Journal of Mathematical Extension
Vol. 16, No. 9, (2022) (1)1-20
URL: https://doi.org/10.30495/JME.2022.1861
ISSN: 1735-8299
Original Research Paper

# Constructing a Heyting Semilattice that has Wajesberg Property by Using Fuzzy Implicative Deductive Systems of Hoops

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**Abstract.** In this paper, we defined the notions of $(\in, \in)$-fuzzy implicative deductive systems and $(\in, \in \vee q)$-fuzzy implicative deductive systems of hoops and studied some traits and tried to define some definitions that are equivalent to them. Thus by using the notion of $(\in, \in)$-fuzzy deductive system of hoop, we defined a new congruence relation on hoop and show that the algebraic structure that is made by it is a Brouwerian semilattice, Heyting algebra and Wajesberg hoop.

Received: November 2020; Accepted: May 2021
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**AMS Subject Classification:** 03G25, 06A12, 06B99, 06D72.
**Keywords and Phrases:** Hoop, fuzzy implicative deductive system, Brouwerian semilattice, Heyting algebra, Wajesberg hoop.

# 1 Introduction

One of the logical algebras that is studied by many mathematicians these days is an algebraic structure which is called hoop and was introduced by Bosbach in [12, 13]. This algebraic structure can easily be considered as an extension for BL-algebras and MV-algebras, and there are many examples that show that this algebraic structure is different from the residuated lattices. To learn more about hoops, we suggest that readers study the articles such as [4, 3, 6, 9, 15, 18, 21]. It is safe to say that most of the studies and researches in the field of hoop algebras have been done by Aaly and Borzooei, who have studied this algebraic structure in various fields. For example, they studied different deductive systems in [6], they have studied how this deductive systems relate to each other, the quotient structure produced by them, and etc., on this algebra. The main idea of using and defining the concept of fuzzy point as fuzzy sets is expressed in the article [20] which was then examined in various articles and in various fields, such as logical algebras. For example, Jun in [17] introduced fuzzy subalgebras in of $BCK/BCI$-algebras and called it $(\alpha, \beta)$-fuzzy subalgerbas of $BCK/BCI$-algebras. In fact, first this concept was studied in the field of sub-algebras and different types of it were introduced and studied, then this idea was studied in the field of special sub-algebras such as ideals and filters. Therefore, its importance and application in various fields led us to examine these concepts in the field of hoop algebras.

In this paper, we defined the notions of $(\in, \in)$-fuzzy implicative deductive systems and $(\in, \in \vee q)$-fuzzy implicative deductive systems of hoops and studied some traits and tried to define some definitions that are equivalent to them. Thus by using the notion of $(\in, \in)$-fuzzy deductive system of hoop, we defined a new congruence relation on hoop and show that the algebraic structure that is made by it is a Brouwerian semilattice, Heyting algebra and Wajesberg hoop.

## 2 Preliminaries

A *hoop* is an algebraic structure $(\mathfrak{X}, \bullet, \twoheadrightarrow, 1)$ where $(\mathfrak{X}, \bullet, 1)$ is a commutative monoid and, for each $\omega, \sigma, \kappa \in \mathfrak{X}$,

$(\mathfrak{X}_1)$ $\omega \twoheadrightarrow \omega = 1$,

$(\mathfrak{X}_2)$ $\omega \bullet (\omega \twoheadrightarrow \sigma) = \sigma \bullet (\sigma \twoheadrightarrow \omega)$,

$(\mathfrak{X}_3)$ $\omega \twoheadrightarrow (\sigma \twoheadrightarrow \kappa) = (\omega \bullet \sigma) \twoheadrightarrow \kappa$.

A hoop $(\mathfrak{X}, \preccurlyeq)$ is a poset where $\omega \preccurlyeq \sigma$ iff $\omega \twoheadrightarrow \sigma = 1$. A *bounded* hoop $\mathfrak{X}$ is an algebraic structure that has the least element such as $0 \in \mathfrak{X}$ such that $0 \preccurlyeq \omega$, for every $\omega \in \mathfrak{X}$. Consider $\omega^0 = 1,\ \omega^n = \omega^{n-1} \bullet \omega$, for each $n \in \mathbb{N}$. The operation " $\sim$ " is defined on a bounded hoop $\mathfrak{X}$ by, $\omega^{\sim} = \omega \twoheadrightarrow 0$, for every $\omega \in \mathfrak{X}$. A non-empty subset $\mathcal{S}$ of $\mathfrak{X}$ is called a *sub-hoop* if for every $\omega, \sigma \in \mathcal{S}$,

$$\omega \bullet \sigma \in \mathcal{S} \text{ and } \omega \twoheadrightarrow \sigma \in \mathcal{S}.$$

Clearly, each sub-hoop contains the constant 1.

**Note.** From now on, the symbol $\mathfrak{X}$ means a hoop such as $(\mathfrak{X}, \bullet, \twoheadrightarrow, 1)$.

**Proposition 2.1.** [12, 14] *For each $\omega, \sigma, \kappa \in \mathfrak{X}$, we have:*
$(i)$ $(\mathfrak{X}, \preccurlyeq)$ *is a meet-semilattice,*
$(ii)$ $\omega \bullet \sigma \preccurlyeq \kappa$ *iff* $\omega \preccurlyeq \sigma \twoheadrightarrow \kappa$,
$(iii)$ $\omega \bullet \sigma \preccurlyeq \omega, \sigma$ *and* $\omega^n \preccurlyeq \omega$, *for any* $n \in \mathbb{N}$,
$(iv)$ $\omega \preccurlyeq \sigma \twoheadrightarrow \omega$,
$(v)$ $1 \twoheadrightarrow \omega = \omega$ *and* $\omega \twoheadrightarrow 1 = 1$,
$(vi)$ $\omega \preccurlyeq (\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma$,
$(vii)$ $\omega \twoheadrightarrow \sigma \preccurlyeq (\sigma \twoheadrightarrow \kappa) \twoheadrightarrow (\omega \twoheadrightarrow \kappa)$,
$(viii)$ $\omega \preccurlyeq \sigma$ *implies* $\omega \bullet \kappa \preccurlyeq \sigma \bullet \kappa$, $\kappa \twoheadrightarrow \omega \preccurlyeq \kappa \twoheadrightarrow \sigma$ *and* $\sigma \twoheadrightarrow \kappa \preccurlyeq \omega \twoheadrightarrow \kappa$,
$(ix)$ $((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \omega = \sigma \twoheadrightarrow \omega$,
$(x)$ *If $\mathfrak{X}$ is bounded, then* $\omega^{\sim} \preccurlyeq \omega \twoheadrightarrow \sigma$ *and* $\omega^{\sim\sim\sim} = \omega^{\sim}$.

**Definition 2.2.** [14] For each $\omega, \sigma \in \mathfrak{X}$, define,

$$\omega \vee \sigma = ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \wedge ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega).$$

Then $\mathfrak{X}$ is said to be a $\vee$-*hoop* if $\vee$ is the join operation and $(\mathfrak{X}, \vee, \wedge)$ is a distributive lattice.

