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Some Fuzzy Multigroups Obtained from Fuzzy Subgroups

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Abstract. In this paper, we first study fuzzy subgroups and fuzzy multigroups of a group G and we obtain some chains of fuzzy subgroups of a fuzzy subgroup and by these sequences, we construct fuzzy multigroups. We show that there is a relationship between a fuzzy multigroup with underlying group G and a fuzzy multigroup with underlying group $Aut(A)$. Moreover, we generate a code by using the defined special fuzzy multigroup automorphisms.

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1 Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 [30]. The theory of fuzzy set has grown tremendously over time giving birth to some algebraic structures like fuzzy group introduced by Rosenfeld in 1971 [25]. In 1975 Negoita and Ralescu [23], considered a generalization of Rosenfeld's definition in which the unit interval $I = [0, 1]$ was replaced by an appropriate lattice structure. In 1979 Anthony and Sherwood [3] redefined a fuzzy subgroup of a group using the concept of triangular norm. Some properties of the fuzzy groups have been discussed in details in [19, 26], etc.

The term multiset was first suggested by N.G. De Bruijn to Knuth in a private correspondence as noted in [18]. Yager [31] applied the idea of multiset, which is an extension of set with repeated elements in a collection to propose fuzzy multiset. That is, fuzzy multiset allows repetition of membership degrees of elements in multiset framework. In fact, fuzzy multiset generalizes fuzzy sets. With this, one can conveniently say that, every fuzzy set is a fuzzy multiset but the reverse is not necessarily true. Fuzzy multisets theory has been extensively studied and applied in real-life problems [7, 8, 20, 21, 24, 28].

The concept of fuzzy multigroups was proposed in [27] as an algebraic structure of fuzzy multisets that generalizes fuzzy groups. This algebraic structure is a multiset of $G \times [0, 1]$ satisfying some set of axioms, where G is a classical group. In fact, since fuzzy multiset is a generalization of fuzzy set, it then follows that fuzzy multigroup is an extension of fuzzy group. The concept of fuzzy multigroups constitutes an application of fuzzy multisets to the notion of group. Fuzzy multigroups and fuzzy groups are different generalizations of classical groups such that, every fuzzy group is a fuzzy multigroup but the converse is not always true. The notion of fuzzy submultigroups of fuzzy multigroups and some properties of fuzzy multigroups were explicated in [13, 15]. The ideas of abelian fuzzy multigroups and order of fuzzy multigroups have been studied [5, 16], and the notion of normal fuzzy submultigroups and Frattinini fuzzy submultigroups of fuzzy multigroups was proposed with some number of results in [9, 14]. The idea of homomorphism in fuzzy multigroups context was extensively explored in [4, 10, 11]. Also, Davvaz in [6, 17] discussed various properties of fuzzy hypergroups and fuzzy multi-hypergroups. In [24], Onasanya and Hoskova-Mayerova defined multi-fuzzy groups and in [1, 2], the authors defined fuzzy multi-polygroups and fuzzy multi-H_v-ideals and studied their properties.

One of the significant aspects of fuzzy subgroup theory is a classification of the fuzzy subgroups of finite groups under a suitable equivalence relation[22, 29]. This fact motivated us to generalize the concept of equivalency to fuzzy multigroup. Moreover, there are many methods for obtaining chains of fuzzy subgroups. This leads us to construct some fuzzy multigroup by these chains.

The paper is organized as follows. In Section 2 we present some preliminary results on the fuzzy sets, fuzzy subgroups, fuzzy multisets and fuzzy multigroups. In section 3, we obtain some suitable sequences of fuzzy subgroups of a fuzzy subgroup with underlying group G and by these sequences, we obtain fuzzy multigroups with underlying group G. In Section 4 we study the notion of equivalence relations on fuzzy multigroups. In Section 5, by the notion of automorphisms on multigroups, we describe a method for producing a fuzzy multigroup with an underlying group of automorphism of a fuzzy multigroup on G. In Section 6 we finally generate one-time passwords (OTP) by fuzzy multigroups.

