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Original Research Paper

Approximate Generalized Additive-Quadratic Functional Equations on Ternary Banach Algebras

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Abstract. In this paper, we introduce the concept of j-hom-derivation, $j \in \{1, 2\}$ and solve the new generalized additive-quadratic functional equations in the sense of ternary Banach algebras. Moreover, using the fixed point method, we prove its Hyers-Ulam stability.

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1 Introduction

A ternary Banach algebra X with $\|.\|$ is a complex Banach algebra equipped with a ternary product $(abc) \to abc$ of X^3 into X. This product is $\mathbb{C}$-linear in the outer variable, conjugate $\mathbb{C}-$linear in the middle variable associative in the sense that $ab(cvu) = a(bcv)u = (abc)vu$ and satisfies $\|abc\| \leq \|a\|.\|b\|.\|c\|$ and $\|aaa\| = \|a\|^3$ (see [23]). Ternary structures and their extensions, known as n-ary algebras have many

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applications in mathematical physics and photonics, such as the quark model and Nambu mechanics [14, 18]. Today, many physical systems can be modeled as a linear system. The principle of additivity has various applications in physics especially in calculating the internal energy in thermodynamic and also the meaning of the superposition principle.

The Hyers-Ulam stability problem which arises from Ulam's question says that for two given fixed functions φ and ψ, the functional equation $\mathcal{F}_1(G) = \mathcal{F}_2(G)$ is stable if for a function g for which $d(\mathcal{F}_1(g), \mathcal{F}_2(g)) \leq \varphi$ holds, there is a function h such that $\mathcal{F}_1(h) = \mathcal{F}_2(h)$ and $d(g, h) \leq \psi$ [8, 10, 20, 22]. In 1941 [10], Hyers solved the approximately additive mappings on the setting of Banach spaces.

The functional equation $f(a+b) = f(a)+f(b)$ is an additive equation and its solution is called an additive mapping.

First, T. M. Rassias [20] and Aoki [1] and then a number of authors extended this result by considering the unbounded Cauchy differences in different spaces. For example see [7, 9, 11, 15]. F. Skof in 1983 [21], proved the stability problem of quadratic functional equation between normed and Banach spaces. The functional equation $g(a+b)+g(a-b) = 2g(a) + 2g(b)$ is called quadratic equation. Cholewa [2] showed that the Skof's theorem is also true for the mappings defined on abelian groups. Later, a lot of research appeared with various generalization of quadratic and different type of functional equations, see [4, 16] and the references therein. Also for further study of the Hyers Ulam stability for a large variety of functional equatios such as, trigonometric type, mean value type , hypergeometric differential equations and the functional equation on a complete metric groups the reader is referred to [3, 5, 12, 13].

Consider the generalized additive-quadratic functional equation

$$\begin{aligned}
&3^j f_j(\frac{a+b+c}{3}) + f_j(a) + f_j(b) + (-1)^j f_j(c) - 2^j f_j(\frac{a+b}{2}) - 2^j f_j(\frac{b+c}{2}) \\
&- (-1)^j 2^j f_j(\frac{a+c}{2}) = \rho[j f_j(a+b+c) + j f_j(a) - f_j(a+b) \\
&- f_j(a+c) - (j-1) f_j(b+c)]
\end{aligned} \tag{1}$$

where $\rho \neq 0, \pm 1$ is a complex number and $j \in \{1, 2\}$. In this paper, we solve (1) and show that for $j = 1$, a function which satisfies (1) is additive and for $j = 2$, it is quadratic. We also prove its Hyers-Ulam stability by using the fixed point method. To do this, we use the Diaz-Margolis fixed point theorem [17].

Theorem 1.1. *[17] Let (A, d) be a complete generalized metric space and let $F : A \to A$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $a \in A$, either*

$$d(F^i(a), F^{i+1}(a)) = \infty$$

for all nonnegative integers i or there exists a positive integer i_0 such that

$1 : d(F^i(a), F^{i+1}(a)) < \infty, \qquad \forall i \geq i_0$;

2 : *the sequence $\{F^i(a)\}$ converges to a unique fixed point b^* of F in the set $B = \{b \in A \mid d(F^{i_0}a, b) < \infty\}$;*

$3 : d(b, b^*) \leq \frac{1}{1-L} d(b, F(b))$ *for all $b \in A$.*

2 Main Results

Extending the concepts of m-homomorphism, $1 \leq m \leq 4$, and hom-derivation which has been introduced by M. Eshaghi Gordji *et. al* [6] and C. Park *et. al* [19], respectively, we have the following definitions in the case of ternary Banach algebras. Throughout the paper, X is a ternary Banach algebra.

We say that a function f is a j-mapping, $j \in \{1, 2\}$ if for $j = 1$ it is additive and for $j = 2$, the function f is quadratic.

