rho(A)$, so there is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries.

Some necessary conditions on the list of real number $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ to be the spectrum of a nonnegative matrix are listed below.
(1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ (Perron-Frobenius theorem).
(2) $s_k = \sum_{i=1}^{n} \lambda_i^k \geq 0$.
(3)$s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \ldots$ (JLL inequality)[7, 8].

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This paper is organized as follows. First, we discuss the specified properties and structure of bisymmetric matrices in section 2, and in the next section find a solution for BSNIEP for order 2,3,5 and finally, we solve BSNIEP for a special given spectrum of arbitrary order n.

2. THE PROPERTIES OF BISYMMETRIC MATRICES

A matrix for which the values on each line parallel to the main diagonal are constant, is called a Toeplitz matrix and Hankel matrix, is a square matrix in which each ascending skew-diagonal from left to right is constant.

Let $A = (a_{ij})$ be an $n \times n$ matrix. A is persymmetric if for all i, j we have

$$a_{ij} = A_{n-j+1,n-i+1}.$$

This can be equivalently expressed as $AJ_n = J_n A^T$ where J_n is the exchange matrix, i.e.$J_n = (e_n, e_{n-1}, ..., e_1)$ and we denote by e_i the ith, $(i = 1, ..., n)$ column of identity matrix I_n. Also it is clear that

$$J_n = J_n^T, \qquad J_n J_n^T = I_n.$$

If a symmetric matrix is rotated by 90 degrees, it becomes a persymmetric matrix. Symmetric persymmetric matrices are sometimes called bisymmetric matrices [1].

Definition 2.1. A real $n \times n$ matrix $A = (a_{i,j})$ is called a bisymmetric matrix if its elements satisfy the properties

$$a_{i,j} = a_{j,i}, \qquad a_{i,j} = a_{n-j+1,n-i+1}.$$

The set of all $n \times n$ bisymmetric real matrices is denoted by $BSR^{n\times n}$.

Clearly, a bisymmetric matrix is a square matrix that is symmetric about both of its main diagonals.

If A is a real bisymmetric matrix with distinct eigenvalues, then the matrices that commute with A must be bisymmetric [2]. The inverse of bisymmetric matrices can be represented by recurrence formulas [3].

Lemma 2.2. *A matrix $A \in BSR^{n\times n}$ if and only if $A^T = A$ and $S_n A S_n = A$.*

Noting that $BSR^{n\times n} \subset SR^{n\times n}$, all eigenvalues of a bisymmetric matrix are real numbers, where $SR^{n\times n}$ denote the set of symmetric matrices of dimnesion n.

Definition 2.3. Given $A \in R^{n\times n}$, if $n-k$ is even, then the k-square central principal submatrix of A, denoted as $A_c\,(k)$, is a k-square submatrix obtained by deleting the first and last $\frac{n-k}{2}$ rows and columns of A, that is

$$A_c\,(k) = (0\, I_k\, 0)\, A (0\, I_k\, 0)^T, \qquad 0 \in R^{(k)\times(\frac{n-k}{2})}$$

central principal submatrices preserves the bisymmetric structure of the given matrix.

The product of two bisymmetric matrices is a centrosymmetric matrix.

3. CONSTRUCTION

3.1. CASE n=2.

Theorem 3.1. *Let $\sigma = \{\lambda_1, \lambda_2\}$ be a set of two real numbers such that $\lambda_1 \geq |\lambda_2|$. Then σ is the set of eigenvalues of a bisymmetric nonnegative matrix.*

Proof. we consider desired matrix is $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ by get roots of characteristic polynomial of A, we have

$$\det(A - \lambda I) = 0 \Longrightarrow \{\lambda_1 = a - b,\ \lambda_2 = a + b\},$$

so $a = \frac{\lambda_1+\lambda_2}{2}$, $b = \frac{\lambda_1-\lambda_2}{2}$ then the matrix

$$A = \begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1-\lambda_2}{2} \\ \frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} \end{bmatrix},$$

solves the problem. □

3.2. CASE n=3.

Theorem 3.2. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of real numbers such that*

(i) $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$,
(ii) $\lambda_1 \in R,\ \lambda_1 \geq |\lambda_i|; i = 2, 3.$,