2 Preliminaries

In this section, we study the fundamental definitions that will be used in the sequel. We use $I = [0, 1]$, the real unit interval, as a chain with the usual ordering, which $\wedge$ stands for infimum (or intersection), $\vee$ stands for supremum (or union) as the degree of membership and by I^n we mean that $I \times I \times \cdots \times I$ for n times. A fuzzy subset of a set X is defined as a mapping $\mu : X \to [0, 1]$. Moreover, we define

$$\alpha_\mu = \bigvee\{\mu(x) \mid x \in X\}, \qquad \beta_\mu = \bigwedge\{\mu(x) \mid x \in X\}.$$

We denote the set of all fuzzy subsets of X by I^X. Further, we denote fuzzy subsets by the Greek letters μ, ν, η, etc. Let $\mu, \nu \in I^X$. If $\mu(x) \leq \nu(x)$, for all $x \in X$, then we say that μ is contained in ν (or ν contains μ) and we write $\mu \subseteq \nu$. Let $\mu \in I^X$, for $a \in I$, then the α-cut (or α-level) subset of μ denoted by μ_α can be defined as $\mu_\alpha = \{x \mid x \in X, \mu(x) \geq \alpha\}$.

The fuzzy subset μ of the group G is called a fuzzy subgroup of G if:

(1) $\mu(xy) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in G$;

(2) $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$.

Let X be a universal set, $\mathbb{W}$ be the set of all nonnegative integers and let C_M: $X \to \mathbb{W}$ be a function. A multiset M over the set X is the following set

$$M = \{(x, C_M(x)) : x \in X, C_M(x) > 0\},$$

where the value $C_M(x)$ is called the multiplicity (count) of x in M, that is, $C_M(x)$ is the number of copies of x occur in the multiset M.

Definition 2.1. [31] Let X be a non-empty set. A fuzzy multiset A of X is characterized by a count membership function $CM_A : X \to Q$, where Q is the set of all multiset drown from the unit interval $[0, 1]$.

From [28], a fuzzy multiset can also be characterized by a high-order function. In particular, a fuzzy multiset A can be characterized by a function

$$CM_A : X \to N^I \ or \ CM_A : X \to [0, 1] \to N,$$

where $I = [0, 1]$ and $N = \mathbb{N} \cup \{0\}$.

By [21], For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$ and it is denoted by:

$$\{\mu_1(x), \mu_2(x), \ldots, \mu_p(x), \ldots : \mu_1(x) \geq \mu_2(x) \geq \cdots \geq \mu_p(x) \geq \ldots\}$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = (\mu_1(x), \mu_2(x), \ldots, \mu_n(x)),$$

where $\mu_1(x), \mu_2(x), \ldots, \mu_n(x) \in [0, 1]$ such that

$$\mu_1(x) \geq \mu_2(x) \geq \cdots \geq \mu_n(x),$$

or simply

$$CM_A = (\mu_i)_{i \in I},$$

for $\mu_i \in [0,1]$ and $i = 1, 2, \ldots, n$. Now, a fuzzy multiset A is given as

$$A = \left\{ \frac{CM_A(x)}{x} : x \in X \right\} \quad \text{or} \quad A = \{(x, CM_A(x)) : x \in X\}.$$

In a simple term, a fuzzy multiset A of X is characterized by the count membership function $CM_A(x)$ for $x \in X$, that takes the value of a multiset of a unit interval $I = [0,1]$.

The set of all fuzzy multisets is depicted by FMS(X).

Remark 2.2. If there is no ambiguity, we will use $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ instead of A.

Example 2.3. Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.4\}$, $CM_A(b) = \{0.9, 0.6\}$ and $CM_A(c) = \{0\}$, we get that A is a fuzzy multiset of X written as

$$A = \left\{ \frac{(1, 0.5, 0.4)}{a}, \frac{(0.9, 0.6)}{b} \right\}.$$

Let A be a fuzzy multiset on X. define the length $L(x; A)$, that is, the length of μ

$$L(x; A) = \vee\{j : \mu_j(x) \neq 0\}.$$

Then it is not difficult to see that $C_A : X \to \mathbb{W}$ by $C_A(x) = L(x, A)$ is a multiset on X.