A non-empty subset $\mathfrak{F}$ of $\mathfrak{X}$ is said to be a *deductive system* of $\mathfrak{X}$ if, for every $\omega, \sigma \in \mathfrak{F}$, $\omega \bullet \sigma \in \mathfrak{F}$ and if for each $\sigma \in \mathfrak{X}$ and $\omega \in \mathfrak{F}$, $\omega \preccurlyeq \sigma$, then $\sigma \in \mathfrak{F}$ (see [14]).
Also, $\emptyset \neq \mathfrak{F} \subseteq \mathfrak{X}$ is said to be an *implicative deductive system* of $\mathfrak{X}$ if $1 \in \mathfrak{F}$ and, for each $\omega, \sigma, \kappa \in \mathfrak{X}$, $\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \in \mathfrak{F}$ and $\omega \in \mathfrak{F}$ imply $\sigma \in \mathfrak{F}$ (see [6]).

A fuzzy set $\varrho$ in a set $Z$ like

$$\varrho(\sigma) := \begin{cases} \varepsilon \in (0,1] & \text{if } \sigma = \omega, \\ 0 & \text{if } \sigma \neq \omega, \end{cases}$$

is called a *fuzzy point* with support $\omega$ and $\varepsilon$ and is shown by $\omega_\varepsilon$.

Suppose $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. Then for a fuzzy point $\omega_\varepsilon$ and a fuzzy set $\varrho$ in $Z$ we define $\omega_\varepsilon \alpha \varrho$ as follows:

For $\omega \in Z$ and $\varepsilon \in [0,1]$, $\omega_\varepsilon \in \varrho$ (resp. $\omega_\varepsilon q \varrho$) which means $\varrho(\omega) \succcurlyeq \varepsilon$ (resp. $\varrho(\omega) + \varepsilon \succ 1$), and $\omega_\varepsilon$ is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\varrho$.

Also, we write $\omega_\varepsilon \in \vee\, q\, \varrho$ (resp. $\omega_\varepsilon \in \wedge\, q\, \varrho$) where $\omega_\varepsilon \in \varrho$ or $\omega_\varepsilon q \varrho$ (resp. $\omega_\varepsilon \in \varrho$ and $\omega_\varepsilon q \varrho$).

For any fuzzy set $\varrho$ in $\mathfrak{X}$ and $\varepsilon \in (0,1]$, we introduce the next subsets of $\mathfrak{X}$ and called them $\in$*-level set*, $q$*-set* and $\in \vee\, q$*-set*, respectively.

$$\begin{aligned} &\mathcal{U}(\varrho; \varepsilon) := \{\omega \in \mathfrak{X} \mid \varrho(\omega) \succcurlyeq \varepsilon\}, \\ &\varrho_q^\varepsilon := \{\omega \in \mathfrak{X} \mid \omega_\varepsilon\, q\, \varrho\}, \\ &\varrho_{\in \vee\, q}^\varepsilon := \{\omega \in \mathfrak{X} \mid \omega_\varepsilon \in \vee\, q \varrho\}. \end{aligned}$$

**Definition 2.3.** [11] Consider $(\alpha, \beta)$ is any one of $(\in, \in)$ and $(\in, \in \vee\, q)$. A fuzzy set $\varrho$ in $\mathfrak{X}$ is said to be an $(\alpha, \beta)$-*fuzzy deductive system* of $\mathfrak{X}$ if

$$(\forall \omega \in \mathfrak{X})(\forall \varepsilon \in (0,1])(\omega_\varepsilon \alpha \varrho \ \Rightarrow \ 1_\varepsilon \beta \varrho),$$

$$(\forall \omega, \sigma \in \mathfrak{X})(\forall \varepsilon, \iota \in (0,1])(\omega_\varepsilon \alpha \varrho, \ (\omega \twoheadrightarrow \sigma)_\iota \alpha \varrho \ \Rightarrow \sigma_{\min\{\varepsilon, \iota\}} \beta \varrho).$$

**Corollary 2.4.** [11] *Each* $(\in, \in)$*-fuzzy deductive system of* $\mathfrak{X}$ *such as* $\varrho$ *satisfies the next condition:*

$$(\forall\ \omega, \sigma \in \mathfrak{X})(\text{if } \omega \preccurlyeq \sigma, \ \text{then } \varrho(\omega) \preccurlyeq \varrho(\sigma))$$

**Theorem 2.5.** [11] *Consider* $\varrho$ *is an* $(\in,\in)$*-fuzzy deductive system of* $\mathfrak{X}$*,* $\omega,\sigma \in \mathfrak{X}$ *and* $\varepsilon,\iota,l,m \in (0,1]$*. Define*

$$\omega \approx_{\varrho} \sigma \text{ iff } (\omega \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho \text{ and } (\sigma \twoheadrightarrow \omega)_{\iota} \in \varrho.$$

*Then the relation* $\approx_{\varrho}$ *is a congruence relation on* $\mathfrak{X}$*. Thus* $\frac{\mathfrak{X}}{\approx_{\varrho}} = \{[e]_{\varrho} \mid e \in \mathfrak{X}\}$ *and operations* $\otimes$ *and* $\rightsquigarrow$ *on* $\frac{\mathfrak{X}}{\approx_{\varrho}}$ *are as follows:*

$$[e]_{\varrho} \otimes [u]_{\varrho} = [e \bullet u]_{\varrho} \text{ and } [e]_{\varrho} \rightsquigarrow [u]_{\varrho} = [e \twoheadrightarrow u]_{\varrho}.$$

*Hence,* $(\frac{\mathfrak{X}}{\approx_{\varrho}}, \otimes, \rightsquigarrow, [1]_{\varrho})$ *is a hoop where*

$$[e]_{\varrho} \preccurlyeq [u]_{\varrho} \text{ iff } (e \twoheadrightarrow u)_{\varepsilon} \in \varrho, \text{ for any } e,u \in \mathfrak{X} \text{ and } \varepsilon \in (0,1].$$

## 3 $(\alpha,\ \beta)$-fuzzy implicative deductive systems of hoops

Here, we introduce $(\alpha,\beta)$-fuzzy implicative deductive systems for $(\alpha,\beta) \in \{(\in,\in),(\in,\in\vee q)\}$ of hoops and we study their traits and find some equivalence definitions of them. Moreover, we study the relation among $(\alpha,\beta)$-fuzzy implicative with $(\alpha,\beta)$-fuzzy deductive system one.

**Note.** Set $\mathfrak{X}$ is a bounded hoop and $\varrho$ is a fuzzy set in $\mathfrak{X}$.

**Definition 3.1.** Assume $(\alpha,\beta)$ is one of $(\in,\in)$ and $(\in,\in\vee q)$. Then $\varrho$ is said to be an $(\alpha,\beta)$-*fuzzy implicative deductive system* of $\mathfrak{X}$ if next assertions are valid.

$$(\forall \omega \in \mathfrak{X})(\forall \varepsilon \in (0,1])(\omega_{\varepsilon}\alpha\varrho \ \Rightarrow \ 1_{\varepsilon}\beta\varrho), \tag{1}$$

$$(\forall \omega,\sigma \in \mathfrak{X})(\forall \varepsilon,\iota \in (0,1])(\omega_{\varepsilon}\alpha\varrho,\ (\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_{\iota}\alpha\varrho \ \Rightarrow \sigma_{\min\{\varepsilon,\iota\}}\beta\varrho). \tag{2}$$

**Example 3.2.** Suppose $\mathfrak{X} = \{0,e,u,1\}$. Then the operations $\bullet$ and $\twoheadrightarrow$ on $\mathfrak{X}$ are defined by the next tables:

| $\twoheadrightarrow$ | 0 | e | u | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| e | e | 1 | 1 | 1 |
| u | 0 | e | 1 | 1 |
| 1 | 0 | e | u | 1 |

| $\bullet$ | 0 | e | u | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| e | 0 | 0 | e | e |
| u | 0 | e | u | u |
| 1 | 0 | e | u | 1 |

Thus $(\mathfrak{X}, \bullet, \twoheadrightarrow, 0, 1)$ is a bounded hoop. Define $\varrho(0) = 0.6$, $\varrho(e) = 0.4$, $\varrho(u) = 0.55$ and $\varrho(1) = 0.8$. Obviously, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$.