Then there exist a bisymmetric nonnegative matrix that realize σ

Proof. If λ_1 is Perron eigenvalues of real set $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ with nonnegative λ_2 and this set is the spectrum of 3×3 nonnegative bisymmetric(also 3×3 centrosymmetric, since all 3×3 bisymmetric matrix is centrosymmetric matrix) matrix then the following trivial solution solve the problem:

$$\begin{bmatrix} \frac{\lambda_1+\lambda_3}{2} & 0 & \frac{\lambda_1-\lambda_3}{2} \\ 0 & \lambda_2 & 0 \\ \frac{\lambda_1-\lambda_3}{2} & 0 & \frac{\lambda_1+\lambda_3}{2} \end{bmatrix},$$

otherwise in σ we have $\lambda_1 > 0 \geq \lambda_3 \geq \lambda_2$ then if assume that the nonnegative matrix solution has the following form:

$$C = \begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix},$$

then its charactristic polynomial is:

$$\begin{aligned} P(\lambda) = \lambda^3 - 3\,\lambda^2 a - \left(c^2 + 2\,b^2 - 3\,a^2\right)\lambda + \\ ac^2 - 2\,cb^2 + 2\,ab^2 - a^3 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3). \end{aligned} \tag{3.1}$$

If in (3.1) we find a, b, and c based on λ_1, λ_2 and λ_3 then we have

$$\begin{aligned} &a = \tfrac{\lambda_1+\lambda_2+\lambda_3}{3}, \\ &c = -2/3\,\lambda_3 + 1/3\,\lambda_2 + 1/3\,\lambda_1, \\ &b = 1/4\,\sqrt{2\,{\lambda_1}^2 - 4\,\lambda_2\lambda_1 + 2\,{\lambda_2}^2 - 2\,c^2}, \end{aligned}$$

so the 3×3 nonnegative bisymmetric matrix is:

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2+\lambda_3}{3} & b & \frac{\lambda_1-2\lambda_2+\lambda_3}{3} \\ b & \frac{\lambda_1+\lambda_2+\lambda_3}{3} & b \\ \frac{\lambda_1-2\lambda_2+\lambda_3}{3} & b & \frac{\lambda_1+\lambda_2+\lambda_3}{3} \end{bmatrix},$$

and it is easy to see that this matrix is nonnegative bisymmetric and has spectrum σ. □

Suppose $\sigma \in \mathbb{Q}$, the following Theorem shows the conditions under which we can have a 3×3 bisymmetric nonnegative matrix from rational numbers.

Theorem 3.3. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of real numbers such that*

(i) $\lambda_1 + \lambda_2 + \lambda_3 \geq 0$,
(ii) $\lambda_1 \in R,\ \lambda_1 \geq |\lambda_i|; i = 2, 3.$,
(iii) $\lambda_1 + 2\lambda_3 \geq 0$,
(iv) $\lambda_1 + \lambda_3 \geq \pm 3\lambda_2$.

Then there exist a bisymmetric nonnegative matrix that realize σ

Proof. We consider desired matrix as

$$A = \begin{bmatrix} a & b & c \\ b & a+c-b & b \\ c & b & a \end{bmatrix},$$

and by finding the roots of charactristic polynomial of matrix A we have:

$$\det(A - \lambda I) = 0 \Longrightarrow \{\lambda_1 = a+b+c, \lambda_2 = a-c, \lambda_3 = a-2b+c\}$$

so $a = \frac{2\lambda_1+3\lambda_2+\lambda_3}{6}$,$b = \frac{\lambda_1-\lambda_3}{3}$, $c = \frac{2\lambda_1-3\lambda_2+\lambda_3}{6}$,$a+c-b = \frac{2\lambda_1+4\lambda_3}{6}$ then the matrix

$$A = \begin{bmatrix} \frac{2\lambda_1+3\lambda_2+\lambda_3}{6} & \frac{\lambda_1-\lambda_3}{3} & \frac{2\lambda_1-3\lambda_2+\lambda_3}{6} \\ \frac{\lambda_1-\lambda_3}{3} & \frac{2\lambda_1+4\lambda_3}{6} & \frac{\lambda_1-\lambda_3}{3} \\ \frac{2\lambda_1-3\lambda_2+\lambda_3}{6} & \frac{\lambda_1-\lambda_3}{3} & \frac{2\lambda_1+3\lambda_2+\lambda_3}{6} \end{bmatrix},$$

is nonnegative bisymmetric matrix and solves the problem. □

Example 3.4. For $\sigma = \{5, -1, -3\}$ find a bisymmetric nonnegative matrix that realizes spectrum σ.