Example 2.4. Consider the fuzzy multiset

$$A = \left\{ \frac{(0.5, 0.3, 0.1, 0.1)}{1}, \frac{(0.5, 0.1)}{2}, \frac{(0.65, 0.2, 0.2, 0.1)}{3} \right\} \; of \; X = \{1, 2, 3\}.$$

Then we have

$$L(x, A) = \begin{cases} 4, & x = 1, 3; \\ 2, & x = 2. \end{cases}$$

Definition 2.5. [27] Let G be a group. A fuzzy multiset A over G is a fuzzy multigroup over G if the count (count membership) of A satisfies the following two conditions.

(1) $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \; \forall x, y \in G$

(2) $CM_A(x^{-1}) \geq CM_A(x)\ \forall x \in G$.

If there is no ambiguity, we will use $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ over G instead of multigroup A.

It can be easily verified that if A is a fuzzy multigroup of G, then

$$CM_A(x^{-1}) = CM_A(x)\ \forall x \in G.$$

Moreover, if $e \in G$ be the identify element of G, then

$$CM_A(e) = \vee_{x \in G} CM_A(x).$$

The set of all fuzzy multigroups over G is depicted by FMG(G).

Example 2.6. Let $(\mathbb{Z}_4, +)$ be a group. Consider

$$A = \left\{ \frac{(0.8, 0.7, 0.7, 0.5)}{2}, \frac{(0.6, 0.4)}{1}, \frac{(0.6, 0.4)}{3}, \frac{(0.9, 0.8, 0.7, 0.5)}{0} \right\}$$

and

$$B = \left\{ \frac{(0.6, 0.4, 0.3, 0.1)}{2}, \frac{(0.9, 0.7, 0.7, 0.5, 0.1, 0.1)}{1}, \frac{(0.8, 0.7, 0.7, 0.5, 0.1, 0.1)}{3}, \frac{(0.9, 0.8, 0.7, 0.5, 0.1, 0.1)}{0} \right\}.$$

Then A is a fuzzy multigroup, but B is not a fuzzy multigroup, because $CM_B(1) \neq CM_B(3)$.

Clearly, a fuzzy multigroup is a fuzzy subgroup that admits repetition of membership values. That is, a fuzzy multigroup collapses into a fuzzy subgroup whenever repetition of membership values is ignored.

Definition 2.7. [11, 12] Let $A \in FMG(G)$. Then for $\alpha \in [0, 1]$, the sets $A_{[\alpha]}$ and $A_{(\alpha)}$ defined by

$$A_{[\alpha]} = \{x \in G | CM_A(x) \geq \alpha\}$$

and

$$A_{(\alpha)} = \{x \in G | CM_A(x) \geq \alpha\}$$

are called strong and weak upper α-cuts of A.

Whenever the count membership values of x is greater than or equal to α, that is,

$$CM_A(x) = (\mu_1(x), \mu_2(x), \ldots, \mu_n(x)) \geq \alpha,$$

Theorem 2.8. *[11] Let $A \in FMG(G)$. Then $A_{[\alpha]}$, $\alpha \in [0,1]$ is a subgroup of G for all $\alpha \leq CM_A(e)$, where e is the identity element of G.*

3 The Construction of Fuzzy Multigroups from Fuzzy Subgroups

In this Section we obtain some suitable chains of fuzzy subgroups of a fuzzy subgroup with underlying group G and by these chains, we obtain fuzzy multigroups with underlying group G.