**Theorem 3.3.** *$\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$ iff*

$$(\forall \omega \in \mathfrak{X})(\varrho(1) \succcurlyeq \varrho(\omega)),$$

$$(\forall \omega, \sigma \in \mathfrak{X})(\varrho(\sigma) \succcurlyeq \min\{\varrho(\omega), \varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))\}).$$

**Proof.** $(\Rightarrow)$ Suppose $\omega \in \mathfrak{X}$ and $\varepsilon \in (0, 1]$ such that $\varrho(\omega) = \varepsilon$. From $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$, we obtain $\varrho(1) \succcurlyeq \varepsilon = \varrho(\omega)$. So, for each $\omega \in \mathfrak{X}$, $\varrho(1) \succcurlyeq \varrho(\omega)$. Consider $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 1]$ such that $\varrho(\omega) \succcurlyeq \varepsilon$ and $\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)) \succcurlyeq \iota$, and so $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_\iota \in \varrho$. Moreover, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$, $\sigma_{\min\{\varepsilon, \iota\}} \in \varrho$, thus $\varrho(\sigma) \succcurlyeq \min\{\varepsilon, \iota\}$. Hence,

$$\min\{\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)), \varrho(\omega)\} \preccurlyeq \varrho(\sigma).$$

$(\Leftarrow)$ Assume $\omega \in \mathfrak{X}$ and $\varepsilon \in (0, 1]$ such that $\omega_\varepsilon \in \varrho$. Then $\varrho(\omega) \succcurlyeq \varepsilon$. From $\varepsilon \preccurlyeq \varrho(\omega) \preccurlyeq \varrho(1)$, we consequence $1_\varepsilon \in \varrho$. Now, suppose $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_\iota \in \varrho$, for every $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 1]$. Then by hypothesis,

$$\min\{\varepsilon, \iota\} \preccurlyeq \min\{\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)), \varrho(\omega)\} \preccurlyeq \varrho(\sigma),$$

thus $\min\{\varepsilon, \iota\} \preccurlyeq \varrho(\sigma)$. Hence, $\sigma_{\min\{\varepsilon, \iota\}} \in \varrho$. Therefore, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$. □

**Theorem 3.4.** *Each $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$.*

**Proof.** If $\omega_\varepsilon \in \varrho$, then $1_\varepsilon \in \varrho$, for every $\omega \in \mathfrak{X}$ and $\varepsilon \in (0, 1]$. Assume $\omega, \sigma \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 1]$ such that $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow \sigma)_\iota \in \varrho$. So, $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow 1) \twoheadrightarrow \sigma))_\iota \in \varrho$. From $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$, $\sigma_{\min\{\varepsilon, \iota\}} \in \varrho$. Thus, $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$. □

Next example shows that the converse of Theorem 3.4, does not hold.

**Example 3.5.** Consider $\mathfrak{X} = \{0, e, u, i, o, 1\}$. Define operations $\bullet$ and $\twoheadrightarrow$ on $\mathfrak{X}$ by next tables:

| $\twoheadrightarrow$ | 0 | e | u | i | o | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| e | i | 1 | u | i | u | 1 |
| u | o | e | 1 | u | e | 1 |
| i | e | e | 1 | 1 | e | 1 |
| o | u | 1 | 1 | u | 1 | 1 |
| 1 | 0 | e | u | i | o | 1 |

| $\bullet$ | 0 | e | u | i | o | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| e | 0 | e | o | 0 | o | e |
| u | 0 | o | i | i | 0 | u |
| i | 0 | 0 | i | i | 0 | i |
| o | 0 | o | 0 | 0 | 0 | o |
| 1 | 0 | e | u | i | o | 1 |

So $(\mathfrak{X}, \bullet, \twoheadrightarrow, 0, 1)$ is a bounded hoop. Define $\varrho$ in $\mathfrak{X}$ as follows:

$$\varrho : \mathfrak{X} \twoheadrightarrow [0,1],\ x \mapsto \begin{cases} 0.5 & \text{if } \omega = 0, \\ 0.7 & \text{if } \omega = e, \\ 0.3 & \text{if } \omega = u, \\ 0.5 & \text{if } \omega = i, \\ 0.3 & \text{if } \omega = o, \\ 0.8 & \text{if } \omega = 1 \end{cases}$$

Obviously, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$ which is not an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$. Because

$$0.3 = \varrho(u) \not\succeq \min\{\varrho(0), \varrho(0 \twoheadrightarrow u)\} = \min\{0.5, 0.8\}.$$

**Corollary 3.6.** *Each $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$ like $\varrho$ satisfies in the next condition:*

$$(\forall\ \omega, \sigma \in \mathfrak{X})(\textit{if } \omega \preccurlyeq \sigma,\ \textit{then } \varrho(\omega) \preccurlyeq \varrho(\sigma)).$$

**Theorem 3.7.** *Suppose $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$. Then, the next equivalent statements hold for any $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 1]$:*
*(i)* *$\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system,*
*(ii)* *if $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_\varepsilon \in \varrho$, then $\omega_\varepsilon \in \varrho$,*
*(iii)* *$(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)_\varepsilon \in \varrho$,*

*(iv)* $((\omega^{\sim} \twoheadrightarrow \omega) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$,
*(v)* *if* $((\omega \bullet \kappa^{\sim}) \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$ *and* $(\sigma \twoheadrightarrow \kappa)_{\iota} \in \varrho$, *then* $(\omega \twoheadrightarrow \kappa)_{\min\{\varepsilon,\iota\}} \in \varrho$,
*(vi)* *if* $((\omega \bullet \sigma^{\sim}) \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$, *then* $(\omega \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$.