By Theorem (3.2) the following bisymmetric nonnegative matrix is solution:

$$\begin{bmatrix} 1/3 & 2/3\sqrt{14} & 4/3 \\ 2/3\sqrt{14} & 1/3 & 2/3\sqrt{14} \\ 4/3 & 2/3\sqrt{14} & 1/3 \end{bmatrix}.$$

and we see that for this spectrum we cannot find a nonnegative bisymmetric matrix according to Theorem (3.3) because the (iii) or (iv) condition of the Theorem (3.3) will not always exist. But if we increase the Perron eigenvalue's to 6, the matrix of the following nonnegative bisymmetric matrix of the set of rational numbers will be the answer:

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

3.3. CASE n=4. If λ_1 is Perron eigenvalues of $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ with nonnegative λ_2 and $\lambda_2 \geq \lambda_4$ is the spectrum of 4×4 nonnegative bisymmetric matrix then the following trivial solution solve the problem:

$$\begin{bmatrix} \frac{\lambda_1+\lambda_3}{2} & 0 & 0 & \frac{\lambda_1-\lambda_3}{2} \\ 0 & \frac{\lambda_2+\lambda_4}{2} & \frac{\lambda_2-\lambda_4}{2} & 0 \\ 0 & \frac{\lambda_2-\lambda_4}{2} & \frac{\lambda_2+\lambda_4}{2} & 0 \\ \frac{\lambda_1-\lambda_3}{2} & 0 & 0 & \frac{\lambda_1+\lambda_3}{2} \end{bmatrix},$$

otherwise we study some special cases in the following Theorem:

Theorem 3.5. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a set of real numbers with following condition*

(i) $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 0$,
(ii) $\lambda_1 \in R$, $\lambda_1 \geq |\lambda_i|; i = 2, 3, 4, \lambda_3 = \lambda_4$,
(iii) $\lambda_1 + \lambda_2 \geq \pm 2\lambda_3$.

Then there exist a bisymmetric nonnegative matrix that realizes σ

Proof. In the first case, at first we assume that $\lambda_3 = \lambda_4 = 0$, then it easy to see that the following circulant nonnegative bisymmetric matrix has eigenvalues $\{\lambda_1, \lambda_2, 0, 0\}$:

$$\begin{bmatrix} \frac{\lambda_1+\lambda_2}{4} & \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2}{4} & \frac{\lambda_1-\lambda_2}{4} \\ \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2}{4} & \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2}{4} \\ \frac{\lambda_1+\lambda_2}{4} & \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2}{4} & \frac{\lambda_1-\lambda_2}{4} \\ \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2}{4} & \frac{\lambda_1-\lambda_2}{4} & \frac{\lambda_1+\lambda_2}{4} \end{bmatrix}.$$

Now we consider the answer matrix as a 4×4 circulant matrix. It is clear that this matrix is a type of Teoplitz matrix.

$$A = \begin{bmatrix} a & b & c & b \\ b & a & b & c \\ c & b & a & b \\ b & c & b & a \end{bmatrix},$$

We find the characteristic polynomials of this matrix and then determine its roots.

$$\det(A - \lambda I) = 0 \Longrightarrow \{\lambda_1 = a + 2b + c, \lambda_2 = a - 2b + c, \lambda_3 = a - c, \lambda_4 = a - c\}$$

so $a = \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4}$, $b = \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4}$, $c = \frac{\lambda_1+\lambda_2+\lambda_4-3\lambda_3}{4}$, then the following bisymmetric nonnegative matrix:

$$A=\begin{bmatrix}\frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4} & \frac{\lambda_1+\lambda_2+\lambda_4-3\lambda_3}{4} & \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4}\\ \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4} & \frac{\lambda_1+\lambda_2+\lambda_4-3\lambda_3}{4}\\ \frac{\lambda_1+\lambda_2+\lambda_4-3\lambda_3}{4} & \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4} & \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4}\\ \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4} & \frac{\lambda_1+\lambda_2+\lambda_4-3\lambda_3}{4} & \frac{\lambda_1+\lambda_4-\lambda_2-\lambda_3}{4} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{4}\end{bmatrix},$$

is the solution, with Perron eigenvalue λ_1.

