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On Commutative Gelfand Rings

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Abstract. By studying and using the quasi-pure part concept, we improve some statements and show that some assumptions in some articles are superfluous. We give some characterizations of Gelfand rings. For example: we prove that R is Gelfand if and only if $m\big(\sum_{\alpha\in A} I_\alpha\big) = \sum_{\alpha\in A} m(I_\alpha)$, for each family $\{I_\alpha\}_{\alpha\in A}$ of ideals of R, in addition if R is semiprimitive and $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$, we show that R is a Gelfand ring if and only if Y is normal. We prove that if R is reduced ring, then R is a von Neumann regular ring if and only if $\mathrm{Spec}(R)$ is regular. It has been shown that if R is a Gelfand ring, then $\mathrm{Max}(R)$ is a quotient of $\mathrm{Spec}(R)$, and sometimes $h_M(a)$'s behave like the zerosets of the space of maximal ideal. Finally, it has been proven that $Z\big(\mathrm{Max}(C(X))\big) = \{h_M(f) : f \in C(X)\}$ if and only if $\{h_M(f) : f \in C(X)\}$ is closed under countable intersection if and only if X is pseudocompact.

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1 Introduction

The pure and quasi-pure part of an ideals of rings play important roles in classifying certain classes of rings. These concepts along with Gelfand concept and connection between them, have been studied in numerous articles, for example the reader can see [6], [1] [9], [8], [7] and [13]. In this article, by study and the use of quasi-pure part along with the spectrum of maximal ideals with Zariski topology, we give some characterizations of Gelfand rings. We show that some assumptions in some articles are redundant and finally we find some connections between $\{h_M(a) : a \in R\}$ and zerosets of the space of maximal ideals.

In the rest of this section, we recall some pertinent definitions and give two elementary lemmas. In Section 2, we recall the quasi-pure part and study this notion. Also, we show that some assumptions in some articles are unneeded. Section 3 is devoted to give some characterizations of Gelfand rings. We devote the final section to role of $h_M(a)$'s in $\mathrm{Max}(R)$. In this section we give some characterizations of von Neumann regular rings and show that sometimes $h_M(a)$'s behave like the zerosets, and the space of maximal ideals of a Gelfand ring is quotient of the space of prime ideals of the ring. Also, we prove that in the ring $C(X)$; zerosets of the space of maximal ideals are coincide with $h_M(f)$'s, if and only if X is pseudocompact, if and only if $h_M(f)$'s are closed under countable intersection.

Throughout the article, R denotes a commutative ring with identity. The family of all maximal ideals of R is denoted by $\mathrm{Max}(R)$. The intersection of all maximal ideals of a ring R, denoted by $\mathrm{Jac}(R)$, is called the *Jacobson radical* of R. If $\mathrm{Jac}(R) = \{0\}$, then R is called *semiprimitive*. For every subset S of R, the set $\{a \in R : aS = \{0\}\}$, denoted by $\mathrm{Ann}(S)$, is called the *annihilator* of S. $\mathrm{Spec}(R)$ denotes the family of all prime ideals of R. Every minimal element of a $\mathrm{Spec}(R)$ is called a *minimal prime ideal*. The family of all minimal prime ideals of a ring R is denoted by $\mathrm{Min}(R)$. For every ring R, $\mathrm{Rad}(R)$ denotes the intersection of all prime ideals of R. If $\mathrm{Rad}(R) = \{0\}$, then R is called *reduced*. A ring is called *Gelfand*, if each prime ideal is contained in a unique maximal ideal. An element a of a ring is called regular element if there is some $b \in R$ such that $a = a^2b$. If each element of a ring is regular, then R is called von Neumann regular ring. By [11, Teorem

1.16]; a ring R is von Neumann regular ring, if and only if R is reduced and each prime ideal of the ring is maximal.

Throughout this article, $C(X)$ (resp., $C^*(X)$) is the ring of all (resp., bounded) real-valued continuous functions on a Tychonoff (i.e., completely regular Hausdorff) space X. For $f \in C(X)$ the *zeroset* of f is the set $Z(f) = \{x \in X : f(x) = 0\}$. We denote the set of all zerosets of X by $Z(X)$. A space is called *pseudocompact*, if $C(X) = C^*(X)$. βX denotes Stone-Čech compactification of X. Suppose that $A \subseteq \beta X$, then by M^A and O^A we mean the sets $\{f \in C(X) : A \subseteq \mathrm{cl}_{\beta X} Z(f)\}$ and $\{f \in C(X) : A \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$, respectively. The m-topology is defined on $C(X)$ by taking sets of the form $B(f, m) = \{g \in C(X) : \forall x \in X \quad |f(x) - g(x)| < m(x)\}$ as a base, where $f \in C(X)$ and m is a positive unit in $C(X)$. By [10, 7Q.2], $\overline{I} = \bigcap_{I \subseteq M \in \mathrm{Max}(R)} M$ is the closure of I in m-topology, for each ideal I of $C(X)$.

An ideal I of a ring R is called *pure* ideal if $aI = I$, for each $a \in I$. By [6, Proposition 8.8], for any ideal I there is the largest pure ideal contained in I. This ideal is called the *pure part* of I and we denote it by $s(I)$; of course, the pure part of an ideal I is denoted by $\mathring{I}$ in [6, Section 8]. Clearly; an ideal I is pure, if and only if $I = s(I)$. If P is a prime ideal of a ring R, then P *component of the zero ideal* is denoted by 0_P and defined by $\{a \in R : \exists b \notin P \quad ab \in \mathrm{Rad}(R)\}$. Also, we denote the set $\{a \in R : \exists b \notin P \quad ab = 0\}$ by $0(P)$. Clearly, $0_P = \{a \in R : (\mathrm{Rad}(R) : a) \not\subseteq P\}$, $0(P) = \{a \in R : \mathrm{Ann}(a) \not\subseteq P\}$, $0(P) \subseteq 0_P$ and if R is a reduced ring, then $0(P) = 0_P$.

Lemma 1.1. *For every prime ideal P of a ring R, $0_P = \bigcap_{P \supseteq Q \in \mathrm{Min}(R)} Q$.*

Proof. Suppose that $P \supseteq Q \in \mathrm{Min}(R)$ and $a \in 0_P$, then $(\mathrm{Rad}(R) : a) \not\subseteq P$, so $(\mathrm{Rad}(R) : a) \not\subseteq Q$ and therefore $a \in Q$. This implies that $0_P \subseteq \bigcap_{P \supseteq Q \in \mathrm{Min}(R)} Q$. Now let $a \notin 0_P$, then $(\mathrm{Rad}(R) : a) \subseteq P$ and thus $S = \{a^n x : n \in \mathbb{N} \text{ and } x \in P^c\}$ is a multiplicatively closed set disjoint from $\mathrm{Rad}(R)$. Hence there is a prime ideal Q containing $\mathrm{Rad}(R)$ disjoint from S, clearly we can assume $Q \in \mathrm{Min}(R)$. Therefore $a \notin Q$ and $Q \subseteq P$, consequently $a \notin \bigcap_{P \supseteq Q \in \mathrm{Min}(R)} Q$. It deduces that $0_P \supseteq \bigcap_{P \supseteq Q \in \mathrm{Min}(R)} Q$ which completes the proof. □

Suppose that $Y \subseteq \mathrm{Spec}(R)$, then $\bigcap Y$ is denoted by $k(Y)$ and for every subset S of a ring R, we denote $\{P \in Y : S \subseteq P\}$ and $\{P \in Y :$

$S \nsubseteq P\}$ by $h_Y(S)$ and $h^c_Y(S)$, respectively. We abbreviate $h_{\mathrm{Spec}(R)}(S)$, $h_{\mathrm{Min}(R)}(S)$ and $h_{\mathrm{Max}(R)}(S)$ by $h(S)$, $h_m(S)$ and $h_M(S)$, respectively. Clearly, $\{h^c_Y(S) : S \subseteq R\}$ is a topology on Y. This topology is called *Zariski topology* on Y. It is clear that $\{h^c_Y(a) : a \in R\}$ is a base for this topology. An ideal I of a ring R is called z-ideal (sz-ideal, sz°-ideal), if for every element a (finite subset F) of R, $kh_M(a) \subseteq R$ ($kh_M(F) \subseteq R$, $kh_m(F) \subseteq R$). We state some well elementary properties of these spaces in the following lemma. One can find these in [3, Section 5].