**Proof.** Let $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 1]$. Then:
$(i) \Rightarrow (ii)$ Suppose $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$. From $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$, $1_{\varepsilon} \in \varrho$, we consequence $(1 \twoheadrightarrow ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega))_{\varepsilon} \in \varrho$ and $1_{\varepsilon} \in \varrho$. Thus, by (i), $\omega_{\varepsilon} \in \varrho$.
$(ii) \Rightarrow (i)$ Let $\omega_{\varepsilon} \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_{\iota} \in \varrho$. Moreover, $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$, we obtain $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$. By (ii), we consequence that $\sigma_{\min\{\varepsilon,\iota\}} \in \varrho$. Therefore, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system .
$(i) \Rightarrow (iii)$ Since $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$, $1_{\varepsilon} \in \varrho$. Moreover, by Proposition 2.1, we have

$$\begin{aligned} & \omega \twoheadrightarrow [((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \kappa) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)] \\ = & ((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \kappa) \twoheadrightarrow [\omega \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)] \\ = & ((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \kappa) \twoheadrightarrow [((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow (\omega \twoheadrightarrow \omega)] \\ = & 1 \end{aligned}$$

Then

$$(\omega \twoheadrightarrow [((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \kappa) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)])_{\varepsilon} = 1_{\varepsilon} \in \varrho.$$

Since $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system, we get that $(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$.
$(iii) \Rightarrow (i)$ Let $\omega_{\varepsilon} \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_{\iota} \in \varrho$. As $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$, we get $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$. Moreover, by (iii), $(((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$, we get that $\sigma_{\min\{\varepsilon,\iota\}} \in \varrho$. Hence, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system .
$(iii) \Rightarrow (iv)$ Set $\sigma = 0$ in (iii).
$(iv) \Rightarrow (iii)$ Assume $((\omega^{\sim} \twoheadrightarrow \omega) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$. By Proposition 2.1(x) and (viii), $\omega^{\sim} \preccurlyeq \omega \twoheadrightarrow \sigma$, and so $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega \preccurlyeq \omega^{\sim} \twoheadrightarrow \omega$ and also we have

$$(\omega^{\sim} \twoheadrightarrow \omega) \twoheadrightarrow \omega \preccurlyeq ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega.$$

From $\varrho$ is an $(\in, \in)$-fuzzy deductive system and $((\omega^{\sim} \twoheadrightarrow \omega) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$, by Corollary 2.4, $(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$.

$(v) \Rightarrow (vi)$ Consider $((\omega \bullet \sigma^{\sim}) \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$. As $(\sigma \twoheadrightarrow \sigma)_{\varepsilon} = 1_{\varepsilon} \in \varrho$, by (v), $(\omega \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$.
$(vi) \Rightarrow (v)$ Assume $((\omega \bullet \kappa^{\sim}) \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$ and $(\sigma \twoheadrightarrow \kappa)_{\iota} \in \varrho$. From $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$, by Proposition 2.1(vii) and Corollary 2.4,

$$((\omega \bullet \kappa^{\sim}) \twoheadrightarrow \sigma) \preccurlyeq (\sigma \twoheadrightarrow \kappa) \twoheadrightarrow ((\omega \bullet \kappa^{\sim}) \twoheadrightarrow \kappa),$$

thus $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow ((\omega \bullet \kappa^{\sim}) \twoheadrightarrow \kappa))_{\varepsilon} \in \varrho$. Hence, $((\omega \bullet \kappa^{\sim}) \twoheadrightarrow \kappa)_{\min\{\varepsilon,\iota\}} \in \varrho$. By (vi), we have $(\omega \twoheadrightarrow \kappa)_{\min\{\varepsilon,\iota\}} \in \varrho$.
$(vi) \Rightarrow (iv)$ As $((\omega^{\sim} \twoheadrightarrow \omega) \twoheadrightarrow (\omega^{\sim} \twoheadrightarrow \omega))_{\varepsilon} = 1_{\varepsilon}$, we obtain $(((\omega^{\sim} \twoheadrightarrow \omega) \bullet \omega^{\sim}) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$. Now, by (vi), $((\omega^{\sim} \twoheadrightarrow \omega) \twoheadrightarrow \omega)_{\varepsilon} \in \varrho$.
$(vi) \Rightarrow (i)$ Assume $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_{\varepsilon} \in \varrho$ and $\omega_{\iota} \in \varrho$. From $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$, we obtain $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$. Moreover, by Proposition 2.1(x), $\sigma^{\sim} \preccurlyeq \sigma \twoheadrightarrow \kappa$. Also, by Proposition 2.1(viii), $(\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma \preccurlyeq \sigma^{\sim} \twoheadrightarrow \sigma$. As $\varrho$ is an $(\in, \in)$-fuzzy deductive system of $\mathfrak{X}$ and $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$, by Corollary 2.4, $(\sigma^{\sim} \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$, and so $(1 \twoheadrightarrow (\sigma^{\sim} \twoheadrightarrow \sigma))_{\min\{\varepsilon,\iota\}} \in \varrho$, then by (vi), $(1 \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} = \sigma_{\min\{\varepsilon,\iota\}} \in \varrho$. Hence, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$.
$(i) \Rightarrow (vi)$ Suppose $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$. If $((\omega \bullet \sigma^{\sim}) \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$, then by Proposition 2.1(iv) and (viii), $\sigma \preccurlyeq \omega \twoheadrightarrow \sigma$ and so $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow 0 \preccurlyeq \sigma \twoheadrightarrow 0$. Thus

$$\sigma^{\sim} \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \preccurlyeq (\omega \twoheadrightarrow \sigma)^{\sim} \twoheadrightarrow (\omega \twoheadrightarrow \sigma).$$

From $((\omega \bullet \sigma^{\sim}) \twoheadrightarrow \sigma)_{\varepsilon} = (\sigma^{\sim} \twoheadrightarrow (\omega \twoheadrightarrow \sigma))_{\varepsilon} \in \varrho$, by Corollary 2.4, $((\omega \twoheadrightarrow \sigma)^{\sim} \twoheadrightarrow (\omega \twoheadrightarrow \sigma))_{\varepsilon} \in \varrho$. Hence,

$$(1 \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow 0) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)))_{\varepsilon} \in \varrho.$$

As $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system and $1_{\varepsilon} \in \varrho$, we consequence $(\omega \twoheadrightarrow \sigma)_{\varepsilon} \in \varrho$. □

**Theorem 3.8.** *If $\varrho$ is a non-zero $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$, then the set*

$$\mathfrak{X}_0 := \{\omega \in \mathfrak{X} \mid \varrho(\omega) \neq 0\}$$

*is an implicative deductive system of $\mathfrak{X}$.*

**Proof.** Assume $\omega \in \mathfrak{X}_0$. From $\varrho(\omega) \neq 0$, we consequence that there is $\varepsilon \in (0,1]$ such that $\varrho(\omega) \succcurlyeq \varepsilon$. As $\varrho$ is an $(\in,\in)$-fuzzy implicative deductive system of $\mathfrak{X}$ and $\omega_\varepsilon \in \varrho$, we obtain $1_\varepsilon \in \varrho$. Thus $\varrho(1) \succcurlyeq \varrho(\omega) = \varepsilon \neq 0$, so $1 \in \mathfrak{X}_0$. Suppose $\omega, \omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \in \mathfrak{X}_0$. So, there is $\varepsilon, \iota \in (0,1]$, where $\varrho(\omega) \succcurlyeq \varepsilon$ and $\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)) \succcurlyeq \iota$. Hence $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_\iota \in \varrho$. By Definition 3.1, $\sigma_{\min\{\varepsilon,\iota\}} \in \varrho$, thus $\varrho(\sigma) \succcurlyeq \min\{\varepsilon, \iota\} \neq 0$. So $\sigma \in \mathfrak{X}_0$. Hence, $\mathfrak{X}_0$ is an implicative deductive system of $\mathfrak{X}$. □

**Proposition 3.9.** *Consider $\varrho$ is an $(\in,\in)$-fuzzy implicative deductive system of $\mathfrak{X}$. For every $\varepsilon \in (0,1]$, $\varrho_q^\varepsilon$ is an implicative deductive system of $\mathfrak{X}$.*