Lemma 3.1. *Let μ be a fuzzy subgroup of G. Then $\{\mu_i^+\}_{i=1}^n$ is a family of fuzzy subgroups of G and*

$$\mu_n^+(x) \geq \mu_{n-1}^+(x) \geq \cdots \geq \mu_1^+(x) = \mu(x)$$

where $\mu_n^+ : G \longrightarrow [0,1]$, for any $n \in \mathbb{N}$ is as follows

$$\mu_1^+(x) = \mu(x) \quad \text{and} \quad \mu_n^+(x) = \min\{\alpha_{\mu_{n-1}^+} + \mu_{n-1}^+(x), 1\}, \quad \text{for} \quad n \geq 2.$$

Proof. If $\mu(xy) \geq \mu(x)$, for all $x, y \in G$, then $\alpha_\mu + \mu(xy) \geq \alpha_\mu + \mu(x)$. Thus

$$\min\{\alpha_\mu + \mu(xy), 1\} \geq \min\{\alpha_\mu + \mu(x), 1\}.$$

So, $\mu_2^+(xy) \geq \mu_2^+(x) \geq \mu_2^+(x) \wedge \mu_2^+(y)$. Similarly, if $\mu(xy) \geq \mu(y)$, one can show that $\mu_2^+(xy) \geq \mu_2^+(y) \geq \mu_2^+(x) \wedge \mu_2^+(y)$. Moreover, it is easy to show that $\mu_2^+(x^{-1}) \geq \mu_2^+(x)$ and hence μ_2^+ is a fuzzy subgroup of G. The rest of the proof is fulfilled by induction on n. □

Lemma 3.2. *Let μ be a fuzzy subgroup of G. Then $\{\mu_i\}_{i\in\mathbb{N}}$ is a family of fuzzy subgroups of G and $\mu_1(x) \geq \mu_2(x) \geq \cdots \geq \mu_n(x)$ where $\mu_n : G \longrightarrow [0,1]$, for any $n \in \mathbb{N}$ is as follows*

$$\mu_1(x) = \mu(x) \quad \text{and} \quad \mu_n(x) = \frac{\mu_{n-1}(x)}{1 + \alpha_{\mu_{n-1}} - \mu_{n-1}(x)}, \quad \text{for} \quad n \geq 2.$$

Proof. It is easy to see that for all $x \in G$ and $n \geq 2$

$$0 \leq \frac{\mu_{n-1}(x)}{1+\alpha_{\mu_{n-1}}-\mu_{n-1}(x)} \leq 1.$$

Now, suppose that x and y are elements of G. If $\mu(xy) \geq \mu(x)$, for all $x, y \in G$, then

$$\begin{aligned} \mu_2(xy) &= \frac{\mu_1(xy)}{1+\alpha_{\mu_1}-\mu_1(xy)} \\ &\geq \frac{\mu_1(x)}{1+\alpha_{\mu_1}-\mu_1(x)} = \mu_2(x) \\ &\geq \mu_2(x) \wedge \mu_2(y). \end{aligned}$$

Similarly, if $\mu(xy) \geq \mu(y)$ for all $x, y \in G$, one can show that $\mu_2(xy) \geq \mu_2(y) \geq \mu_2(x) \wedge \mu_2(y)$. Moreover, it is easy to show that $\mu_2(x^{-1}) \geq \mu_2(x)$ and hence μ_2 is a fuzzy subgroup of G. According to definition, we have

$$\mu_2(x) = \frac{\mu_1(x)}{1+\alpha_{\mu_1}-\mu_1(x)}.$$

This implies that $1+\alpha_\mu-\mu_1(x) \geq 1$. This means that $\frac{1}{1+\alpha_{\mu_1}-\mu_1(x)} \leq 1$. Thus $\mu_2(x) \leq \mu_1(x)$, for all $x \in G$. The rest of the proof is fulfilled by induction on n. $\square$

The next results show that every fuzzy subgroup gives us fuzzy multigroups.

