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Original Research Paper

Symmetry Classification and Invariance of the Reynolds Equation

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Abstract. In this essay, extensions to the results of Lie symmetry classification of Reynolds equation are proposed. The infinitesimal technique is used to derive symmetry groups of the Reynolds equation. One-dimensional optimal system is constructed for symmetry sub-algebras gained through Lie point symmetry. At the end, the general symmetry group of the non-conservative generalized thin-film equation are determined.

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1 Introduction

As you know the symmetry property is a natural phenomenon. By using partial differential equations having symmetry properties, we can

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describe many physical, biological and chemical processes. After creating the group classification method by Sophus Lie in 19th century [8], Lie symmetry analysis has always been an interesting method for mathematicians in dealing with differential equations. The Lie group approach proposes a useful procedure for integrability, reducing equations and finding out the exact solutions of differential equations. Its algorithm is as follows that group of transformations transforms solutions of the system of differential equations to other solutions of them [3, 11, 12]. In the first article published on this subject [7], Lie calculated symmetry group of one dimensional heat equation and then reduced this equation by symmetry reduction method to find solution of it. In [14], partially invariant solutions has extended by Ovsiannikov. In this attempt, the significance of the equivalence group has investigated. An equivalence group or Lie transformation group acts on the generalization space of independent variables, dependent variables, and their derivatives while keeps the class of partial differential equations.

This article is assigned to studying and finding out analytical solutions of the one-dimensional Reynolds equation. This partial differential equation by describing pressure generation of thin viscous fluid films is one of the important equations in fluid dynamics and lubrication theory. The initial version of this equation was proposed by Osborne Reynolds in 1886. The extension of rupture singularities in this equation is studied in [4]. In the one- dimensional Reynolds equation x belongs to bounded interval $[0, l]$,

$$u_t = \partial_x \left(u^3 p_x \right) - J,$$

where u is fluid film thickness and $J = -\gamma p(u)/(u + K_0)$ is the non-conservative flux. γ is a scaling constant, $K_0 > 0$ and $p(u) \equiv f(u) - u_{xx}$ is the fluid pressure. $f(u)$ and u_{xx} respectively are disjoining pressure function and the linearised curvature. A physical model stimulates the form of $f(u)$. While the case $\gamma \leq 0$ in J corresponds to physical models where intermolecular forces can complete against the evaporative flux, while the case $\gamma > 0$ leads to a mathematically-interesting PDE where singularity formation can accur (see references [4, 5, 6] for more details). For $\gamma = 0$, we have the fourth-order differential equation for one dimensional covering flows [4] as follows:

$$u_t = \partial_x \left(u^3 \partial_x \left[f(u) - u_{xx} \right] \right). \tag{1}$$

Rupture bearing in a non conservative generalized thin film equation will allow to compete evaporation and dewetting and the competition between them conduce finite time rupture.

2 Lie Symmetry Group Analysis

Before considering Lie symmetry groups of Reynolds equation, it is essential that we recall a system differential equation(see [11, 12]).

Definition 2.1. *A system of n-th order differential equations in p independent and q dependent variables is given as a system of equations*

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l,$$

involving $x = (x^1, \cdots, x^p), u = (u^1, \cdots, x^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ representations all the derivatives of u of all orders from 0 to n.

Suppose a partial differential equation including one dependent variable and p independent variables. A one-parameter Lie group of point transformations

$$\overline{x}_i = x_i + \epsilon\xi_i(x, u) + O(\epsilon^2); \qquad \overline{u} = u + \epsilon\varphi(x, u) + O(\epsilon^2),$$

where $i = 1, \cdots, p$, $\epsilon \ll 1$, and $\xi_i = \partial_\epsilon \overline{x}_i|_{\epsilon=0}$, operating on $(x, u)-space$.