**Proof.** Assume $\omega \in \varrho_q^\varepsilon$, for each $\omega \in \mathfrak{X}$ and $\varepsilon \in (0,1]$. Then $\omega_\varepsilon q \varrho$, and so $\varrho(\omega) + \varepsilon \succ 1$. Thus, $\varrho(\omega) \succ 1 - \varepsilon$. By hypothesis, from $\omega_{1-\varepsilon} \in \varrho$, we obtain $1_{1-\varepsilon} \in \varrho$, so $\varrho(1) \succ 1 - \varepsilon$. Thus, $\varrho(1) + \varepsilon \succ 1$ and $1 \in \varrho_q^\varepsilon$. Consider $\omega, \omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \in \varrho_q^\varepsilon$, for each $\omega, \sigma, \kappa \in \mathfrak{X}$. So

$$\varrho(\omega) + \varepsilon \succ 1 \quad , \quad \varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)) + \varepsilon \succ 1.$$

Thus

$$\varrho(\omega) \succ 1 - \varepsilon \quad , \quad \varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)) \succ 1 - \varepsilon.$$

As $\varrho$ is an $(\in,\in)$-fuzzy implicative deductive system of $\mathfrak{X}$, we obtain $\varrho(\sigma) \succ 1-\varepsilon$ and $\varrho(\sigma)+\varepsilon \succ 1$. Thus, $\sigma \in \varrho_q^\varepsilon$. Hence, $\varrho_q^\varepsilon$ is an implicative deductive system of $\mathfrak{X}$. □

**Corollary 3.10.** *Suppose $\varrho$ is an $(\in, \in\vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. Thus for every $\varepsilon \in (0,1]$, $\varrho_{\in\vee q}^\varepsilon$ is an implicative deductive system of $\mathfrak{X}$.*

**Proof.** Using Theorem 3.8 and Proposition 3.9 □

**Proposition 3.11.** *Assume $\varrho$ is an $(\in,\in)$-fuzzy implicative deductive system of $\mathfrak{X}$. Thus, for every $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon \in (0,1]$,*
*(i) If $(\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma))_\varepsilon \in \varrho$, then $(\omega \twoheadrightarrow \sigma)_\varepsilon \in \varrho$.*
*(ii) If $(\kappa \twoheadrightarrow (\sigma \twoheadrightarrow \omega))_\varepsilon \in \varrho$, then $((\kappa \twoheadrightarrow \sigma) \twoheadrightarrow (\kappa \twoheadrightarrow \omega))_\varepsilon \in \varrho$.*

**Proof.** $(i)$ Consider $(\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma))_\varepsilon \in \varrho$, for each $\omega, \sigma \in \mathfrak{X}$ and $\varepsilon \in (0,1]$. By Proposition 2.1(vii),

$$\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \preccurlyeq ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma).$$

Also, by Corollary 2.4,

$$((\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)))_\varepsilon = 1_\varepsilon \in \varrho.$$

Moreover, $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$, so $(\omega \twoheadrightarrow \sigma)_\varepsilon \in \varrho$.

(*ii*) Suppose $(\kappa \twoheadrightarrow (\sigma \twoheadrightarrow \omega))_\varepsilon \in \varrho$, for each $\omega, \sigma \in \mathfrak{X}$ and $\varepsilon \in (0, 1]$. Thus $(\sigma \twoheadrightarrow (\kappa \twoheadrightarrow \omega))_\varepsilon \in \varrho$. From $\kappa \bullet (\kappa \twoheadrightarrow \sigma) \preccurlyeq \sigma$, by Proposition 2.1(viii),

$$\sigma \twoheadrightarrow (\kappa \twoheadrightarrow \omega) \preccurlyeq (\kappa \bullet (\kappa \twoheadrightarrow \sigma)) \twoheadrightarrow (\kappa \twoheadrightarrow \omega).$$

Moreover, $\varrho$ is an $(\in, \in)$-fuzzy deductive system, then by Corollary 2.4, $(\kappa \twoheadrightarrow ((\kappa \twoheadrightarrow \sigma) \twoheadrightarrow (\kappa \twoheadrightarrow \omega)))_\varepsilon \in \varrho$, thus $(\kappa \twoheadrightarrow (\kappa \twoheadrightarrow ((\kappa \twoheadrightarrow \sigma) \twoheadrightarrow \omega)))_\varepsilon \in \varrho$. Using (i), $(\kappa \twoheadrightarrow ((\kappa \twoheadrightarrow \sigma) \twoheadrightarrow \omega))_\varepsilon \in \varrho$, it follows $((\kappa \twoheadrightarrow \sigma) \twoheadrightarrow (\kappa \twoheadrightarrow \omega))_\varepsilon \in \varrho$. □

**Proposition 3.12.** *Assume $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$. If $(\sigma \twoheadrightarrow \omega)_\varepsilon \in \varrho$, for every $\omega, \sigma \in \mathfrak{X}$ and $\varepsilon \in (0, 1]$, then*

$$(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega))_\varepsilon \in \varrho.$$

**Proof.** Let $\omega, \sigma \in \mathfrak{X}$, $\varepsilon \in (0, 1]$ and $\varrho$ be an $(\in, \in)$-fuzzy implicative deductive system of $\mathfrak{X}$. Suppose $(\sigma \twoheadrightarrow \omega)_\varepsilon \in \varrho$. By Proposition 2.1(iv), $\omega \preccurlyeq ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega$, thus by Proposition 2.1(viii), $(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \sigma \preccurlyeq \omega \twoheadrightarrow \sigma$, and so by Proposition 2.1(vii) and (viii),

$$\begin{aligned} \sigma \twoheadrightarrow \omega &\preccurlyeq ((\omega \twoheadrightarrow \sigma) \bullet ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma)) \twoheadrightarrow \omega \\ &= (\omega \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \\ &\preccurlyeq ((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \end{aligned}$$

Moreover, since $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system, then by Theorems 3.7 and 3.4, and Corollary 2.4, we have

$$\begin{aligned} & \varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \\ \succcurlyeq\ & \varrho(((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega)) \\ \succcurlyeq\ & \varrho((\omega \twoheadrightarrow \sigma) \twoheadrightarrow ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \\ =\ & \varrho((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)) \\ \succcurlyeq\ & \varrho(\sigma \twoheadrightarrow \omega) \end{aligned}$$

Hence, by Corollary 2.4, $(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_\varepsilon \in \varrho$. Also, by Proposition 2.1(vi) and (viii), $\omega \preccurlyeq (\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega$, and so

$$(\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma \preccurlyeq (((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma.$$

Thus,

$$\begin{aligned} & ((((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega) \\ \preccurlyeq \; & ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega). \end{aligned}$$

By Theorem 3.4, $\varrho$ is an $(\in, \in)$-fuzzy deductive system, then by Theorem 3.3,

$$\begin{aligned} & \varrho(((((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega)) \\ \preccurlyeq \; & \varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega)). \end{aligned}$$