Theorem 3.3. *Let μ be a fuzzy subgroup of G and $n \in \mathbb{N}$. If $CM_{G_\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, then $G_\mu = \left\{ \frac{CM_{G_\mu}(x)}{x} | x \in G \right\}$ is a fuzzy multigroup over G.*

Proof. Apply Lemma 3.2. $\square$

Corollary 3.4. *Let μ be a fuzzy subgroup of G. Set $\nu_{C_\mu}(x) = \frac{C_\mu(x)}{1+\alpha_{C_\mu}}$ where $C_\mu(x) = 1 + \left|\mu_{\beta_\mu}\right| - \left|\mu_{\mu(x)}\right|$ and $\alpha_{C_\mu} = \vee_{x \in G} C_\mu(x)$. Then $G_{\nu_{C_\mu}}$ is a fuzzy multigroup of G.*

Proof. For any $x, y \in G$, we have

$$C_\mu(xy) = 1 + \left|\mu_{\beta_\mu}\right| - \left|\mu_{\mu(xy)}\right|.$$

If $\mu(xy) \geq \mu(x)$, for all $x, y \in G$, then $\mu_{\mu(xy)} \subseteq \mu_{\mu(x)}$. Thus

$$1 + \left|\mu_{\beta_\mu}\right| - \left|\mu_{\mu(xy)}\right| \geq 1 + \left|\mu_{\beta_\mu}\right| - \left|\mu_{\mu(x)}\right|.$$

This shows that $C_\mu(xy) \geq C_\mu(x) \geq C_\mu(x) \wedge C_\mu(y)$. If $\mu(xy) \geq \mu(y)$, for all $x, y \in G$, similarly we can see that $C_\mu(xy) \geq C_\mu(y) \geq C_\mu(x) \wedge C_\mu(y)$. Moreover, for any $x \in G$, we have

$$C_\mu(x^{-1}) = 1 + \left|\mu_{\beta_\mu}\right| - \left|\mu_{\mu(x^{-1})}\right| = C_\mu(x).$$

Therefore, ν_{C_μ} is a fuzzy subgroup. On the other hand, $\nu_{C_{\mu_1}}(x) \geq \nu_{C_{\mu_2}}(x) \geq \cdots$. This implies that $G_{\nu_{C_\mu}}$ is a fuzzy multigroup of G. $\square$

Example 3.5. Let $G = (\mathbb{Z}_4, +)$ be a group, $n = 3$ and $\mu : G \longrightarrow [0,1]$ be as follows:

$$\mu(x) = \begin{cases} 1, & x \in \{0,2\}; \\ \frac{1}{2}, & x \in \mathbb{Z}_4 - \{0,2\}. \end{cases}$$

Then

$$\mu_1(x) = \begin{cases} 1, & x \in \{0,2\}; \\ \frac{1}{2}, & x \in \mathbb{Z}_4 - \{0,2\}, \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & x \in \{0,2\}; \\ \frac{1}{3}, & x \in \mathbb{Z}_4 - \{0,2\}, \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 1, & x \in \{0,2\}; \\ \frac{1}{5}, & x \in \mathbb{Z}_4 - \{0,2\}. \end{cases}$$

So $CM_{G_\mu} = (\mu_1, \mu_2, \mu_3)$ is a fuzzy multigroup and we have

$$G_\mu = \left\{ \frac{(1,1,1)}{0}, \frac{(1,1,1)}{2}, \frac{(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})}{3}, \frac{(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})}{1} \right\}.$$

Proposition 3.6. *Let μ be a fuzzy subgroup of G. If*

$$CM_{G_{\mu_n^+}} = \left(\mu_n^+, \mu_{n-1}^+, \ldots, \mu_1^+, \mu_1, \mu_2, \ldots, \mu_n\right),$$

then $G_{\mu_n^+} = \{\frac{CM_{G_{\mu_n^+}}(x)}{x} | x \in G\}$ is a fuzzy multigroup of G.

Proof. Apply Lemma 3.1 and Theorem 3.3. $\square$

Theorem 3.7. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup of G. Let $\pi_k : G \longrightarrow \mathbb{W}$ be as follows:*

$$\pi_k(x) = \alpha_{\mu_1}\alpha_{\mu_2}\cdots\alpha_{\mu_k}.\mu_k(x),$$

then, $CM_{A^k_\mu} = (\pi_k, \ldots, \pi_{k+n})$ is a fuzzy multigroup of G, where $k \in \mathbb{N}_n$.