Lemma 1.2. *Let R be a ring and $Y \subseteq \mathrm{Spec}(R)$. The following statements holds.*

(a) A subset D of Y is dense in Y, if and only if $k(D) = k(Y)$.

(b) $h^c_Y(I) \subseteq h_Y(J)$, if and only if $IJ = k(Y)$.

(c) If $\mathrm{Max}(R) \subseteq Y$, then

  (i) $h_Y(I) \subseteq h^c_Y(J)$, if and only if $I + J = R$.

  (ii) Y is a compact space.

  (iii) Y is a T_1-space, if and only if $Y = \mathrm{Max}(R)$.

Proof. It is straightforward. □

A map ϕ from an ordered set S into S is called *dual closure map*, if ϕ preserves relation, $\phi(s) \leqslant s$, for every $s \in S$ and $\phi^2 = \phi$. For every ordered set S, the greatest and smallest element of S(if exist) are called the *top* and *bottom* element, respectively.

The reader is referred to [10], [4], [5] and [14] for undefined terms and notations.

2 Quasi-Pure Part Of An Ideal

Let us introduce the quasi-pure part by the following proposition.

Proposition 2.1. *For any ideal I of a ring R, all the following sets are equal ideal. This ideal is called* quasi-pure part *of I*

(a) $\bigcup_{i\in I} \mathrm{Ann}(1 - i)$.

(b) $\{a \in R : \exists i \in I \quad a = ai\}$.

(c) $\{a \in R : I + \mathrm{Ann}(a) = R\}$.

Proof. It is easy to see that the set (a) is an ideal and the sets (a)-(c) are equal. $\square$

We will denote the quasi-pure part of I by $m(I)$. This ideal is denoted by $\mathcal{E}(I)$ in [6, Definitiuon 8.10] and is called unit part of I. It is easy to see that $s(I) \subseteq m(I)$, and also, I is pure, if and only if $I = m(I)$, for each ideal I of a ring R. In [6, Proposition 8.30], it has been shown that in each Gelfand ring the quasi-pure part and the pure part of each ideal are equal.

In [12, Lemma 1.8], it has been shown that if R is a semiprimitive ring, then $\{a \in R : \exists x \in R \, \exists b \in I \quad h_M(b) \subseteq h^c_M(x) \subseteq h_M(a)\} = m(I)$. Now we develop this fact in the following proposition.

Proposition 2.2. *Let R be a ring and* $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. *Then*

(a) $m(I) \subseteq \{a \in R : \exists i \in I \quad h_Y(i) \subseteq h_Y(a)^\circ\} = \{a \in R : h_Y(I) \subseteq h_Y(a)^\circ\}$.

(b) In addition if $k(Y) = \{0\}$, then $m(I) = \{a \in R : \exists i \in I \quad h_Y(i) \subseteq h_Y(a)^\circ\} = \{a \in R : h_Y(I) \subseteq h_Y(a)^\circ\}$.

Proof. Suppose that $J = \{a \in R : \exists i \in I \quad h_Y(i) \subseteq h_Y(a)^\circ\}$ and $K = \{a \in R : h_Y(I) \subseteq h_Y(a)^\circ\}$.

(a). If $a \in m(I)$, then, by Proposition 2.1, $I + \mathrm{Ann}(a) = R$, so there are $i \in I$ and $x \in \mathrm{Ann}(a)$ such that $i + x = 1$ and $ax = 0$. Now Lemma 1.2 shows that $h_Y(i) \subseteq h^c_Y(x) \subseteq h_Y(a)$ and thus $h_Y(i) \subseteq h_Y(a)^\circ$. This implies that $m(I) \subseteq J$. Clearly, $J \subseteq K$. Now suppose that $a \in K$, then an ideal I' of R exists such that $h_Y(I) \subseteq h^c_Y(I') \subseteq h_Y(a)$. Now Lemma 1.2 concludes that $I + I' = 1$, thus there are $i \in I$ and $i' \in I'$ such that $i + i' = 1$, by Lemma 1.2. Hence $h_Y(i) \subseteq h^c_Y(i') \subseteq h^c_Y(I') \subseteq h_Y(a)$ and consequently $a \in J$. This shows that $J = K$.

(b). Now suppose that $a \in K$, then $h_Y(I) \subseteq h_Y(a)^\circ$, so an ideal I' exists such that $h_Y(I) \subseteq h^c_Y(I') \subseteq h_Y(a)$, hence $I + I' = R$ and $aI' = \{0\}$, by Lemma 1.2. They imply that $i \in I$ and $i' \in I$ exist such that $i + i' = 1$ and $ai' = 0$, so $a \in \mathrm{Ann}(1 - i)$ and therefore $a \in m(I)$, by Proposition 2.1. Consequently, $K \subseteq m(I)$. □

In [12, Theorem 1.12], it has been shown that "in semiprimitive rings, if $\{h_Y(a) : a \in I\}$ is closed under finite intersection; then I is a pure ideal, if and only if $h_Y(a)$ is a neighborhood of $h_Y(I)$, for each $a \in I$". Now by the above proposition, we can see that the fact "$\{h_Y(a) : a \in I\}$ is closed under finite intersection" is redundant.

If M is a maximal ideal of a ring, then by Proposition 2.1,

$$\begin{aligned} m(M) &= \{a \in R : M + \mathrm{Ann}(a) = R\} \\ &= \{a \in R : \mathrm{Ann}(a) \nsubseteq M\} = 0(M) \subseteq 0_M \end{aligned} \tag{1}$$

Hence a maximal ideal M of a ring is pure if and only if $0(M) = M$. Thus a maximal ideal M of a reduced ring is pure if and only if $M = 0_M$. Hence this consequence is a generalization of [1, Theroem 1.5] and shows that the assumption "Gelfand ring" is redundant. This fact is also satisfies for the following corollary and [1, Corollary 1.6], analogously.

Corollary 2.3. *A reduced ring is von Neumann regular, if and only if every maximal ideal of the ring is pure.*

In [6, Proposition 8.17], it has been shown that in Gelfand rings, $m(I) = \bigcap_{M \in h_M(I)} m(M)$. In the following we develop that proposition in commutative rings and show that the assumption "R is a Gelfand ring" is superfluous.