As we prove that if $(\alpha \twoheadrightarrow \beta)_\varepsilon \in \varrho$, then $((\beta \twoheadrightarrow \alpha) \twoheadrightarrow \alpha) \twoheadrightarrow \beta)_\varepsilon \in \varrho$. Let $\beta = (\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega$ and $\alpha = \sigma$. Since $(\alpha \twoheadrightarrow \beta)_\varepsilon = (\sigma \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega))_\varepsilon = 1_\varepsilon \in \varrho$ and $\varrho$ is an $(\in, \in)$-fuzzy implicative deductive system, we consequence

$$\begin{aligned} & \varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega)) \\ \succcurlyeq \; & \varrho(((((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega)). \end{aligned}$$

Hence,

$$(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega))_\varepsilon \in \varrho.$$

□

**Definition 3.13.** [14, 5] *(i)* A hoop $(\mathfrak{X}, \bullet, \twoheadrightarrow, 1)$ is called a *Wajsberg* if for any $\omega, \sigma \in \mathfrak{X}$ we have, $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma = (\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega$.
*(ii)* A *Heyting algebra* is an algebra $(H, \rightarrow, \wedge, \vee, 1)$ of type $(2, 2, 2, 0)$ where $(H, \wedge, \vee, 1)$ is a lattice with the greatest element and the binary operation $\rightarrow$ on $H$ verifies, for any $x, y, z \in H$, by $x \leq y \rightarrow z$ if and only if $x \wedge y \leq z$.
*(iii)* *Brouwerian semilattice* is an algebraic structure $(B, \wedge, \rightarrow, 1)$ is $\wedge$-semilattices with a top element 1 and an implication operation $\rightarrow$ for any $x, y, z \in B$, satisfying $x \leq y \rightarrow z$ if and only if $x \wedge y \leq z$.

**Theorem 3.14.** *Let $\varrho$ be an $(\in,\in)$-fuzzy deductive system of $\mathfrak{X}$. If $\varrho$ is an $(\in,\in)$-fuzzy implicative deductive system, then $\frac{\mathfrak{X}}{\approx_\varrho}$ is a Heyting semilattice that has Wajesberg property.*

**Proof.** $(\Rightarrow)$ By Theorem 2.5, $\frac{\mathfrak{X}}{\approx_\varrho}$ is well-define and is a hoop. Since $\frac{\mathfrak{X}}{\approx_\varrho}$ is a hoop, by using Proposition 2.1(i), we have $\frac{\mathfrak{X}}{\approx_\varrho}$ is a $\wedge$-semilattice. Thus it is enough to prove that

$$[\omega]_\varrho \wedge [\sigma]_\varrho \preccurlyeq [\kappa]_\varrho \text{ iff } [\omega]_\varrho \preccurlyeq [\sigma]_\varrho \rightsquigarrow [\kappa]_\varrho, \text{ for all } \omega,\sigma,\kappa \in \mathfrak{X}.$$

Assume $[\omega]_\varrho \wedge [\sigma]_\varrho \preccurlyeq [\kappa]_\varrho$. By using Proposition 2.1(iii), $[\omega]_\varrho \otimes [\sigma]_\varrho \preccurlyeq [\omega]_\varrho \wedge [\sigma]_\varrho \preccurlyeq [\kappa]_\varrho$. Thus, $[\omega]_\varrho \otimes [\sigma]_\varrho \preccurlyeq [\kappa]_\varrho$. As $\frac{\mathfrak{X}}{\approx_\varrho}$ is a hoop, from Proposition 2.1(ii), $[\omega]_\varrho \preccurlyeq [\sigma]_\varrho \rightsquigarrow [\kappa]_\varrho$.

$(\Leftarrow)$ Assume $[\omega]_\varrho \preccurlyeq [\sigma]_\varrho \rightsquigarrow [\kappa]_\varrho$. From Theorem 2.5, $(\omega \twoheadrightarrow (\sigma \twoheadrightarrow \kappa))_\varepsilon \in \varrho$, for $\varepsilon \in (0,1]$. As $\varrho$ is an $(\in,\in)$-fuzzy implicative deductive system, by Proposition 3.11(ii), $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \kappa))_\varepsilon \in \varrho$. So, $[\omega \twoheadrightarrow \sigma]_\varrho \preccurlyeq [\omega \twoheadrightarrow \kappa]_\varrho$. Thus, $[\omega]_\varrho \rightsquigarrow [\sigma]_\varrho \preccurlyeq [\omega]_\varrho \rightsquigarrow [\kappa]_\varrho$. Moreover, $\frac{\mathfrak{X}}{\approx_\varrho}$ is a hoop and by Proposition 2.1(ii) and (i),

$$[\omega]_\varrho \wedge [\sigma]_\varrho = [\omega]_\varrho \otimes ([\omega]_\varrho \rightsquigarrow [\sigma]_\varrho) \preccurlyeq [\kappa]_\varrho.$$

Hence, $\frac{\mathfrak{X}}{\approx_\varrho}$ is a Brouwerian semilattice. On the other side, by Proposition 3.12, for all $\omega,\sigma \in \mathfrak{X}$ and $\varepsilon \in (0,1]$, $(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega))_\varepsilon \in \varrho$. Thus, by Theorem 2.5, $[(\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma]_\varrho \preccurlyeq [(\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega]_\varrho$. By the similar way, $[(\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega]_\varrho \preccurlyeq [(\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma]_\varrho$. Then

$$([\omega]_\varrho \rightsquigarrow [\sigma]_\varrho) \rightsquigarrow [\sigma]_\varrho = ([\sigma]_\varrho \rightsquigarrow [\omega]_\varrho) \rightsquigarrow [\omega]_\varrho.$$

Therefore, $\frac{\mathfrak{X}}{\approx_\varrho}$ is a Wajesberg hoop. Thus, by Definition 2.3, we define

$$[\omega]_\varrho \vee [\sigma]_\varrho = ([\omega]_\varrho \rightsquigarrow [\sigma]_\varrho) \rightsquigarrow [\sigma]_\varrho.$$

Hence, $\vee$ is join operation, and so by Definition 2.3, $\frac{\mathfrak{X}}{\approx_\varrho}$ is a distributive lattice. Thus, $\frac{\mathfrak{X}}{\approx_\varrho}$ is a Heyting semilattice. $\square$

**Note.** According to [10, Theorem 3.10], every $(\in,\in)$-fuzzy subhoop is an $(\in,\in \vee q)$-fuzzy subhoop of $\mathfrak{X}$. As each deductive system is a subhoop, obviously each $(\in,\in)$-fuzzy implicative deductive system of $\mathfrak{X}$ is an $(\in,\in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. The converse is not true always and we can check it by different examples such as [11, Example 3.9]. It means that there is $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$ that is not an $(\in, \in)$-fuzzy deductive system.