Proof. Since μ_i is a fuzzy subgroup of G, then, for each constant $\alpha \in [0,1]$, one can see that $\pi_i(x) = \alpha.\mu_i(x)$ is a fuzzy subgroup of G. Specially, suppose that $\alpha = \alpha_{\mu_1}\alpha_{\mu_2}\cdots\alpha_{\mu_i}$. On the other hand, $\pi_i(x) \geq \pi_{i+1}(x)$, for all $i \in \mathbb{N}_n$. Thus $CM_{A^k_\mu} = (\pi_k, \pi_{k+1}, \ldots, \pi_k + n)$ is a fuzzy multigroup of G. $\square$

Theorem 3.8. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup of G. Define*

$$\mu_{ij}(x) = \frac{\mu_j(x)}{\max\left\{\alpha_{\mu_i}, \alpha_{\mu_j}\right\} + \alpha_{\mu_j} - \mu_j(x)},$$

for all $x \in G$ and $i, j \in \mathbb{N}_n$. Then for every $k \in \mathbb{N}_n$, $CM_{A^k_{\mu_i}} = (\mu_{k(i+1)}, \mu_{k(i+2)}, \ldots, \mu_{k(i+n)})$ is a fuzzy multigroup of G, where $i \in \mathbb{N}_n$.

Proof. It is easy to see that for all $x \in G,\ n \geq 2$, we have

$$0 \leq \frac{\mu_j(x)}{\max\left\{\alpha_{\mu_i}, \alpha_{\mu_j}\right\} + \alpha_{\mu_j} - \mu_j(x)} \leq 1$$

Now suppose that x and y are elements of G. Let $CM_A(xy) \geq CM_A(x)$, for all $x, y \in G$. If $\mu_j(xy) \geq \mu_j(x)$, then

$$\max\{\alpha_{\mu_i}, \alpha_{\mu_j}\} + \alpha_{\mu_j} - \mu_j(xy) \leq \max\{\alpha_{\mu_i}, \alpha_{\mu_j}\} + \alpha_{\mu_j} - \mu_j(x).$$

So, we have for all $x \in G$,

$$\frac{\mu_j(xy)}{\max\{\alpha_{\mu_i}, \alpha_{\mu_j}\} + \alpha_{\mu_j} - \mu_j(xy)} \geq \frac{\mu_j(x)}{\max\{\alpha_{\mu_i}, \alpha_{\mu_j}\} + \alpha_{\mu_j} - \mu_j(x)}.$$

Thus

$$\mu_{ij}(xy) \geq \mu_{ij}(x) \geq \mu_{ij}(x) \wedge \mu_{ij}(y).$$

Moreover,

$$\mu_{ij}(x^{-1}) = \frac{\mu_j(x^{-1})}{\max\{\alpha_{\mu_i}, \alpha_{\mu_j}\} + \alpha_{\mu_j} - \mu_j(x^{-1})} = \mu_{ij}(x).$$

Hence μ_{ij} is a fuzzy subgroup of G. Also, we have $\mu_{k(i+1)}(x) \geq \mu_{k(i+2)}(x) \geq \cdots$. Thus $CM_{A^k_{\mu_i}} = (\mu_{k(i+1)}, \mu_{k(i+2)}, \mu_{k(i+3)}, \ldots, \mu_{k(i+n)})$ is a fuzzy multigroup of G. $\square$

4 Equivalence of Fuzzy Multigroups

Here, we define a suitable relation on fuzzy multigroups of a finite group G and show that this relation is an equivalence one. The fuzzy multigroups of a group G can be classified up to some natural equivalence relations on the set consisting of all fuzzy multisets of G.

Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ and $CM_{A'} = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ be two fuzzy multisets on underlying groups G and G', respectively. We say that A and A' are equivalent and we write $A \equiv A'$, if there is a bijective function $\phi : G \longrightarrow G'$ such that

$$CM_A(x) = CM_{A'}(\phi(x)),\ \forall x \in G.$$

Moreover, we call the map ϕ is an equivalence function from A onto A'. If $\phi : G \longrightarrow G'$ is a group isomorphism and ϕ is an equivalence function from A onto A', then ϕ is called an isomorphism from A onto A' and we write $A \cong A'$.

Theorem 4.1. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ and $CM_{A'} = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ be two fuzzy multisets on underlying groups G and G', respectively and $A \cong A'$. If A is a fuzzy multigroup of G, then A' is a fuzzy multigroup of G'.*

Proof. There is a bijective function $\phi : G \longrightarrow G'$ such that

$$CM_A(x) = CM_{A'}(\phi(x)),$$

for all $x \in G$. Let $x', y' \in G'$, $i \in \mathbb{N}_n$, then there are $x, y \in G$ such that $\phi(x) = x'$ and $\phi(y) = y'$. Thus

$$\begin{aligned} \mu'_i(x'y') &= \mu'_i(\phi(x)\phi(y)) = \mu'_i(\phi(xy)) = \mu_i(xy) \\ &\geq \min\{\mu_i(x), \mu_i(y)\} \\ &= \min \mu'_i(\phi(x)), \mu'_i(\phi(y)), \end{aligned}$$

and

$$\begin{aligned} \mu'_i(x'^{-1}) &= \mu'_i(\phi(x)^{-1}) = \mu'_i(\phi(x^{-1})) \\ &= \mu_i(x^{-1}) = \mu_i(x) = \mu'_i(\phi(x)) \\ &= \mu'_i(x'). \end{aligned}$$

Moreover,

$$\begin{aligned} \mu'_i(x') &= \mu'_i(\phi(x)) = \mu_i(x) \\ &\geq \mu_{i+1}(x) = \mu'_{i+1}(x'). \end{aligned}$$

Therefore A' is a fuzzy multigroup of G'. $\square$
The next Example show that if $A \equiv A'$, then Theorem 4.1 is not true.

Example 4.2. Let $CM_A = (\mu)$ and $CM'_A = (\mu')$ be two fuzzy multisets on underlying group G (fuzzy subgroups), where $\mu : \mathbb{Z}_{12} \longrightarrow [0,1]$ and $\mu' : \mathbb{Z}_{12} \longrightarrow [0,1]$ are defined as follows:

$$\mu(x) = \begin{cases} \frac{1}{13} & x = 0 \\ \frac{1}{13} & x = 4, 8 \\ \frac{1}{15} & e.w. \end{cases} \quad \text{and} \quad \mu'(x) = \begin{cases} \frac{1}{13} & x = 0 \\ \frac{1}{13} & x = 4, 11 \\ \frac{1}{15} & e.w. \end{cases}$$

where $f : G \longrightarrow G'$ is defined by

$$f(x) = \begin{cases} 11 & x = 4 \\ 8 & x = 11 \\ 4 & x = 8 \\ x & e.w. \end{cases}$$

Clearly, f is an equivalence function and A and A' are equivalent. Then μ is a fuzzy subgroup of G but μ' is not a fuzzy subgroup of G'.

Proposition 4.3. *Let μ and μ' be two fuzzy subgroups of groups G and G', respectively. If there is a bijective function $\phi : G \longrightarrow G'$ such that $\mu(x) = \mu'(\phi(x))$, then $G_\mu \cong G_{\mu'}$.*

Proof. Suppose that $x \in G$. Then

$$\mu'_1(\phi(x)) = \mu'(\phi(x)) = \mu(x) = \mu'_1(x),$$

and for $i \geq 2$,

$$\mu'_i(\phi(x)) = \frac{\mu'_{i-1}(\phi(x))}{1 + \alpha_{\mu'_{i-1}} - \mu'_{i-1}(\phi(x))} = \frac{\mu_{i-1}(x)}{1 + \alpha_{\mu_{i-1}} - \mu_{i-1}(x)} = \mu_i(x).$$