Definition 2.1. A mapping $h : X \to X$ is called a ternary j-homomorphism, $j \in \{1, 2\}$, if h is a j-mapping and

$$h(abc) = h(a)h(b)h(c) \quad \forall\ a, b, c \in X.$$

Definition 2.2. Let $h : X \to X$ be a ternary j-homomorphism, $j \in \{1, 2\}$. A j-mapping $D : X \to X$ is called a ternary j-hom-derivation if D satisfies

$$D(abc) = D(a)h(b)h(c) + h(a)D(b)h(c) + h(a)h(b)D(c)$$

for all $a, b, c \in X$.

For $j \in \{1, 2\}$ and given mappings $f_j : X \to X$ we define the difference equation

$$\mathcal{E}^j f_j(a,b,c) = 3^j f_j(\frac{a+b+c}{3}) + f_j(a) + f_j(b) + (-1)^j f_j(c) - 2^j f_j(\frac{a+b}{2})$$
$$- 2^j f_j(\frac{b+c}{2}) - (-1)^j 2^j f_j(\frac{a+c}{2}) - \rho[j f_j(a+b+c) + j f_j(a) - f_j(a+b)$$
$$- f_j(a+c) - (j-1) f_j(b+c)]. \tag{2}$$

In the following we give the solution of the functional equation (1).

Proposition 2.3. *Let $f_j : X \to X$ be a mapping satisfying $\mathcal{E}^j f_j(a,b,c) = 0$, $j \in \{1,2\}$. Then $f_j(0) = 0$ and*
(i) for $j = 1$, f_j is an additive mapping;
(ii) for $j = 2$, f_j is a quadratic mapping if f_j is even.

Proof. First of all, note that $\mathcal{E}^j f_j(0,0,0) = 0$ implies that $f_j(0) = 0$.
(i) Suppose $j = 1$. Let $a = 0$ and $b = 0$ in (2). Then

$$3f_1(\frac{c}{3}) = f_1(c) \tag{3}$$

Putting $a = 0$ and replacing c by b in (2) and using (3), we obtain $f_1(2b) - 2f_1(b) = \rho[f_1(2b) - 2f_1(b)]$. Since $\rho \neq 0, \pm 1$,

$$f_1(2b) = 2f_1(b). \tag{4}$$

Again putting $a = 0$ in (2) and using (3), (4) we get $\rho[f_1(b+c) - f_1(b) - f_1(c)] = 0$. So

$$f_1(b+c) = f_1(b) + f_1(c),$$

i.e., f_1 is additive.
(ii) Suppose $j = 2$ and f_2 is an even mapping satisfying $\mathcal{E}^2 f_2(a,b,c) = 0$. So from $\mathcal{E}^2 f_2(a,b,-a) = 0$ we have

$$9f_2(\frac{b}{3}) + f_2(b) + 2f_2(a) - 4f_2(\frac{a+b}{2}) - 4f_2(\frac{b-a}{2})$$
$$= \rho[2f_2(b) + 2f_2(a) - f_2(a+b) - f_2(b-a)]. \tag{5}$$

Letting and $b = 0$ replacing a by $2a$ in (5), we get

$$f_2(2a) = 4f_2(a). \tag{6}$$

Again in (5) putting $a = 0$ and using (6), we have

$$9f_2(\frac{b}{3}) = f_2(b). \tag{7}$$

Finally, by applying (6) and (7) in $\mathcal{E}^2 f_2(a,b,c) = 0$,

$$f_2(a+b) + f_2(a-b) = 2f_2(a) + 2f_2(b).$$

This completes the proof. □

Assume $j \in \{1,2\}$ and $\delta, \sigma : X^3 \to [0,\infty)$ are two functions satisfying conditions

$$\delta(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \le \frac{k}{2^j}\delta(a,b,c) \tag{8}$$

$$\sigma(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \le \frac{k}{2^{3j}}\sigma(a,b,c) \tag{9}$$

for all $a, b, c \in X$ and some $0 < k < 1$. Therefore $\delta(0,0,0) = 0$ and $\sigma(0,0,0) = 0$. Clearly, by induction one can obtain that

$$2^{nj}\delta(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \le k^n \delta(a,b,c),$$

$$2^{3nj}\sigma(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \le k^n \sigma(a,b,c)$$

for all $n \in \mathbb{N}$. If $f_j : X \to X$ is a function such that $\|\mathcal{E}^j f_j(a,b,c)\| \le \delta(a,b,c)$ then we have $f_j(0) = 0$.

To prove the following results, we consider two cases. For $j = 1$, suppose that f_j and g_j are odd and in case $j = 2$ assume f_j and g_j are even.