**Theorem 3.15.** *A fuzzy set $\varrho$ in $\mathfrak{X}$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$ iff for all $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon \in (0, 0.5]$, it satisfies:*

$$\varrho(1) \succcurlyeq \varrho(\omega),$$

$$\varrho(\sigma) \succcurlyeq \min\{\varrho(\omega), \varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))\}.$$

**Proof.** ($\Rightarrow$) Consider $\omega \in \mathfrak{X}$ and $\varepsilon \in (0, 0.5]$ where $\varrho(\omega) = \varepsilon$, so $\omega_\varepsilon \in \varrho$. From $\varrho$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$, we obtain $1_\varepsilon \in \vee q\varrho$. If $\varrho(1) \succcurlyeq \varepsilon$, then $\varrho(1) \succcurlyeq \varrho(\omega)$. Also, if $1_\varepsilon q\varrho$, then $\varrho(1) + \varepsilon \succ 1$, thus $\varrho(1) \succ 1 - \varepsilon$. As $\varepsilon \in (0, 0.5]$, we get $\varrho(1) \succ 1 - \varepsilon \succ \varepsilon = \varrho(\omega)$. So, in both cases, for every $\omega \in \mathfrak{X}$, $\varrho(1) \succcurlyeq \varrho(\omega)$. Assume $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 0.5]$ where $\varrho(\omega) \succcurlyeq \varepsilon$ and $\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \succcurlyeq \iota$. Thus $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_\iota \in \varrho$. Moreover, $\varrho$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$, we consequence $\sigma_{\min\{\varepsilon,\iota\}} \in \vee q\varrho$. If $\sigma_{\min\{\varepsilon,\iota\}} \in \varrho$, then the sentence holds. If $\sigma_{\min\{\varepsilon,\iota\}} q\varrho$, then $\varrho(\sigma) + \min\{\varepsilon, \iota\} \succ 1$, thus $\varrho(\sigma) \succ 1 - \min\{\varepsilon, \iota\}$. From $\varepsilon, \iota \in (0, 0.5]$, we obtain $\min\{\varepsilon, \iota\} \in (0, 0.5]$. So, $\varrho(\sigma) \succ 1 - \min\{\varepsilon, \iota\} > \min\{\varepsilon, \iota\}$. Hence, in both cases, for each $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 0.5]$, we get

$$\min\{\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma), \varrho(\omega)\} \preccurlyeq \varrho(\sigma).$$

($\Leftarrow$) Similar to the proof of Theorem 3.3. □

**Corollary 3.16.** *Every $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$ such as $\varrho$ satisfies in the next condition:*

$$(\forall\ \omega, \sigma \in \mathfrak{X})(\forall\ \varepsilon \in (0, 0.5])(\textit{if}\ \omega \preccurlyeq \sigma,\ \textit{then}\ \varrho(\omega) \preccurlyeq \varrho(\sigma)). \tag{3}$$

**Proof.** By using Theorem 3.15, from $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$, we obtain $\varrho(\omega) \preccurlyeq \varrho(1)$, for $\omega \in \mathfrak{X}$ and $\varepsilon \in (0, 0.5]$. Moreover, $\omega \preccurlyeq \sigma$, so $\omega \twoheadrightarrow \sigma = 1$. Thus by Theorem 3.15,

$$\begin{aligned}
\varrho(\sigma) &\succcurlyeq \min\{\varrho(\omega), \varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow 1) \twoheadrightarrow \sigma))\} \\
&= \min\{\varrho(\omega), \varrho(\omega \twoheadrightarrow \sigma)\} \\
&= \min\{\varrho(\omega), \varrho(1)\} \\
&= \varrho(\omega).
\end{aligned}$$

□

**Theorem 3.17.** *Each $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$ is an $(\in, \in \vee q)$-fuzzy deductive system of $\mathfrak{X}$.*

**Proof.** It follows by Theorem 3.4. □

**Theorem 3.18.** *Consider $\varrho$ is an $(\in, \in \vee q)$-fuzzy deductive system of $\mathfrak{X}$. The next equivalent conditions hold for every $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 0.5]$.*
*(i)* $\varrho$ *is an $(\in, \in \vee q)$-fuzzy implicative deductive system,*
*(ii)* $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_\varepsilon \in \varrho$ *implies* $\omega_\varepsilon \in \vee q\varrho$*,*
*(iii)* $(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \omega)_\varepsilon \in \vee q\varrho$*,*
*(iv)* $((\omega^\sim \twoheadrightarrow \omega) \twoheadrightarrow \omega)_\varepsilon \in \vee q\varrho$*,*
*(v)* $((\omega \bullet \kappa^\sim) \twoheadrightarrow \sigma)_\varepsilon \in \varrho$ *and* $(\sigma \twoheadrightarrow \kappa)_\iota \in \varrho$ *imply* $(\omega \twoheadrightarrow \kappa)_{\min\{\varepsilon,\iota\}} \in \vee q\varrho$*,*
*(vi)* $((\omega \bullet \sigma^\sim) \twoheadrightarrow \sigma)_\varepsilon \in \varrho$ *implies* $(\omega \twoheadrightarrow \sigma)_\varepsilon \in \vee q\varrho$*.*

**Proof.** Assume $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon, \iota \in (0, 0.5]$. Thus
$(i) \Rightarrow (ii)$ Suppose $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_\varepsilon \in \varrho$. Since $\varrho$ is an $(\in, \in \vee q)$-fuzzy deductive system of $\mathfrak{X}$, $1_\varepsilon \in \vee q\varrho$. If $1_\varepsilon \in \varrho$, then since $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_\varepsilon \in \varrho$, by (i), $\omega_\varepsilon \in \vee q\varrho$. If $1_\varepsilon q\varrho$, then $\varrho(1) + \varepsilon \succ 1$, and so $\varrho(1) \succ 1 - \varepsilon$. As $\varepsilon \in (0, 0.5]$, we obtain $\varrho(1) \succ 1 - \varepsilon \succ \varepsilon$. Thus, $1_\varepsilon \in \varrho$. Moreover, from $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)_\varepsilon \in \varrho$ and $1_\varepsilon \in \varrho$, by (i), $\omega_\varepsilon \in \vee q\varrho$. So in both cases, we consequence that, $\omega_\varepsilon \in \vee q\varrho$.
$(ii) \Rightarrow (i)$ Let $\omega_\varepsilon \in \varrho$ and $(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma))_\iota \in \varrho$. Since $\varrho$ is an $(\in, \in \vee q)$-fuzzy deductive system of $\mathfrak{X}$, we obtain $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \vee q\varrho$. If $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} \in \varrho$, then by (ii), we consequence that $\sigma_{\min\{\varepsilon,\iota\}} \in \vee q\varrho$. If $((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)_{\min\{\varepsilon,\iota\}} q\varrho$, then $\varrho((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) + \min\{\varepsilon, \iota\} \succ 1$, and so $\varrho((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \succ 1 - \min\{\varepsilon, \iota\}$. As $\varepsilon, \iota \in (0, 0.5]$, we get $\min\{\varepsilon, \iota\} \in (0, 0.5]$, and so $\varrho((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) > \min\{\varepsilon, \iota\}$. Hence by (ii), $\sigma_{\min\{\varepsilon,\iota\}} \in \vee q\varrho$. Thus, in both cases, $\varrho$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system .