Proposition 2.4. *Let R be a ring and* $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. *Then*

$$m(I) = \bigcap_{P \in h_Y(I)} m(P)$$

Proof. One can see easily that $m(I) \subseteq \bigcap_{P \in h_Y(I)} m(P)$. Thus it is sufficient to show that $\bigcap_{P \in h_Y(I)} m(P) \subseteq m(I)$. Suppose that $x \notin m(I)$, so $I + \mathrm{Ann}(x) \neq R$, hence $M \in \mathrm{Max}(R)$ exists such that $I + \mathrm{Ann}(x) \subseteq M$ and therefore $I \subseteq M$ and $M + \mathrm{Ann}(x) \subseteq M \neq R$. They imply that $M \in h_Y(I)$ and $x \notin m(M)$ and consequently $x \notin \bigcap_{P \in h_Y(I)} m(P)$. This shows that $\bigcap_{P \in h_Y(I)} m(P) \subseteq m(I)$. □

By the above proposition and fact (1), we can see that

$$m(I) = \bigcap_{M \in h_M(I)} m(M) = \bigcap_{M \in h_M(I)} 0(M) \tag{2}$$

In addition if R is a reduced ring then $m(I) = \bigcap_{M \in h_M(I)} 0_M$, for each ideal I of a ring R. In [12], since the author was studying the z-ideals, he only focused on the semiprimitive rings and gave this fact for the semiprimitive rings, in [12, Lemma 1.10(a)].

In [1, Theorem 1.8], it has been shown that if I is a pure ideal of a reduced Gelfand ring, then $I = \bigcap_{M \in h_M(I)} 0_M$. In the following corollary we develop this theorem and can conclude that the converse of the theorem is also true. Also, by the following corollary the "Gelfand ring" supposition is unneeded in [1, Theorem 1.8].

Corollary 2.5. *For each ideal I of a ring, I is pure if and only if $I = \bigcap_{M \in h_M(I)} 0(M)$.*

Proof. It concludes immediately from fact (2). □

Corollary 2.6. *For each pair ideals I and J of a ring R, $m(I \cap J) = m(I) \cap m(J)$.*

Proof. By Proposition 2.4,

$$\begin{aligned} m(I \cap J) &= \bigcap_{M \in h_M(I \cap J)} m(M) = \bigcap_{M \in h_M(I) \cup h_M(J)} m(M) \\ &= \Big(\bigcap_{M \in h_M(I)} m(M) \Big) \cap \Big(\bigcap_{M \in h_M(J)} m(M) \Big) = m(I) \cap m(J). \end{aligned}$$

□

In [12] from [12, Lemma 1.8], it has been deduced that in semiprimitive rings, $m(I)$ is z-ideal, for each ideal I of the ring. The following proposition and corollary are improvements of this fact.

Proposition 2.7. *R is a reduced ring if and only if $m(I)$ is a sz°-ideal, for every ideal I of a ring R.*

Proof. $\Rightarrow$). By fact (2) and Lemma 1.1, $m(I)$ is an intersection of minimal prime ideals, since every minimal prime ideal is sz°-ideal, it follows that $m(I)$ is a sz°-ideal.

$\Leftarrow$). Since $m(\{0\}) = \{0\}$ is a sz°-ideal and $\mathrm{Rad}(R)$ is the smallest sz°-ideal, it follows R is reduced. □

Corollary 2.8. *R is a semiprimitive ring if and only if $m(I)$ is sz-ideal, for every ideal I of R.*

Proof. It follows from the above proposition, [2, Proposition 2.9] and this fact that the smallest sz-ideal is $\mathrm{Jac}(R)$. □

In [3], inspired of notations O^A and M^A in context of the real-valued continuous functions, $\{a \in R : A \subseteq h_Y(a)^{\circ}\}$ and $\{a \in R : A \subseteq h_Y(a)\}$ have been denoted by $O^A(Y)$ and $M^A(Y)$, respectively; for each $A \subseteq Y \subseteq \mathrm{Spec}(R)$. Clearly, $M^A(Y) = k(A)$ and $h_Y(M^A(Y)) = \mathrm{cl}_Y A$.

Corollary 2.9. *Let R be a ring, I be an ideal of R and $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. Then*

(a) $m(I) \subseteq O^{h_Y(I)}(Y) \subseteq M^{h_Y(I)}(Y)$.

(b) If $k(Y) = \{0\}$, *then* $m(I) = O^{h_Y(I)}(Y)$.

Proof. It is evident by Proposition 2.2. □

Proposition 2.10. *Suppose that $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. If I is an ideal of R, then $m(kh_Y(I)) = m(I)$.*

Proof. By Proposition 2.4,

$$m(kh_Y(I)) = \bigcap_{P \in h_Y\big(kh_Y(I)\big)} m(P) = \bigcap_{P \in h_Y(I)} m(P) = m(I)$$

□

Corollary 2.11. *If I is an ideal of a ring R, then $m(I) = m(\sqrt{I}) = m(I_{sz}) = m(I_z) = m(\overline{I})$; in which $\overline{I}$ is the m-closure of I.*

Proof. By the above proposition, if we assume that $Y = \mathrm{Max}(R)$, then $m(\overline{I}) = m(I)$, since $m(I) \subseteq m(\sqrt{I}) \subseteq m(I_{sz}) \subseteq m(I_z) \subseteq m(\overline{I})$, it follows that $m(I) = m(\sqrt{I}) = m(I_{sz}) = m(I_z) = m(\overline{I})$. □

3 Some Characterization Of Gelfand Rings

In [6, Theorem 8.13], it has been proven that R is a Gelfand ring if and only if for each family $\{I_\alpha\}_{\alpha\in}$ of ideals of R, $s\big(\sum_{\alpha\in A} I_\alpha\big) = \sum_{\alpha\in A} s(I_\alpha)$. In the following theorem we give some characterizations of Gelfand rings by the quasi-pure part notion and show that the above fact is also true for the quasi-pure part.

Theorem 3.1. *The following statements are equivalent for a ring R.*

(a) R is a Gelfand ring.

(b) For each ideal I of R, $h_M(m(I)) = h_M(I)$.

(c) For each ideal I of R, $\overline{m(I)} = \overline{I}$.

(d) For each family $\{I_\alpha\}_{\alpha\in A}$ of ideals of R, we have

$$m\left(\sum_{\alpha\in A} I_\alpha\right) = \sum_{\alpha\in A} m\left(I_\alpha\right) = m\left(\sum_{\alpha\in A} m(I_\alpha)\right).$$

(e) For each family $\{I_\alpha\}_{\alpha\in A}$ of ideals of R, $\sum_{\alpha\in A} I_\alpha = R$ if and only if $m\big(\sum_{\alpha\in A} I_\alpha\big) = R$ if and only if $\sum_{\alpha\in A} m(I_\alpha) = R$.

(f) For each pair ideals I and J of R, $I + J = R$ if and only if $m(I) + m(J) = R$.

(g) For each pair distinct maximal ideals M and N of R, $m(M) + m(N) = R$.

Proof. (a) $\Rightarrow$ (b). Suppose that $I \not\subseteq M$, then $i \in I \setminus M$ and $m \in M$ exist such that $i+m = 1$. Hence [8, Theroem 4.1] deduces that there are $a, b \in R$ such that $(1-am)(1-bi) = 0$, so $1-am \in \mathrm{Ann}(1-bi) \subseteq m(I)$, thus $1 - am \in m(I) \setminus M$. It implies that $h_M(m(I)) \subseteq h_M(I)$ and therefore $h_M(m(I)) = h_M(I)$.