$$\mathbf{x} = \xi_i \partial_{x_i} + \varphi \partial_u, \qquad i = 1, \cdots, p, \tag{2}$$

is infinitesimal generator. Therefore the vector field $\mathbf{x}$ have characteristic function $Q(x, u^1) = \varphi(x, u) - \sum_{i=1}^{p} \xi_i(x, u) u_{x_i}$. The symmetry generator associated with (2) is presented by

$$\mathbf{x} = \xi\partial_x + \tau\partial_t + \varphi\partial_u.$$

The n-th prolongation of $\mathbf{x}$ is the vector field

$$\mathbf{x}^{(n)} = \mathbf{x} + \sum_{\alpha=1}^{q} \sum_{J} \varphi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha}, \tag{3}$$

$$J = (j_1, \cdots, j_k), \qquad 1 \leq j_k \leq p, \qquad 1 \leq k \leq n,$$

where the coefficients φ_α^J are found by the following:

$$\varphi_\alpha^J(x,u^{(n)}) = D_J(\varphi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha,$$

where $u_i^\alpha := \partial u^\alpha/\partial x^i$, and $u_{J,i}^\alpha := \partial u_J^\alpha/\partial x^i$ [12].
Therefore the vector field

$$\begin{aligned}\mathbf{x}^{(4)} = \mathbf{x} + \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \varphi^{xt}\partial_{u_{xt}} + \varphi^{xx}\partial_{u_{xx}} + \\ \cdots + \varphi^{xxxt}\partial_{u_{xxxt}} + \varphi^{xxxx}\partial_{u_{xxxx}},\end{aligned} \tag{4}$$

is fourth prolongation of $\mathbf{x}$. Its coefficient functions are:

$$\begin{aligned}\varphi^x &= D_x\varphi - u_x D_x\xi - u_t D_x\tau,\\ \varphi^t &= D_t\varphi - u_x D_t\xi - u_t D_t\tau,\\ \varphi^{xt} &= D_t\varphi^x - u_{xx} D_t\xi - u_{xt} D_t\tau,\\ \varphi^{xx} &= D_x\varphi^x - u_{xx} D_x\xi - u_{xt} D_x\tau,\\ &\vdots\\ \varphi^{xxxx} &= D_x\varphi^{xxx} - u_{xxxx} D_x\xi - u_{txxx} D_x\tau.\end{aligned} \tag{5}$$

Where the total derivatives operators D_x and D_t are as follows:

$$\begin{aligned}D_x &= \partial_x + u_x\partial_u + u_{xx}\partial u_x + u_{xt}\partial_{u_t} + \cdots,\\ D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{tx}\partial_{u_x} + \cdots.\end{aligned}$$

If vector field is admitted by Eq.(1), then $\mathbf{x}$ must satisfy in the condition $\mathbf{x}^{(4)}[u_t - \partial_x(u^3\partial_x(f(u) - u_{xx}))] = 0$, whenever $u_t = \partial_x(u^3\partial_x[f(u) - u_{xx}])$. Substituting (5) into infinitesimal criterion and equating the coefficients of the various monomials in the various power partial derivatives of u, we obtain the compelet set of determining equations:

$$\begin{aligned}\tau_{tt} = \varphi_x = \varphi_t = \xi_t = \xi_u = 0,\\ 2u\varphi f_{uu} + uF_t f_u + 3\varphi = 0,\\ 4u\xi_x = 3\varphi + uF_t,\\ u\varphi_u = \varphi,\end{aligned}$$

wherer ξ, τ, φ are depend on x, t, u and f depends on u and F is arbitrary function. With solving determining equations we obtain:

$$F = (4c_2 - 3c_1)t + c_4, \qquad \varphi = c_1 u, \qquad \xi = c_2 x + c_3.$$

Where c_i, $i = 1, \cdots, 4$ are arbitrary constants. Therefore we have:

$$c_1 u f_{uu} + 2c_2 f_u = 0. \tag{6}$$

3 Lie Group Classification of Reynolds Equation

In the following section, according to the above discussion about Lie theory, considering (6) and we classify the symmetries of the Eq.(1). We consider four general Cases(Idea from[8, 9, 10]).