□

Theorem 3.6. *Let $\sigma=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ be a set of real numbers with following condition*

(i) $\lambda_1+\lambda_2+\lambda_3+\lambda_4\geq 0$,
(ii) $\lambda_1\in R$, $\lambda_1\geq|\lambda_i|; i=2,3,4, \lambda_3=\lambda_4$,
(iii) $\lambda_1+\lambda_2\geq\pm(\lambda_3+\lambda_4)$.

Then there exist a bisymmetric nonnegative matrix that realizes σ

Proof. We consider the nonnagative bisymmetric matrix as following Hankel form:

$$\begin{bmatrix} a & b & c & b\\ b & c & b & c\\ c & b & c & b\\ b & c & b & a\end{bmatrix}.$$

The charactristic polynomial of above matrix is obtained as :

$$P(\lambda)=\lambda^4+(-2\,a-2\,c)\,\lambda^3+\left(-4\,b^2-c^2+4\,ca+a^2\right)\lambda^2+$$
$$\left(-4\,cb^2+4\,ab^2+2\,c^3-2\,ca^2\right)\lambda-a^2b^2+2\,ab^2c-b^2c^2+c^4-2\,c^3a+a^2c^2,$$

we find the roots of above polynomial and denote them by $\lambda_1,\lambda_2,\lambda_2$ and λ_4, so we have

$$\left\{\begin{array}{l}\lambda_1=b+1/2\,a+1/2\,c+1/2\,\sqrt{4\,b^2+8\,bc+a^2-2\,ca+5\,c^2},\\ \lambda_2=b+1/2\,a+1/2\,c-1/2\,\sqrt{4\,b^2+8\,bc+a^2-2\,ca+5\,c^2},\\ \lambda_3=-b+1/2\,a+1/2\,c+1/2\,\sqrt{4\,b^2-8\,bc+a^2-2\,ca+5\,c^2},\\ \lambda_4=-b+1/2\,a+1/2\,c-1/2\,\sqrt{4\,b^2-8\,bc+a^2-2\,ca+5\,c^2}.\end{array}\right. \tag{3.2}$$

Now from (3.2) we find a,b and c and then we will provide the bisymmetric nonnegative matrix in this case. To do this, from the first two equations and then from the last two equations of (3.2) respectively we have:

$$\begin{array}{c}\lambda_1+\lambda_2=2b+a+c,\\ \lambda_3+\lambda_4=-2b+a+c,\end{array} \tag{3.3}$$

then

$$b=\frac{(\lambda_1+\lambda_2)-(\lambda_3+\lambda_4)}{4}, \tag{3.4}$$

also from (3.2) we have

$$\begin{array}{c}\lambda_1-\lambda_2=\sqrt{(2\,b+2\,c)^2+(a-\,c)^2},\\ \lambda_3-\lambda_4=\sqrt{(2\,b-2\,c)^2+(a-\,c)^2},\end{array} \tag{3.5}$$

so

$$c = \frac{(\lambda_1 - \lambda_2)^2 - (\lambda_3 - \lambda_4)^2}{16b} = \frac{(\lambda_1 - \lambda_2)^2 - (\lambda_3 - \lambda_4)^2}{4[(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4)]}.$$

By (3.5)we have $\lambda_1 - \lambda_2 \geq \lambda_3 - \lambda_4$ then by hypthesis we have $b, c \geq 0$. Now, by combinig and simplifying the relations (3.3) and (3.4), we can obtain a. So

$$a = \frac{(\lambda_1 + \lambda_2)^2 - (\lambda_3 + \lambda_4)^2 + 4\lambda_1\lambda_2 - 4\lambda_3\lambda_4}{4[(\lambda_1 + \lambda_2) - (\lambda_3 + \lambda_4)]}, \tag{3.6}$$

By placing a, b and c in Hankel matrix, the desired matrix will be obtained. □

Example 3.7. Assume given

$$\sigma = \left\{9/2 + 1/2\sqrt{65}, 9/2 - 1/2\sqrt{65}, 7/2 + 1/2\sqrt{65}, 7/2 - 1/2\sqrt{65}\right\},$$

by above Theorem we see that $\lambda_1 + \lambda_2 \geq (\lambda_3 + \lambda_4)$. Then we have $a = 8, b = \frac{1}{2}$ and $c = 0$ and so the following Hankel matrix is bisymmetric matrix and has eigenvalues σ:

$$\begin{bmatrix} 8 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 8 \end{bmatrix}.$$

3.4. CASE n=5. In this subsection at first we try to get a extention of problem that related to above subsection.

Theorem 3.8. *Let $\sigma_1 = \{\lambda_1, \lambda_2, \lambda_3\}$ be the spectrum of nonnegative bisymmetric matrix*

$$\begin{bmatrix} a & b & c \\ b & d & b \\ c & b & a \end{bmatrix},$$

and λ_4 and λ_5 are two real numbers such that $\lambda_4 \geq 0$ and $\lambda_4 + \lambda_5 \geq 0$, then $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ is realized by a nonnegative bisymmetric 5×5 matrix.