(b) $\Rightarrow$ (a). Suppose that $M \in \mathrm{Max}(R)$, then $h_M(m(M)) = h_M(M) = \{M\}$. Now Let $P \in \mathrm{Spec}(R)$ and $M \in h_M(P)$. By fact (1) and Lemma 1.2, we have $m(M) \subseteq 0_M \subseteq P$, so $h_M(P) \subseteq h_M(m(M)) = \{M\}$ and therefore R is Gelfand.

(b) $\Rightarrow$ (c). By the hypothesis, $\overline{m(I)} = kh_M(m(I)) = kh_M(I) = \overline{I}$.

(c) $\Rightarrow$ (b). Since $\overline{m(I)} = \overline{I}$, it follows that $h_M(\overline{m(I)}) = h_M(\overline{I})$, thus $h_M(kh_M(m(I))) = h_M(kh_M(I))$ and therefore

$$\begin{aligned} h_M(m(I)) &= \mathrm{cl}_Y h_M(m(I)) = h_M(kh_M(m(I))) \\ &= h_M(kh_M(I)) = \mathrm{cl}_Y h_M(I) = h_M(I) \end{aligned}$$

(b) $\Rightarrow$ (d). By the supposition and Proposition 2.4,

$$\begin{aligned} \sum_{\alpha\in A} m(I_\alpha) \subseteq m\Big(\sum_{\alpha\in A} I_\alpha\Big) &= \bigcap \Big\{ m(M) : \sum_{\alpha\in A} I_\alpha \subseteq M \in \mathrm{Max}(R)\Big\} \\ &= \bigcap \big\{ m(M) : \forall \alpha \in A \quad I_\alpha \subseteq M \in \mathrm{Max}(R)\big\} \\ &= \bigcap \big\{ m(M) : \forall \alpha \in A \quad m(I_\alpha) \subseteq M \in \mathrm{Max}(R)\big\} \\ &= \bigcap \Big\{ m(M) : \sum_{\alpha\in A} m(I_\alpha) \subseteq M \in \mathrm{Max}(R)\Big\} \\ &= m\Big(\sum_{\alpha\in A} m(I_\alpha)\Big) \subseteq \sum_{\alpha\in A} m(I_\alpha) \end{aligned}$$

(d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g). They are straightforward.

(g) $\Rightarrow$ (a). Suppose that R is not a Gelfand ring, so by [9, Theorem 4.1], there are distinct maximal ideals M and N such that $m(M) \subseteq N$, then $m(M) + m(N) \subseteq N \neq R$. $\square$

In [3, Lemma 5.1], it has been shown that a reduced ring R is a Gelfand ring if and only if $h_M(m(I)) = h(m(I))$, for each ideal I of R. Now the above theorem shows that the assumption "reduced" is redundant.

Theorem 3.2. *Let R be a Gelfand ring. Then the following hold.*

(a) Let $\mathscr{I}(R)$ be the lattice of ideals of R with respect to the inclusion order. Then the function $m : \mathscr{I}(R) \to \mathscr{I}(R)$, defined by $I \mapsto m(I)$, preserves the finite meets and arbitrary joins. Also, m is a dual closure map on the ordered set $\mathscr{I}(R)$.

(b) For every pair ideals I and J of R,

$$m(I) = m(J) \Leftrightarrow h_M(I) = h_M(J) \Leftrightarrow \overline{I} = \overline{J}$$

(c) *Suppose* $\sim$ *is the following defined relation on* $\mathscr{I}(R)$.

$$I \sim J \Leftrightarrow m(I) = m(J)$$

Then $\sim$ *is an equivalent relation and for each* $I \in \mathscr{I}$, $m(I)$ *and* $\overline{I}$ *are the bottom and top element of the equivalent class* $[I]$, *respectively.*

(d) $I \subseteq \mathrm{Jac}(R)$ *if and only if* $m(I) \subseteq \mathrm{Rad}(R)$, *for each ideal* I *of* R.

Proof. (a). By Corollary 2.6 and Theorem 3.1(d), m preserves the finite meets and arbitrary joins. We know that $m(I) \subseteq I$, for every $I \in \mathscr{I}(R)$; $I \subseteq J$ implies $m(I) \subseteq m(J)$, for every $I, J \in \mathscr{I}(R)$; and $m\big(m(I)\big) = m(I)$, for every $I \in \mathscr{I}(R)$. So we can conclude that m is a dual closure map on the ordered set $\mathscr{I}(R)$.

(b). By Proposition 2.4, $h_M(I) = h_M(J)$ implies that $m(I) = m(J)$ and by Theorem 3.1(b), $m(I) = m(J)$ implies that $h_M(I) = h_M(J)$. Also, by Theorem 3.1(d), $m(I) = m(J)$ deduces that $\overline{I} = \overline{J}$ and by Corollary 2.11, $\overline{I} = \overline{J}$ implies that $m(I) = m(J)$.

(c). It is evident that $\sim$ is an equivalent class. Since $m(I)$ is a pure ideal, $m(m(I)) = m(I)$, so $m(I) \in [I]$. By Corollary 2.11, $\overline{I} \in [I]$. Also, it is clear $m(I) = m(J) \subseteq J \subseteq \overline{J} = \overline{I}$, for every $J \in [I]$, thus $m(I)$ and $\overline{I}$ are the bottom and top element of $[I]$, respectively.

(d). By Proposition 2.4, $I \subseteq \mathrm{Jac}(R)$ implies that

$$m(I) = \bigcap_{M \in \mathrm{Max}(R)} 0(M)$$

and therefore, by Lemma 1.2,

$$m(I) \subseteq \bigcap_{M \in \mathrm{Max}(R)} 0_M = \mathrm{Rad}(R).$$

Conversely, by Theorem 3.1(b), $h_M(I) = h_M(m(I)) = \mathrm{Max}(R)$ and therefore $I \subseteq \mathrm{Jac}(R)$. $\square$

Lemma 3.3. *If* X *is a topological space, then the following statements are equivalent.*

(a) X *is regular.*

(b) If $\mathcal{B}$ is base for topology, then $\{\overline{B} : B \in \mathcal{B}\}$ is a neighborhood base of closed sets.

(c) X has a neighborhood base of closed sets.

(d) Each closed set F is the intersection of all closed neighborhoods of F.

(e) If $\mathcal{F}$ is a base for closed sets, then for each closed set E of X, we have

$$E = \bigcap\{F \in \mathcal{F} : F \text{ is a neighborhood of } E\}$$

Proof. (a) ⇒ (b) ⇒ (c) ⇒ (a). They are straightforward.

(a) ⇒ (d). Suppose that E is the intersection of all closed neighborhoods of F. Clearly, $F \subseteq E$. Now let $a \notin F$. Since X is regular, there is an open set U such that $a \in U \subseteq \overline{U} \subseteq F^c$, so $F \subseteq U^{c^{\circ}} \subseteq U^c$, thus $a \notin E$, hence $E \subseteq F$ and therefore $E = F$.

(d) ⇒ (e). Set $K = \bigcap\{F \in \mathcal{F} : F$ is a neighborhood of $E\}$. Clearly, $E \subseteq K$. Suppose that $x \notin E$, by the assumption, a closed set L of X exists such that $E \subseteq L^{\circ}$ and $x \notin L$. So $F \in \mathcal{F}$ exits such that $L \subseteq F$ and $x \notin F$, hence $E \subseteq F^{\circ}$ and $x \notin F$ and therefore $x \notin K$. This implies that $K \subseteq E$ and thus $K = E$.