Case 1: If $f' = 0$, then $f = a$ is constants. Therefore we have

$$u_t = -3u^2 u_x u_{x^3} - u^3 u_{x^4}. \tag{7}$$

For the equation (7) the Lie group infinitesimals are $\xi = c_1 t + c_2$, $\tau = c_3 x + c_4$, $\varphi = u(4c_3 - c_1)/3$. Thus the infinitesimal generator of symmetry algebra is resulted as

$$\mathbf{x}_1 = \partial_x, \quad \mathbf{x}_2 = \partial_t, \quad \mathbf{x}_3 = t\partial_t - \frac{1}{3}u\partial_u, \quad \mathbf{x}_4 = x\partial_x + \frac{4}{3}u\partial_u. \tag{8}$$

The characteristic equation associated with $\mathbf{x}_3$ is $dx/0 = dt/t = du/(-u/3)$. By integrating we obtain

$$r = x, \qquad s = \ln t, \qquad v(r) = ut^{1/3}. \tag{9}$$

Substituting (9) in (7) leads to $9vv_{rrr}v_r + 3v^2 v_{rrrr} = 1$. The reduced equation is $ut^{1/3} = 0$. Therefore $u = 0$.

The characteristic equation associated with $\mathbf{x}_4$ is $dx/x = dt/0 = du/(4u/3)$. By integrating we obtain $r = t$, $s = \ln x$, $v(r) = ux^{-4/3}$. Therefore $u = c_1 = 0$.

The invariants associated with $\mathbf{x}_1$ are t and u, and its symmetry group is $g_1^{\epsilon} = (x + \epsilon, t, u)$.

Table 1: Commutator table for Case 1 and Case 3.

$[\mathbf{x}_i, \mathbf{x}_j]$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\mathbf{x}_1$	0	0	0	$\mathbf{x}_1$
$\mathbf{x}_2$	0	0	$\mathbf{x}_2$	0
$\mathbf{x}_3$	0	$-\mathbf{x}_2$	0	0
$\mathbf{x}_4$	$-\mathbf{x}_1$	0	0	0

$[\mathbf{x}_i, \mathbf{x}_j]$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$
$\mathbf{x}_1$	0	0	$-\alpha^{-1}\mathbf{x}_1$
$\mathbf{x}_2$	0	0	$\mathbf{x}_2$
$\mathbf{x}_3$	$\alpha^{-1}\mathbf{x}_1$	$-\mathbf{x}_2$	0

The invariants associated with $\mathbf{x}_2$ are x and u, and its symmetry group is $g_2^\epsilon = (x, t+\epsilon, u)$.

The invariants associated with $\mathbf{x}_3$ are x and $ut^{1/3}$, and its symmetry group is $g_3^\epsilon = (x, e^\epsilon t, e^{-\epsilon/3}u)$.

The invariants associated with $\mathbf{x}_4$ are t and $x^{-4/3}u$ and its symmetry group is $g_4^\epsilon = (e^\epsilon x, t, e^{4\epsilon/3}u)$.

Case 2: If $f'' = 0$ and $f' \neq 0$ then $f = au + b$ $(a, b \in \mathbb{R}, a \neq 0)$. Therefore we have

$$u_t = 3u^2 u_x(u_x - u_{xxx}) + u^3(au_{xx} - u_{xxxx}). \tag{10}$$

The symmetry algebra is generated by the Lie symmetry generators $\mathbf{x}_1 = \partial_x$, $\mathbf{x}_2 = \partial_t$, $\mathbf{x}_3 = t\partial_t - (u/3)\partial_u$. The characteristic equation associated with $\mathbf{x}_3$ is
$dx/0 = dt/t = du/(-u/3)$. By integrating we obtain $r = x$, $s = \ln t$, $v(r) = ut^{1/3}$. Substituting (9) in (10) leads to

$$3v^2 v_{r^4} + 9vv_r v_{r^3} = 3av^2 v_{rr} + 9avv_r^2 + 1.$$

The reduced equation is $ut^{1/3} = 0$. Therefore $u = 0$. The invariants associated and symmetry groups with $\mathbf{x}_1$ and $\mathbf{x}_2$ and $\mathbf{x}_3$ aforesaid.

Case 3: If $f'' \neq 0$ then $f''/f' = -2c_2u/c_1$. By integrating we obtain $f = au^b + c$, where $a, c \in \mathbb{R}$, $b \in \mathbb{N}$, $a \neq 0$, and $b > 1$.