Theorem 2.4. *Let $j \in \{1,2\}$ and $f_j : X \to X$ be a function satisfying*

$$\|\mathcal{E}^j f_j(a,b,c)\| \le \delta(a,b,c) \tag{10}$$

where $\delta : X^3 \to [0,\infty)$ fulfills (8). Then there exist unique ternary j-mappings $h_j : X \to X$ such that

$$\|f_1(a) - h_1(a)\| \le \frac{k}{2(1-k)}\delta(0,a,-a)$$

$$\|f_2(a) - h_2(a)\| \le \frac{k}{2(1-k)}\delta(a,0,-a).$$

Proof. Let $\mathfrak{A}$ be the set of all functions $g : X \to X$ with $g(0) = 0$. Define the mapping $Q_j : \mathfrak{A} \to \mathfrak{A}$ by $Q_j(g)(a) = 2^j g(\frac{a}{2})$, $j \in \{1, 2\}$ and for every $g, h \in \mathfrak{A}$ and $x \in X$ define

$$d_1(g,h) = \inf\{\alpha > 0 \ : \ \|g(a) - h(a)\| \leq \alpha\delta(0, a, -a)\}$$

$$d_2(g,h) = \inf\{\alpha > 0 \ : \ \|g(a) - h(a)\| \leq \alpha\delta(a, 0, -a)\}$$

where $\inf \emptyset = +\infty$. It is easy to show that for each $j \in \{1, 2\}$, d_j is a generalized metric on $\mathfrak{A}$ and $(\mathfrak{A}, d_j)$ is a complete generalized metric space. Let $g, h \in \mathfrak{A}$. Then

$$\|Q_1(g)(a) - Q_1(h)(a)\| \leq 2d_1(g,h)\frac{k}{2}\delta(0, a, -a)$$

$$\|Q_2(g)(a) - Q_2(h)(a)\| \leq 2^2 d_2(g,h)\frac{k}{2^2}\delta(a, 0, -a).$$

Then $d_j\Big(Q_j(g), Q_j(h)\Big) \leq kd_j(g,h)$, i.e., Q_j is a contraction mapping. In case $j = 1$, put $a = 0$ and $c = -b$ in (10), then

$$\|f_1(b) - 2f_1(\frac{b}{2})\| \leq \frac{1}{2}\delta(0, b, -b).$$

In case $j = 2$, by setting $b = 0$ and $c = -a$ in (10) we have

$$\|f_2(a) - 2^2 f_2(\frac{a}{2})\| \leq \frac{1}{2}\delta(a, 0, -a).$$

Above relations imply that $d_j(f_j, Q_j(f_j)) \leq \frac{1}{2}$. Hence by Theorem 1.1, there exist a positive integer n_0 and a unique fixed point h_j of Q_j in the set $\Omega = \{g \in \mathfrak{A} : \ d(Q_j^{n_0}(f_j), g) < \infty\}$ and $\lim_{n\to\infty} Q_j^n(f_j)(a) = h_j(a)$. So, for all $a \in X$, $Q_j(h_j)(a) = h_j(a)$ and $\lim_{n\to\infty} 2^{jn} f_j(\frac{a}{2^n}) = h_j(a)$. Also we have $d_j(f_j, h_j) \leq \frac{k}{2(1-k)}$. This implies that

$$\|f_1(a) - h_1(a)\| \leq \frac{k}{2(1-k)}\delta(0, a, -a)$$

$$\|f_2(a) - h_2(a)\| \leq \frac{k}{2(1-k)}\delta(a, 0, -a).$$

But for each $j \in \{1, 2\}$,

$$\begin{aligned}\|\mathcal{E}^j h_j(a,b,c)\| &= \lim_{n\to\infty} 2^{jn}\|\mathcal{E}^j f_j(\frac{a}{2^n},\frac{b}{2^n},\frac{c}{2^n})\| \\ &\le \lim_{n\to\infty} 2^{jn}\delta(\frac{a}{2^n},\frac{b}{2^n},\frac{c}{2^n}) \\ &\le \lim_{n\to\infty} k^n\delta(a,b,c) \\ &= 0.\end{aligned}$$

Hence by Proposition 2.3, for each $j \in \{1, 2\}$, h_j is a j-mappings and the proof is complete. □

Now, we are going to prove Hyers-Ulam stability of ternary j-hom-derivations in ternary Banach algebras corresponding to the functional equation (1), by using the fixed point method.