(e) ⇒ (a). Suppose that G is open and $a \in G$. Then $a \notin G^c$, so by the assumption there is a closed set $F \in \mathcal{F}$ such that $a \notin F$ and $G^c \subseteq F^{\circ}$, thus $a \in F^c \subseteq \overline{F^c} \subseteq G$ and therefore X is regular. □

If R is a Gelfand ring, then $\mathrm{Max}(R)$ is regular, by [8, Proposition 1.2] and Lemma 1.2. Thus we can conclude [3, Lemmas 5.2 and 5.3 and Proposition 5.4], from the above lemma.

Proposition 3.4. *Let R be a ring and $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. Then the following are equivalent.*

(a) Y is a regular space.

(b) $h_Y(O^A(Y)) = \mathrm{cl}_Y A$, for every $A \subseteq Y$.

(c) $h_Y(O^A(Y)) = A$, for every closed subset A of Y.

Proof. (a) ⇒ (b). Since $O^A(Y) \subseteq M^A(Y) = k(A)$, $\mathrm{cl}_Y A = h_Y k(A) \subseteq h_Y(O^A(Y))$. Now suppose that $P \notin \mathrm{cl}_Y A$. Since Y is regular, thus Lemma 3.3 follows that there is some $a \in R$ such that $A \subseteq h_Y(a)^\circ \subseteq h_Y(a)$ and $P \notin h_Y(a)$, hence $a \in O^A(Y) \setminus P$ and therefore $P \notin h_Y(O^A(Y))$. This shows that $h_Y(O^A(Y)) \subseteq \mathrm{cl}_Y A$ and thus the equality holds.

(b) ⇒ (c). It is evident.

(c) ⇒ (a). Suppose A is a closed subset of Y. Since $A = h_Y(O^A(Y)) = \bigcap_{a\in O^A(Y)} h_Y(a) = \bigcap_{A\subseteq h_Y(a)^\circ} h_Y(a)$, by Lemma 3.3, it follows that Y is regular. □

In [9, Theorem 1.2], it has been shown that a ring R is Gelfand if and only if $\mathrm{Spec}(R)$ is normal. In the following theorem we improve this fact for semiprimitive rings.

Theorem 3.5. *Let R be a semiprimitive ring and $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. Then the following are equivalent.*

(a) R is a Gelfand ring.

(b) $\mathrm{Max}(R)$ is a regular space.

(c) For each ideal I of R, there is a unique closed subset A of $\mathrm{Max}(R)$ such that $O^A \subseteq I \subseteq M^A$.

(d) Y is normal.

(e) For each distinct maximal ideals M and N, there are $a, b \in R$ such that $M \in h^c_Y(a)$, $N \in h^c_Y(b)$ and $h^c_Y(a) \cap h^c_Y(b) = \emptyset$.

Proof. (a) ⇒ (b). By [8, Proposition 1.2] and Lemma 1.2, $\mathrm{Max}(R)$ is T_4 and therefore $\mathrm{Max}(R)$ is regular.

(b) ⇒ (c). By Corollary 2.9, $O^{h_M(I)} = m(I) \subseteq I \subseteq M^{h_M(I)}$. Suppose that A is a closed subset of $\mathrm{Max}(R)$ and $O^A \subseteq I \subseteq M^A$, thus Propositions 3.4 implies that $A = h_M(O^A) \supseteq h_M(I) \supseteq h_M(M^A) = h_M(k(h_M(A))) = \overline{A} = A$, hence $A = h_M(I)$ and therefore $h_M(I)$ is unique.

(c) ⇒ (a). Suppose that $m(M) \subseteq N$, then by Corollary 2.9, $O^{\{M\}} = O^{h_M(M)} = m(M) \subseteq M = M^{h_M(M)} = M^{\{M\}}$ and $O^{\{M,N\}} = O^{h_M(M\cap N)} = m(M \cap N) \subseteq m(M) \subseteq M \cap N = M^{\{M,N\}}$, since $\{M\} = h_M(M)$ and

$\{M, N\} = h_M(M \cap N)$ are closed subsets of $\operatorname{Max}(R)$, the assumption follows that $\{M, N\} = \{M^\circ\}$, i.e. $M = N$. Consequently R is a Gelfand ring, by [9, Theorem 1.2].

(a) $\Rightarrow$ (d). Suppose that A and B are disjoint closed sets of Y, then there are closed sets A' and B' of $\operatorname{Spec}(R)$ such that $A = A' \cap Y$ and $B = B' \cap Y$. We claim A' and B' are disjoint. On contrary, if $P \in A' \cap B'$ exists, so there is a maximal ideal M containing P and $k(A), k(B) \subseteq P \subseteq M$, since A' and B' are closed, it follows that $M \in hk(A') \cap hk(B') = A' \cap B' \cap Y$ and thus $M \in A \cap B = \emptyset$, which is a contradiction. Consequently A' and B' are disjoint closed sets of $\operatorname{Spec}(R)$. By [9, Theorem 1.2], $\operatorname{Spec}(R)$ is normal, so A' and B' can be separated by two disjoint open subsets of $\operatorname{Spec}(R)$, so A and B can be separated by two disjoint open sets of Y and therefore Y is normal.

(d) $\Rightarrow$ (e). Suppose that M and N are distinct maximal ideals of R. Then $\{M\} = h_Y(M)$ and $\{N\} = h_Y(N)$ are disjoint closed subsets of Y, so there are some ideals I and J of R such that $\{M\} \subseteq h^c_Y(I)$, $\{N\} \subseteq h^c_Y(J)$ and $h^c_Y(I) \cap h^c_Y(J) = \emptyset$. Thus $a \in I \setminus M$ and $b \in J \setminus N$ exist. Clearly, $M \in h^c_Y(a)$, $N \in h^c_Y(b)$ and $h^c_Y(a) \cap h^c_Y(b) = \emptyset$.

(e) $\Rightarrow$ (b). Suppose that A is a closed subset of $\operatorname{Max}(R)$ and a maximal ideal $M \notin A$, then an ideal I of R exists such that $A = h_M(I)$. For each maximal ideal $N \in h_Y(I)$, we have $N \neq M$, so by the assumption there are $a_N, b_N \in R$ such that $M \in h^c_Y(a_N)$, $N \in h^c_Y(b_N)$ and $h^c_Y(a_N) \cap h^c_Y(b_N) = \emptyset$. Clearly, $h_M(I) \subseteq \bigcup_{N \in h_M(I)} h^c_M(a_N)$. By Lemma 1.2, $h_M(I)$ is compact, so $a_{N_1}, a_{N_2}, \ldots, a_{N_n}$ exist such that $h_M(I) \subseteq \bigcup_{i=1}^n h^c_M(a_{N_i})$. Set $U = \bigcup_{i=1}^n h^c_M(a_{N_i})$ and $V = \bigcap_{i=1}^n h^c_M(b_{N_i})$, then $A \subseteq U$, $M \in V$ and U and V are two disjoint open sets in $\operatorname{Max}(R)$, so $\operatorname{Max}(R)$ is regular. □

Corollary 3.6. *If R is a Gelfand ring, then there is a one-to-one correspondence between the family of all pure ideals of R and the family of all ideals of R of the form O^A, where A is a closed subset of* $\operatorname{Max}(R)$.