The equation is

$$u_t = abu^{b+1}((b+2)u_x^2 + uu_{xx}) - 3u^2 u_x u_{x^3} - u^3 u_{x^4}.$$

The Lie algebra is generated by the Lie symmetry vectors $\mathbf{x}_1 = \partial_x$, $\mathbf{x}_2 = \partial_t$, $\mathbf{x}_3 = (4b+2)^{-1}\big((b-1)x\partial_x + t(4b+2)\partial_t - 2u\partial_u\big)$. By integrating the characteristic equation associated with $\mathbf{x}_3$ we obtain $r = tx^{\alpha}$, $s = -\alpha \ln x$, and $v(r) = ux^{-\alpha/(2b+9)}$, where $\alpha = (4b+2)/(1-b)$.

The invariants associated and symmetry groups with $\mathbf{x}_1$ and $\mathbf{x}_2$ aforesaid. The invariants associated with $\mathbf{x}_3$ are tx^{α} and $ux^{2/(b-1)}$.

Case 4: Otherwise $c_1 = c_2 = 0$. Herein, the Lie symmetry algorithm leads us to the generators $\mathbf{x}_1 = \partial_x$ and $\mathbf{x}_2 = \partial_t$. The invariants and the symmetry group associated with $\mathbf{x}_1$ and $\mathbf{x}_2$ aforesaid. In this Case $u = 0$.
The commutatore table for Case 1 and Case 3 are listed in the table (1).

Theorem 3.1. *The Reynolds equation have maximum four generator* (8) *and minimum two generator ∂_x and ∂_t.*

The physiologys can make some connections between these cases and the corresponding physical lubrication models (see [2, 13]).

4 Optimal Control System of the Reynolds Equation

In what follows we perform the optimal system for one dimensional subalgebra of the Reynolds equation. In order to obtain a complete optimal system, we classify the orbits for the adjoint representation. For this purpose, we take an element of the Lie algebra and simplify it by adjoint transformation.

Definition 4.1. *Suppose $\mathfrak{g}$ be Lie algebra corresponding to Lie group G. An optimal system of r-parameter subgroups is a list of conjugacy non-equivalent r-parameter subalgebras which are not related by transformations that is to say any other subgroup is conjugate to exactly one subgroup in the list. In asimilar way, a list of r-parameter subalgebra constitutes an optimal system if between every r-parameter subalgebra of $\mathfrak{g}$ with a unique element of the list there is an equivalence relation, under some elements of the adjoint representation $\overline{\mathfrak{h}} = \mathrm{Ad}(g(\mathfrak{h}))$[12].*

Table 2: Adjoint representation table of the symmetry generators for Case 1.

$\mathrm{Ad}(\exp(\epsilon\mathbf{x}_i)\mathbf{x}_j)$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\mathbf{x}_1$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4-\epsilon\mathbf{x}_1$
$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3-\epsilon\mathbf{x}_2$	$\mathbf{x}_4$
$\mathbf{x}_3$	$\mathbf{x}_1$	$e^{\epsilon}\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\mathbf{x}_4$	$e^{\epsilon}\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$

Theorem 4.2. (See [12]) *Suppose G be Lie group with corresponding Lie algebra $\mathfrak{g}$ and H and $\overline{H}$ be $s-$dimensional Lie subgroups of the Lie group G that are connected to each other and corresponding Lie subalgebras are $\mathfrak{h}$ and $\overline{\mathfrak{h}}$ respectively. Then $\overline{H}=gHg^{-1}$ are conjugate subgroups if and only if $\overline{\mathfrak{h}}=\mathrm{Ad}(g(\mathfrak{h}))$.*

We apply the following Lie series to computing the adjoint representation
$\mathrm{Ad}(\exp(\epsilon\mathbf{x}_i)\mathbf{x}_j)=\mathbf{x}_j-\epsilon[\mathbf{x}_i,\mathbf{x}_j]+(\epsilon^2/2)[\mathbf{x}_i,[\mathbf{x}_i,\mathbf{x}_j]]-\cdots,$
where $[\mathbf{x}_i,\mathbf{x}_j]$ is the Lie bracket for the Lie algebra, ϵ is a parameter, $i,j=1,2,3,4$.
The adjoint actions of the symmetry generators for Case 1 are listed in the table (2).