Theorem 2.5. *Assume $\delta, \sigma : X^3 \to [0, \infty)$ are two functions which satisfying conditions (8) and (9) for some constant $k \in (0, 1)$. Suppose that f_j and g_j fulfill the following relations*

$$\|\mathcal{E}^j f_j(a,b,c)\| \le \delta(a,b,c) \tag{11}$$

$$\|\mathcal{E}^j g_j(a,b,c)\| \le \delta(a,b,c) \tag{12}$$

$$\|f_j(abc) - f_j(a)f_j(b)f_j(c)\| \le \sigma(a,b,c) \tag{13}$$

$$\begin{aligned}&\|g_j(abc) - g_j(a)f_j(b)f_j(c) - f_j(a)g_j(b)f_j(c) \\ &\quad - f_j(a)f_j(b)g_j(c)\| \le \sigma(a,b,c).\end{aligned} \tag{14}$$

for all $a, b, c \in X$. Then there exist unique ternary j-homomorphisms $H_j : X \to X$ and unique ternary j-hom-derivations $D_j : X \to X$ such that

$$\|f_1(a) - H_1(a)\| \le \frac{k}{2(1-k)}\ \delta(0,a,-a) \tag{15}$$

$$\|f_2(a) - H_2(a)\| \le \frac{k}{2(1-k)}\ \delta(a,0,-a) \tag{16}$$

$$\|g_1(a) - D_1(a)\| \le \frac{k}{2(1-k)}\ \delta(0,a,-a) \tag{17}$$

$$\|g_2(a) - D_2(a)\| \le \frac{k}{2(1-k)}\ \delta(a, 0, -a) \tag{18}$$

for all $a \in X$ *and some* $0 < k < 1$.

Proof. Let $\mathfrak{A}, d_j$ and Q_j, $j \in \{1, 2\}$, be those as defined in the proof of Theorem 2.4. Similar to the proof of Theorem 2.4, there exist unique j-mappings H_j, D_j from X into X such that

$$H_j(a) = \lim_{n\to\infty} Q_j^n(f_j)(a) = \lim_{n\to\infty} 2^{jn} f_j(\frac{a}{2^n}) \tag{19}$$

$$D_j(a) = \lim_{n\to\infty} Q_j^n(g_j)(a) = \lim_{n\to\infty} 2^{jn} g_j(\frac{a}{2^n}) \tag{20}$$

and satisfying (15), (16), (17) and (18) as desired. Mappings H_j, $j = \{1, 2\}$, are ternary j-homomorphisms. In fact, by (13) and (19), we have

$$\begin{aligned}
&\|H_j(abc) - H_j(a)H_j(b)H_j(c)\| \\
&= \lim_{n\to\infty} 2^{3nj} \|f_j(\frac{abc}{2^{3n}}) - f_j(\frac{a}{2^n}) f_j(\frac{b}{2^n}) f_j(\frac{c}{2^n})\| \\
&\le \lim_{n\to\infty} 2^{3nj} \sigma(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}) \\
&\le \lim_{n\to\infty} k^n \sigma(a, b, c) \\
&= 0.
\end{aligned}$$

It follows (14) and (20) imply that D_j is a ternary j-hom-derivations. In fact,

$$\begin{aligned}
&\|D_j(abc) - D_j(a)H_j(b)H_j(c) - H_j(a)D_j(b)H_j(c) - H_j(a)H_j(b)D_j(c)\| \\
&= \lim_{n\to\infty} 2^{3nj} \|g_j(\frac{abc}{2^{3n}}) - g_j(\frac{a}{2^n}) f_j(\frac{b}{2^n}) f_j(\frac{c}{2^n}) \\
&- f_j(\frac{a}{2^n}) g_j(\frac{b}{2^n}) f_j(\frac{c}{2^n}) - f_j(\frac{a}{2^n}) f_j(\frac{b}{2^n}) g_j(\frac{c}{2^n})\| \\
&\le \lim_{n\to\infty} 2^{3nj} \sigma(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}) \\
&\le \lim_{n\to\infty} k^n \sigma(a, b, c) \\
&= 0.
\end{aligned}$$

Now, the proof is complete. □

In Theorem 2.4 and Theorem 2.5, by taking $k = 2^{j-r}, r \in \mathbb{R}$ and

$$\delta(a,b,c) = \sigma(a,b,c) = s(\|a\|^r + \|b\|^r + \|c\|^r)$$

where $a, b, c \in X$ and s is a nonnegative real number, we obtain the following result.

Corollary 2.6. *Let $j \in \{1,2\}$, $r > j$ and s be two elements of $\mathbb{R}_+$. Suppose that $\delta(a,b,c) = \sigma(a,b,c) = s(\|a\|^r + \|b\|^r + \|c\|^r)$. Assume $f_j, g_j : X \to X$, are functions satisfying (11), (12), (13) and (14). Then there exist ternary j-homomorphisms H_j and ternary j-hom-derivations D_j such that*

$$\begin{cases} \|f_j(a) - H_j(a)\| \leq \frac{2^j s}{2^r - 2^j}\|a\|^r \\ \\ \|g_j(a) - D_j(a)\| \leq \frac{2^j s}{2^r - 2^j}\|a\|^r. \end{cases}$$

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