Proof. Set φ the map from the family of all quasi-pure parts of ideals of R to the family of all ideal of R of the form O^A, where A is closed subset of $\operatorname{Max}(R)$, defined by $\varphi(m(I)) = O^{h_M(I)}$. Suppose that $m(I) = m(J)$, then by Theorems 3.1, $h_M(I) = h_M(m(I)) = h_M(m(J)) = h_M(J)$ and thus φ is well-defined. Clearly, φ is onto. By

[8, Proposition 1.2] and Lemma 1.2, $\mathrm{Max}(R)$ is regular. Now suppose that $O^{h_M(I)} = O^{h_M(J)}$, then by Proposition 3.4, $h_M(I) = h_M\big(O^{h_M(I)}\big) = h_M\big(O^{h_M(J)}\big) = h_M(J)$, thus by Proposition 2.4, $m(I) = m(J)$ and therefore φ is one-to-one. □

4 The Role Of $h_M(a)$ In $\mathrm{Max}(R)$

In the following theorems we give some characterizations of rings in which each prime ideal is maximal.

Theorem 4.1. *Each prime ideal of a ring is maximal if and only if $h(m(I)) = h(I)$, for every ideal I of the ring.*

Proof. (⇒). By the assumption, R is a Gelfand ring and $h_M(I) = h(I)$, so $h(m(I)) = h(I)$, by Theorem 3.1.

(⇐). Suppose that there is a prime ideal P which is not maximal and M is a maximal ideal containing P, then $m(M) \subseteq 0_M \subseteq P$, by fact (1) and Lemma 1.1. Thus $P \in h(m(M)) \setminus h(M)$. □

Theorem 4.2. *Let R be a reduced ring and $Y = \mathrm{Spec}(R)$. Then the following statements are equivalent.*

(a) R is a von Neumann regular ring.

(b) $\mathrm{Spec}(R)$ is a regular space.

(c) For each ideal I of R, there is a unique closed subset A of $\mathrm{Max}(R)$ such that $O^A(Y) \subseteq I \subseteq M^A(Y)$.

Proof. (a) ⇒ (b). Since R is a semiprimitive Gelfand ring, by Theorem 3.5, it follows that $\mathrm{Spec}(R) = \mathrm{Max}(R)$ is regular.

(b) ⇒ (c). It is similar to the proof of (b) ⇒ (c) of Theorem 3.5.

(c) ⇒ (a). Suppose that M is a maximal ideal of R. By Corollary 2.9, $O^{h_Y(M)}(Y) = m(M) \subseteq M^{h_Y(M)}(Y)$ and $O^{h_Y(m(M))}(Y) = m(M) \subseteq M^{h_Y(m(M))}(Y)$. Since $h(M)$ and $h(m(M))$ are closed sets in $\mathrm{Spec}(R)$, by the assumption, it follows that $h(m(M)) = h(M) = \{M\}$ and therefore $h(0_M) = \{M\}$, by fact (1). Now Lemma 1.1 concludes that M is a minimal prime ideal and therefore R is a von Neumann regular ring. □

In the following theorem we give another characterizations of Gelfand rings and show that sometimes $h_M(a)$'s behave like the zerosets in the space of maximal ideals of Gelfand rings.

Theorem 4.3. *Suppose that R is a semiprimitive ring and $\mathrm{Max}(R) \subseteq Y \subseteq \mathrm{Spec}(R)$. Then the following are equivalent.*

(a) R is a Gelfand ring.

(b) $\{h_M(a) : a \in R\}$ is a neighborhood base for $\mathrm{Max}(R)$.

(c) $\{\overline{h^c_M(a)} : a \in R\}$ is a neighborhood base for $\mathrm{Max}(R)$.

(d) For each closed set A in $\mathrm{Max}(R)$, we have $A = \bigcap_{A \subseteq h_M(a)^\circ} h_M(a)$.

(e) For each pair of disjoint closed subsets A and B of $\mathrm{Max}(R)$, there are a, b in R such that $A \subseteq h^c_Y(a)$, $B \subseteq h^c_Y(b)$ and $h^c_Y(a) \cap h^c_Y(b) = \emptyset$.

(f) For each pair of disjoint closed subsets A and B of $\mathrm{Max}(R)$, there are a, b in R such that $A \subseteq h_Y(a)^\circ$, $B \subseteq h_Y(b)^\circ$ and $h_Y(a) \cap h_Y(b) = \emptyset$.

Proof. (a) $\Rightarrow$ (b). Suppose that $M \in h^c_M(I)$, so $M \in h^c_M(m(I))$, by Theorem 3.1. Thus $a \in m(I) \setminus M$ exists, hence there is an $i \in I$ such that $a(1-i) = 0$. Then, by Lemma 1.2, $M \in h^c_M(a) \subseteq h_M(1-i) \subseteq h^c_M(I)$. This implies that $M \in h_M(1-i)^\circ \subseteq h_M(1-i) \subseteq h^c_M(I)$.

(d) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). They follow from Lemma 3.3 and Theorem 3.5.

(a) $\Rightarrow$ (e). By the assumption there are ideals I' and J' of R such that $A = h_Y(I')$ and $B = h_Y(J')$. By Theorem 3.5, there are ideals I and J of R such that $h_Y(I') = A \subseteq h^c_Y(I)$, $h_Y(J') = B \subseteq h^c_Y(J)$ and $h^c_Y(I) \cap h^c_Y(J) = \emptyset$. By Lemma 1.2, $I' + I = R$, thus $a' \in I'$ and $a \in I$ exist such that $a + a' = 1$ and therefore, by Lemma 1.2, $A = h_Y(I') \subseteq h_Y(a') \subseteq h^c_Y(a) \subseteq h^c_Y(I)$. Similarly, we can show that there is some $b \in J$ such that $B \subseteq h^c_Y(b) \subseteq h^c_Y(J)$ and thus $h^c_Y(a) \cap h^c_Y(b) \subseteq h^c_Y(I) \cap h^c_Y(J) = \emptyset$.

(a) $\Rightarrow$ (f). By Theorem 3.5, there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. By part (b), there is a family $\{a_\alpha\}_{\alpha \in I}$

of elements of R such that $U = \bigcup_{\alpha\in I} h_Y(a_\alpha)^\circ = \bigcup_{\alpha\in I} h_Y(a_\alpha)$, thus $A \subseteq \bigcup_{\alpha\in I} h_Y(a_\alpha)^\circ$. By Lemma 1.2, A is compact, so there is a finite family $\{a_{\alpha_i}\}_{i=1}^n$ such that $A \subseteq \bigcup_{i=1}^n h_Y(a_{\alpha_i})^\circ$. Set $a = \prod_{i=1}^n a_{\alpha_i}$, then $A \subseteq \bigcup_{i=1}^n h_Y(a_{\alpha_i})^\circ \subseteq \bigcup_{i=1}^n h_Y(a_i) = h_Y(a)$, so $A \subseteq h_Y(a)^\circ \subseteq h_Y(a) = \bigcup_{i=1}^n h_Y(a_i) \subseteq U$. Similarly, we can show that there is some $b \in R$ such that $B \subseteq h_Y(b)^\circ \subseteq V$. Hence $h_Y(a)^\circ \cap h_Y(b)^\circ \subseteq U \cap V = \emptyset$.