The proof of other cases are similar to Theorem 3.7 and $(i) \Leftrightarrow (ii)$. □

**Theorem 3.19.** *Assume $\varrho \neq 0$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. Thus*

$$\mathfrak{X}_0 := \{\omega \in \mathfrak{X} \mid \varrho(\omega) \neq 0\}$$

*is an implicative deductive system of $\mathfrak{X}$.*

**Proof.** It follows by Theorem 3.8. $\square$

**Proposition 3.20.** *Consider $\varrho$ is an $(\in, \in\vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. Thus $\varrho_q^\varepsilon$ is an implicative deductive system of $\mathfrak{X}$, for every $\varepsilon \in (0.5, 1]$.*

**Proof.** Assume $\omega \in \varrho_q^\varepsilon$, for each $\omega \in \mathfrak{X}$ and $\varepsilon \in (0.5, 1]$. Then $\omega_\varepsilon q \varrho$, and so $\varrho(\omega) + \varepsilon \succ 1$ Thus, $\varrho(\omega) \succ 1 - \varepsilon$. By hypothesis, from $\omega_{1-\varepsilon} \in \varrho$, we obtain $1_{1-\varepsilon} \in \vee q \varrho$. If $\varrho(1) \succ 1 - \varepsilon$, then $\varrho(1) + \varepsilon \succ 1$, so $1 \in \varrho_q^\varepsilon$. If $\varrho(1) + 1 - \varepsilon \succ 1$, then $\varrho(1) \succ \varepsilon$. As $\varepsilon \in (0.5, 1]$, we consequence $\varrho(1) + \varepsilon \succ 2\varepsilon \succ 1$. Thus $\varrho(1) + \varepsilon \succ 1$ and $1 \in \varrho_q^\varepsilon$. Assume $\omega, \omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma) \in \varrho_q^\varepsilon$, for every $\omega, \sigma, \kappa \in \mathfrak{X}$ and $\varepsilon \in (0.5, 1]$. So

$$\varrho(\omega) + \varepsilon \succ 1 \quad , \quad \varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)) + \varepsilon \succ 1.$$

Hence $\varrho(\omega) \succ 1 - \varepsilon$ and $\varrho(\omega \twoheadrightarrow ((\sigma \twoheadrightarrow \kappa) \twoheadrightarrow \sigma)) \succ 1 - \varepsilon$. From $\varrho$ is an $(\in, \in\vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$, we get $\varrho(\sigma) \succ 1 - \varepsilon$ or $\varrho(\sigma) + 1 - \varepsilon \succ 1$. If $\varrho(\sigma) \succ 1 - \varepsilon$, then $\varrho(\sigma) + \varepsilon \succ 1$ and $\varrho(\sigma) \succ \varepsilon$ implies $\varrho(\sigma) + \varepsilon \succ 2\varepsilon \succ 1$, from $\varepsilon \in (0.5, 1]$. In two cases $\varrho(\sigma) + \varepsilon \succ 1$. So, $\sigma \in \varrho_q^\varepsilon$. Therefore, $\varrho_q^\varepsilon$ is an implicative deductive system of $\mathfrak{X}$. $\square$

**Corollary 3.21.** *Consider $\varrho$ is an $(\in, \in\vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. Then $\varrho_{\in\vee q}^\varepsilon$ is an implicative deductive system of $\mathfrak{X}$, for every $\varepsilon \in (0, 1]$.*

**Proof.** By Theorem 3.8 and Propositions 3.9 and 3.20, the proof is clear. $\square$

**Proposition 3.22.** *Each $(\in, \in\vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$ satisfies in the next conditions, for every $\omega, \sigma \in \mathfrak{X}$:*
*$(i)$* $\min\{\varrho(\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma)), 0.5\} \preccurlyeq \varrho(\omega \twoheadrightarrow \sigma)$,
*$(ii)$* $\min\{\varrho(\kappa \twoheadrightarrow (\sigma \twoheadrightarrow \omega)), 0.5\} \preccurlyeq \varrho((\kappa \twoheadrightarrow \sigma) \twoheadrightarrow (\kappa \twoheadrightarrow \omega))$.

**Proof.** $(i)$ Assume $\varrho$ is an $(\in, \in\vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. From Proposition 2.1(vii),

$$\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \preccurlyeq ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma).$$

From Corollary 3.16, we obtain

$$\min\{\varrho(\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma)), 0.5\} \preccurlyeq \varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)).$$

As $\varrho$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$, we have

$$\min\{\varrho(\omega \twoheadrightarrow (\omega \twoheadrightarrow \sigma)), 0.5\} \preccurlyeq \min\{\varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)), 0.5\} \preccurlyeq \varrho(\omega \twoheadrightarrow \sigma).$$

$(ii)$ Using Proposition 3.11 and (i). □

**Proposition 3.23.** *Each $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$ implies the next conditions, for every $\omega, \sigma \in \mathfrak{X}$:*
*$(i)$ $\min\{\varrho(\sigma \twoheadrightarrow \omega), 0.5\} \preccurlyeq \varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega)$,*
*$(ii)$ $\min\{\varrho(1), 0.5\} \preccurlyeq \varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \omega))$.*

**Proof.** $(i)$ Let $\omega, \sigma \in \mathfrak{X}$, $\varepsilon \in (0, 1]$ and $\varrho$ be an $(\in, \in \vee q)$-fuzzy implicative deductive system of $\mathfrak{X}$. By Proposition 2.1(iv), $\omega \preccurlyeq ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega$, thus by Proposition 2.1(viii),

$$(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \sigma \preccurlyeq \omega \twoheadrightarrow \sigma,$$

and so by Proposition 2.1(viii) and (vii),

$$\begin{aligned} \sigma \twoheadrightarrow \omega &\preccurlyeq ((\omega \twoheadrightarrow \sigma) \bullet ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma)) \twoheadrightarrow \omega \\ &= (\omega \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \\ &\preccurlyeq ((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega). \end{aligned}$$

Moreover, since $\varrho$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system, we obtain

$$\begin{aligned} &\varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \\ \succcurlyeq\ &\min\{\varrho(((((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega)), 0.5\} \\ \succcurlyeq\ &\min\{\varrho((\omega \twoheadrightarrow \sigma) \twoheadrightarrow (((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega)), 0.5\} \\ =\ &\min\{\varrho(((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \omega)), 0.5\} \\ \succcurlyeq\ &\min\{\varrho(\sigma \twoheadrightarrow \omega), 0.5\}. \end{aligned}$$

$(ii)$ Similar to Proposition 3.12 and (i). □

**Theorem 3.24.** *Assume $\varrho$ is an $(\in, \in \vee q)$-fuzzy deductive system of $\mathfrak{X}$. If $\varrho$ is an $(\in, \in \vee q)$-fuzzy implicative deductive system, then $\frac{\mathfrak{X}}{\approx_\varrho}$ is a Heyting semilattice that has Wajesberg property.*

## 4 Conclusion

In this paper, the notions of $(\in, \in)$-fuzzy implicative deductive systems and $(\in, \in \vee q)$-fuzzy implicative deductive systems of hoops are defined and studied some traits and defined some definitions that are equivalent. Thus by using the concept of $(\in, \in)$-fuzzy of hoop, a new congruence relation on hoop is introduced, and showed that the algebraic structure that is made by it is a Brouwerian semilattice, Heyting algebra and Wajesberg hoop. In the future, we try to introduce $(\alpha, \beta)$-fuzzy positive implicative deductive systems and $(\alpha, \beta)$-fuzzy fantastic deductive systems for $(\alpha, \beta) \in \{(\in, \in), (\in, \in \vee q)\}$ of hoops and investigate their traits of them. Also, we study the relation among $(\alpha, \beta)$-fuzzy (positive) implicative deductive system and $(\alpha, \beta)$-fuzzy fantastic deductive system . Moreover, we can study about the quotient that is made by them.

**Acknowledgments.** This research is supported by a grant of National Natural Science Foundation of China (11971384).

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