Theorem 4.3. *The one-dimensional optimal system of Lie subalgebras of the equation (7) is as $\{\boldsymbol{x}_1,\ \beta\boldsymbol{x}_1+\boldsymbol{x}_3,\ \gamma\boldsymbol{x}_1+\boldsymbol{x}_2,\ \delta\boldsymbol{x}_3+\boldsymbol{x}_4\}$.*

Proof. A nonzero vector $\mathbf{x}=a_1\mathbf{x}_1+a_2\mathbf{x}_2+a_3\mathbf{x}_3+a_4\mathbf{x}_4$ is given. We start by Simplification of the coefficient a_i as far as possible through judicious applications of adjoint maps to $\mathbf{x}$.

Let $a_4\neq 0$. Scaling $\mathbf{x}$ if necessary, we let $a_4=1$. Considering table 3 and the vanishing coefficients $\mathbf{x}_1$, $\mathbf{x}_2$ then vector $\mathbf{x}$ is equivalent to $\delta\mathbf{x}_3+\mathbf{x}_4$.

If $a_4=0$, $a_3\neq 0$, then we can consider that $a_3=1$ and then the coefficients of $\mathbf{x}_2$ vanish. Thus the vector $\mathbf{x}$ is equivalent to $\beta\mathbf{x}_1+\mathbf{x}_3$.

If $a_4=a_3=0$, $a_2\neq 0$, then we can assume that $a_2=1$. Then the vector $\mathbf{x}$ is equivalent to $\gamma\mathbf{x}_1+\mathbf{x}_2$.

If $a_4 = a_3 = a_2 = 0$, then we can assume that $a_1 = 1$ and the vector $\mathbf{x}$ is equivalent to $\mathbf{x}_1$. $\square$

Once we have calculated the one-dimensional optimal system, we can go on to talk about higher dimensions. Lack of space precludes us from following this interesting problem any further here. So we refer the reader to [14], for some of the techniques accessible.

5 Generalization

Consider the following generalized thin-film equation describing the effect of surface tension

$$u_t = -(u^n u_{xxx})_x. \tag{11}$$

If we let $n = 3$ in the above equation we get to Case1. Bernis and Friedman in their work obtained the solutions of Eq.(11) for $n \geq 4$ on bounded domains [1]. In [4] influence of the non-conservative to create rupture to dominate the intermolecular forces has been investigated.

The behavior of solution of the non-conservative generalized thin-film equation

$$u_t = -\partial_x(u^n \partial_x[u^{-4} + u_{xx}]) - u^{-(m+4)} + u^{-m} u_{xx}, \tag{12}$$

depend on the values of the parameters m, n that control the competing non-conservative influences and arranging conservative surface tension influences. The dynamism coefficients (n, m) are the conservative and evaporative terms respectively [5, 6].

The Lie symmetry algebra of (12) is generated by vector fields ∂_x and ∂_t. By using the characteristic equation associated with ∂_x and integrating and substituting in (12) we obtain $v - r = -v^{-m-4}$ that $r = t$, $s = x$, $v(r) = u$. The reduced equation is $u = (-5t - tm + c_1)^{-m-5}$. u is independent on the values of the parameter n.

6 Conclusion and Motivation

In what presented, we consider the Reynolds equations in the 4 Cases, then using the classical Lie symmetric method we determined Lie symmetries group and their invariants associated to the Reynolds equation

$u_t = \partial_x(u^3\partial_x[f(u) - u_{xx}])$ where $f(u)$ is some determined function on u for some physical models, in the different Cases. The commutator table and adjoint representation table of the Lie symmetry generator is constructed. Ultimately, using the optimal control theory we achieve the optimal system of the Reynolds equation.

Classical and nonclassical symmetries for similar and generalized equations can be obtained, for example $u_t = \partial_x(u^3\partial_x[f(x) - u_{xx}])$, with $f = f(t)$, $f = f(x,t)$ or $f = f(x,t,u)$.

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