(e) $\Rightarrow$ (a) and (f) $\Rightarrow$ (a). They deduce from Theorem 3.5. □

Proposition 4.4. *If R is a Gelfand ring, then* $\mathrm{Max}(R)$ *is a quotient of* $\mathrm{Spec}(R)$.

Proof. Suppose that $\eta : \mathrm{Spec}(R) \to \mathrm{Max}(R)$ is given by, $\eta(P)$ is the unique maximal ideal containing P, for each $P \in \mathrm{Spec}(R)$. Suppose that $a \in R$ and $I = \bigcap_{a\in\eta(P)} P$. If $P \in \eta^{-1}\big(h_M(a)\big)$, then $a \in \eta(P)$, so $I \subseteq P$ and therefore $P \in h(I)$. This shows that $\eta^{-1}\big(h_M(a)\big) \subseteq h(I)$. In the other hand, by Lemma 1.1 and fact (2),

$$\begin{aligned} m(Ra) &= \bigcap_{a\in M\in\mathrm{Max}(R)} 0(M) \subseteq \bigcap_{a\in M\in\mathrm{Max}(R)} 0_M \\ &= \bigcap_{\substack{P\subseteq M\\ a\in M\in\mathrm{Max}(R)}} P = \bigcap_{a\in\eta(P)} P = I \end{aligned}$$

Now Theorem 3.1 concludes that $h_M(I) \subseteq h_M(m(Ra)) = h_M(Ra) = h_M(a)$. If $P \in h(I)$, then $I \subseteq P \subseteq \eta(P)$, so $\eta(P) \in h_M(I) \subseteq h_M(a)$ and thus $P \in \eta^{-1}\big(h_M(a)\big)$. This shows that $h(I) \subseteq \eta^{-1}\big(h_M(a)\big)$ and consequently $h(I) = \eta^{-1}\big(h_M(a)\big)$. Hence η is continuous. By Lemma 1.2, $\mathrm{Spec}(R)$ is compact and by [8, Proposition 1.2], $\mathrm{Max}(R)$ is Hausdorff, thus η is closed and therefore $\mathrm{Max}(R)$ is a quotient space of $\mathrm{Spec}(R)$. □

As applications of the main results obtained so far, we study the relationship and interaction between $\{h_M(a) : a \in R\}$ and $Z\big(\mathrm{Max}(C(X))\big)$. For convenience, let us denote $\mathcal{H}_M$ and $\mathcal{M}$ the sets $\{h_M(a) : a \in R\}$ and $\mathrm{Max}(C(X))$, respectively, and use the homeomorphism $\phi : \beta X \to \mathcal{M}$, given by $\phi(p) = M^p$; for every $p \in \beta X$. It is easy to see that $\phi^{-1}\big(h_M(f)\big) = \mathrm{cl}_{\beta X} Z(f)$

Proposition 4.5. *Let X be a Tychonoff space. Then $\mathcal{H}_M \subseteq Z(\mathcal{M})$ if and only if $\mathrm{cl}_{\beta X} Z \in Z(\beta X)$, for every $Z \in Z(X)$.*

Proof. Clearly, $h_M(f) \in Z(\mathcal{M})$ if and only if $\phi^{-1}\big(h_M(f)\big) \in Z(\beta X)$, thus $h_M(f) \in Z(\mathcal{M})$ is equivalent to say that $\mathrm{cl}_{\beta X} Z(f) \in Z(\beta X)$. Thus we can claim that $\mathcal{H} \subseteq Z(\mathcal{M})$ if and only if $\mathrm{cl}_{\beta X} Z(f) \in Z(\beta X)$, for every $Z(f) \in Z(X)$. □

One can see in [10, Exercie 8B.5] that "If X is pseudocompact space, then $\mathrm{cl}_{\beta X} Z \in Z(\beta X)$, for every $Z \in Z(X)$". The following lemma is an improvement of this fact.

Lemma 4.6. *For a Tychonoff space X the following statements are equivalent.*

(a) X is pseudocompact.

(b) $\mathrm{cl}_{\beta X} Z(f) = Z(f^\beta)$, for every $f \in C^*(X)$.

(c) If $\emptyset \neq A \in Z(\beta X)$, then $A \cap X \neq \emptyset$.

Proof. (a) $\Rightarrow$ (b). Suppose that there is some $f \in C^*(X)$ such that $Z(f^\beta) \neq \mathrm{cl}_{\beta X} Z(f)$, so $p \in Z(f^\beta) \setminus \mathrm{cl}_{\beta X} Z(f)$ exists and thus there is some $g \in O^p \cap C^*(X)$ such that $Z(g) \cap Z(f) = \emptyset$, so $Z(f^2 + g^2) = \emptyset$ and therefore $\frac{1}{f^2+g^2} \in C(X)$. Clearly, $f^\beta(p) = g^\beta(p) = 0$, so $\frac{1}{f^2+g^2} \in C(X) \setminus C^*(X)$ and consequently, X is not pseudocompact.

(b) $\Rightarrow$ (c). It is clear.

(c) $\Rightarrow$ (a). Suppose that X is not pseudocompact, then $f \in C(X) \setminus C^*(X)$ exists. Clearly, $g = \frac{1}{f^2+1} \in C^*(X)$, $Z(g^\beta) \neq \emptyset$ and $Z(g^\beta) \cap X = Z(g) = \emptyset$. □

Proposition 4.7. *Let X be a Tychonoff space. Then the following statements are equivalent.*

(a) $Z(\mathcal{M}) \subseteq \mathcal{H}_M$.

(b) For every $A \in Z(\beta X)$ there exits some $Z \in Z(X)$ such that $\mathrm{cl}_{\beta X} Z = A$.

(c) X is pseudocompact.

(d) $Z(\mathcal{M}) = \mathcal{H}_M$.

Proof. (a) $\Rightarrow$ (b). Suppose that $A \in Z(\beta X)$, then $\phi(A) \in Z(\mathcal{M}) \subseteq \mathcal{H}_M$, thus $f \in C(X)$ exits such that $\phi(A) = h_M(f)$ and therefore $A = \phi^{-1}(h_M(f)) = \mathrm{cl}_{\beta X} Z(f)$.

(b) $\Rightarrow$ (c). On contrary, suppose that X is not pseudocompact, then $\emptyset \neq A \in Z(\beta X)$ exists such that $A \cap X = \emptyset$, by Lemma 4.6. By the assumption there is some $Z \in Z(X)$ such that $\mathrm{cl}_{\beta X} Z = A$, so $\mathrm{cl}_{\beta X} Z \cap X = \emptyset$, hence $Z = \emptyset$ and therefore $A = \emptyset$, which is a contradiction.

(c) $\Rightarrow$ (d). Suppose that $A \in Z(\mathcal{M})$, then $\phi^{-1}(A) \in Z(\beta X)$, thus by the Lemma 4.6, there is some $f \in C(X)$ such that $\mathrm{cl}_{\beta X} Z(f) = \phi^{-1}(A)$, so $h_M(f) = \phi\big(\mathrm{cl}_{\beta X} Z(f)\big) = \phi\big(\phi^{-1}(A)\big) = A$ and therefore $A \in \mathcal{H}_M$. It follows that $Z(\mathcal{M}) \subseteq \mathcal{H}_M$. Now suppose that $h_M(f) \in \mathcal{H}$, then by Lemma 4.6, we have $\phi^{-1}\big(h_M(f)\big) = \mathrm{cl}_{\beta X} Z(f) = Z(f^\beta) \in Z(\beta X)$ and thus $h_M(f) = \phi\big(\phi^{-1}(h_M(f)\big) = \phi\big(Z(f^\beta)\big) \in Z(\mathcal{M})$. Hence $Z(\mathcal{M}) \supseteq \mathcal{H}_M$ and therefore $Z(\mathcal{M}) = \mathcal{H}_M$.