Proof. It is tivial that the following matrix

$$\begin{bmatrix} \frac{\lambda_4+\lambda_5}{2} & 0 & 0 & 0 & \frac{\lambda_4-\lambda_5}{2} \\ 0 & a & b & c & 0 \\ 0 & b & d & b & 0 \\ 0 & c & b & a & 0 \\ \frac{\lambda_4-\lambda_5}{2} & 0 & 0 & 0 & \frac{\lambda_4+\lambda_5}{2} \end{bmatrix},$$

is nonnegative and bisymmeytric and has spectrum σ. □

Theorem 3.9. *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be the spectrum of nonnegative bisymmetric matrix*

$$\begin{bmatrix} a & b & c & d \\ b & c & b & c \\ c & b & c & b \\ d & c & b & a \end{bmatrix},$$

and $\lambda_5 \geq 0$, then the following nonnegative bisymmetric matrix is realized the spectrum $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$

$$\begin{bmatrix} a & b & 0 & c & d \\ b & c & 0 & b & c \\ 0 & 0 & \lambda_5 & 0 & 0 \\ c & b & 0 & c & b \\ d & c & 0 & b & a \end{bmatrix}.$$

4. SPECAL CASES OF PROBLEM

Theorem 4.1. *Let's given $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with the following conditions*

(i) $\lambda_1 \geq |\lambda_i|$, *and* $\sum_{i=1}^{n} \lambda_i \geq 0$,
(ii) $\lambda_i = \lambda_j, \ \ i, j = 2, 3, \cdots, n$

then the following nonnegative bisymmatric matrix is realzied spectrum σ

$$C = \begin{bmatrix} \frac{\lambda_1+(n-1)\lambda_2}{n} & \frac{\lambda_1-\lambda_2}{n-1} & \cdots & \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1-\lambda_2}{n-1} \\ \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1+(n-1)\lambda_2}{n} & \cdots & \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1-\lambda_2}{n-1} \\ \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1-\lambda_2}{n-1} & \cdots & \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1-\lambda_2}{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1-\lambda_2}{n-1} & \cdots & \frac{\lambda_1-\lambda_2}{n-1} & \frac{\lambda_1+(n-1)\lambda_2}{n} \end{bmatrix}.$$

Proof. We select the matrices A and L as the following:

$$A = \begin{bmatrix} \lambda_2 & 0 & 0 & 0 & \cdots & 0 & \frac{\lambda_i}{n-1} \\ 0 & \lambda_2 & 0 & 0 & \cdots & 0 & 2\frac{\lambda_1}{n-1} \\ 0 & 0 & \lambda_2 & 0 & \cdots & 0 & 3\frac{\lambda_1}{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda_2 & (n-1)\frac{\lambda_1}{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_1 \end{bmatrix},$$

and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

It is easy to see that we have $C = L^{-1}AL$. □

Example 4.2. Let given $\sigma = \{70, -15, -15, -15, -15\}$ by Theorem (4.1) we find a bisymmetric matrix such that σ is its spectrum.

We assume that

$$A = \begin{bmatrix} -15 & 0 & 0 & 0 & 17 \\ 0 & -15 & 0 & 0 & 34 \\ 0 & 0 & -15 & 0 & 51 \\ 0 & 0 & 0 & -15 & 68 \\ 0 & 0 & 0 & 0 & 70 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

then we have

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

So the soultion matrix is:

$$C = L^{-1}AL = \begin{bmatrix} 2 & 17 & 17 & 17 & 17 \\ 17 & 2 & 17 & 17 & 17 \\ 17 & 17 & 2 & 17 & 17 \\ 17 & 17 & 17 & 2 & 17 \\ 17 & 17 & 17 & 17 & 2 \end{bmatrix}.$$

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[1] Department of Mathematics, Arak University, P. O. Box 38156-8-8349, Arak, Iran..
E-mail address: a-nazari@araku.ac.ir