(d) $\Rightarrow$ (a). It is evident. $\square$

Let X be not a pseudocompact space. The above proposition shows that $C(X)$ (that is a semiprimitive Gelfand ring) is an example of a ring for which the zerosets of $\mathrm{Max}(R)$ are not of the form $h_M(a)$, for some $a \in R$. Therefore, it is natural to ask the following question: "Does $\mathcal{H}_M \subseteq Z(\mathcal{M})$ imply that $\mathcal{H}_M = Z(\mathcal{M})$?". The first part of the following example shows that the answer is "No". Another question which comes from Proposition 4.5 is, " Is there a semiprimitive Gelfand ring R and $a \in R$, for which $h_M(a)$ is not a zeroset in $\mathrm{Max}(R)$?". The second part of the next example shows that the answer is "Yes".

Example 4.8. (a) Assume that X is an infinite discrete space. Since for every $f \in C(X)$, $\mathrm{cl}_{\beta X} Z(f)$ is a closed and open subset of βX, it follows that $\mathrm{cl}_{\beta X} Z(f) \in Z(\beta X)$. Therefore, every $h_M(f)$ is a zeroset in $\mathcal{M}$, whereas by Proposition 4.7, a zeroset in $\mathcal{M}$ is not necessarily of the form $h_M(f)$.

(b) Since $\mathbb{N}$ is closed in the metric space $\mathbb{R}$, by [10, Section 1.10], there is some $f \in C(\mathbb{R})$ such that $Z(f) = \mathbb{N}$. By Proposition 4.5, to show that $\mathcal{H}_M \not\subseteq Z(\mathcal{M})$, it is sufficient to show that $\mathrm{cl}_{\beta\mathbb{R}} Z(f) \notin Z(\beta\mathbb{R})$. On contrary, suppose $\mathrm{cl}_{\beta\mathbb{R}} Z(f) = Z(g^\beta)$, for some $g \in C^*(\mathbb{R})$. Clearly, for each $n \in \mathbb{N}$, there is some $a_n \in \mathbb{R}$ such that $0 < |a_n - n| < 1/n$ and $|g(a_n)| < 1/n$. Set $A = \{a_n : n \in \mathbb{N}\}$, then $p \in \mathrm{cl}_{\beta\mathbb{R}} A$ exists such that $g^\beta(p) = 0$. It is readily seen that A and $\mathbb{N}$ are two disjoint zerosets and thus $\mathrm{cl}_{\beta\mathbb{R}}A \cap \mathrm{cl}_{\beta\mathbb{R}}\mathbb{N} = \emptyset$, by [10, Theorem 6.5]. Hence $p \in Z(g^\beta) = \mathrm{cl}_{\beta\mathbb{R}}Z(f) = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{N}$, which is a contradiction.

Now in the following proposition, we give a sufficient condition on Gelfand rings that $Z(\mathrm{Max}(R)) \subseteq \mathcal{H}_M$.

Proposition 4.9. *Let R be a Gelfand ring. If $\mathcal{H}_M$ is closed under countable intersection, then $Z\big(\mathrm{Max}(R)\big) \subseteq \mathcal{H}_M$.*

Proof. Suppose that $Z \in Z\big(\mathrm{Max}(R)\big)$, then Z is a G_δ-set, by [10, Section 1.10]. So there is a countable family $\{U_n : n \in \mathbb{N}\}$ of open sets such that $Z = \bigcap_{n\in\mathbb{N}} U_n$. Since R is Gelfand, similar to the proof (a) $\Rightarrow$ (b), it follows that $\{h_M(a) : a \in R\}$ is a neighborhood base for $\mathrm{Max}(R)$. Now suppose that $n \in \mathbb{N}$. For each $M \in U_n$, there is some $a_M \in R$ such that $M \in h_M(a_M)^\circ \subseteq h_M(a_M) \subseteq U_n$ and thus $Z \subseteq \bigcup_{M\in U_n} h_M(a_M)^\circ$. By Lemma 1.2, $\mathrm{Max}(R)$ is compact, thus Z is compact, hence there are $a_{M_1}, a_{M_2}, \ldots, a_{M_k}$ such that $Z \subseteq \bigcup_{i=1}^k h_M\big(a_{M_i}\big)^\circ \subseteq \bigcup_{i=1}^k h_M\big(a_{M_i}\big) = h_M(a_1a_2\ldots a_k)$. If we set $a_n = \prod_{i=1}^k a_{M_i}$, then $Z \subseteq h_M(a_n) \subseteq U_n$. Consequently, $Z = \bigcap_{n\in\mathbb{N}} h_M(a_n) \in \mathcal{H}_M$, by the assumption. □

Now we can conclude the following corollary from Propositions 4.7 and 4.9.

Corollary 4.10. *A space X is pseudocompact if and only if $\mathcal{H}_M$ is closed under countable intersection.*

References

[1] H. Al-Ezeh, Pure Ideals In Commutative Reduced Gelfand Rings With Unity, *Arch. Math.*, 53(3) (1989) 266–269.

[2] A.R. Aliabad and R. Mohamadian, On sz°-Ideals In Polynomial Rings, *Comm. Algebra*, 39(2) (2011) 701–717.

[3] A.R. Aliabad, R. Mohamadian, and S. Nazari. On Regular Ideals In Reduced Rings, *Filomat*, 31(12) (2017) 3715–3726.

[4] M. F. Atiyah and I. G. Macdonald, *Introduction To Commutative Algebra*, volume 2. Addison-Wesley Reading, MA, 1969.

[5] T. S. Blyth, *Lattices And Ordered Algebraic Structures*. London: Springer, 2005.

[6] F. Borceux and G. Van den Bossche, *Algebra In A Localic Topos With Applications To Ring Theory*, volume 1038, Springer, 2006.

[7] M. Contessa, On Cerrain Classes Of pe-Rings, *Comm. Algebra*, 12(12) (1984) 1447–1469.

[8] Maria Contessa, On PM-Rings, *Comm. Algebra*, 10(1) (1982) :93–108.

[9] G. De Marco and A. Orsatti, Commutative Rings In Which Every Prime Ideal Is Contained In A Unique Maximal Ideal, *Proc. Amer. Math. Soc.*, 30(3) (1971) 459–466.

[10] L. Gillman and M. Jerison, *Rings Of Continuous Functions*. Van. Nostrand Reinhold, New York, 1960.

[11] Kenneth R. Goodearl, *Von Neumann Regular Rings*, 2nd ed. Malabar, Fl: Krieger Publishing Company, 2nd ed. edition, 1991.

[12] G. Mason, z-Ideals and Prime ideals, *J. Algebra*, 26(2) (1973) 280–297.

[13] Ch. J. Mulvey, A Generalisation Of Gelfand Duality, *J. Algebra*, 56(2) (1979) 499–505.

[14] S. Willard, *General Topology*. Addison Wesley Publishing Company, New York, 1970.

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