Hence $G_\mu \cong G_{\mu'}$. □

Proposition 4.4. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ and $CM_{A'} = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ be two equivalent fuzzy multigroups on underlying groups G, G', respectively. Then A^k_μ and ${A'}^k_{\mu'}$ are equivalent fuzzy multigroups on underlying groups G, G', respectively.*

Proof. The proof is similar to the proof of Proposition 4.3. □

Proposition 4.5. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ and $CM_{A'} = (\mu'_1, \mu'_2, \ldots, \mu'_n)$ be two equivalent fuzzy multigroups on underlying groups G, G', respectively. Then $A^k_{\mu_i}$ and ${A'}^k_{\mu'_i}$ are equivalent fuzzy multigroups on underlying groups G, G', respectively.*

Proof. The proof is similar to the proof of Proposition 4.3. □

Proposition 4.6. *The relation $\cong$ between fuzzy multigroups is an equivalence relation.*

Proof. Reflexivity: consider the identity map $\phi : G \to G$ such that $\phi(x) = x$, for all $x \in G$. Clearly, ϕ is a bijective map satisfying $\mu_i(x) = \mu_i(\phi(x)$, for all $x \in G$. Therefore ϕ is an equivalence function of a fuzzy multiset into itself. Hence $\cong$ is a reflexive relation.

Symmetry: $A \cong A'$ and ϕ is an equivalence function of A onto A', then ϕ^{-1} is an equivalence function of A' onto A. Assume that $A \cong A'$. Let $\phi : G \to G'$ be a bijective map satisfying $\mu_i(x) = \mu_i(\phi(x)$ for all $x \in G$. As ϕ is bijective, so its inverse exists, i.e., $\phi^{-1}(x') = x$, for all $x' \in G'$. Hence $\mu_i(\phi^{-1}(x')) = \mu_i'((\phi\phi^{-1}(x'))) = \mu_i'(x')$. Thus $\phi^{-1} : G' \to G$ is a bijective map which is an equivalence function. Therefore, $\cong$ is a symmetric relation.

Transitivity: let $A \cong A'$ and $A' \cong A''$. suppose that $\phi : G \to G'$ be an equivalence function from A onto A' and $\psi : G' \to G''$ is an equivalence function from A' onto A''. Then, for all $x \in G$, $\mu_i(x) = \mu_i'(\phi(x)) = \mu_i''(\psi(\phi(x))$. Hence $A \cong A''$. □

5 Automorphisms of Fuzzy Multigroups

In this Section, we study the notion of automorphisms on multigroup A of finite group G. We use the methods in Section 2 and construct fuzzy multigroups of automorphism group $Aut(A)$.

Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on an underlying group G. An automorphism on A is an isomorphism $\phi : G \longrightarrow G$. By $\mathrm{Aut}(A)$ we denote the set of all automorphisms on A.

Proposition 5.1. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multiset on underlying group G. Then $(\mathrm{Aut}(A), \circ)$ forms a group.*

Proof. Let $\phi, \psi \in \mathrm{Aut}(A)$ and $x \in G$. Then

$$\mu_i(\phi \circ \psi)(x) = \mu_i(\phi(\psi(x))) = \mu_i(\psi(x)) = \mu_i(x).$$

Thus $\phi \circ \psi \in \mathrm{Aut}(A)$. Clearly, $\mathrm{Aut}(A)$ satisfies associativity under the operation $\circ$. The map $id : G \to G$ by $id(x) = x$ is an identity element of $Aut(A)$. If $\phi : G \to G$ is an automorphism, then it is bijective and so $\phi^{-1} : G \to G$ exists and is a bijection. Let $\phi^{-1}(x) = x'$. Then $\mu_i(\phi^{-1}(x)) = \mu_i(x') = \mu_i(x)$. Hence, $(\mathrm{Aut}(A), \circ)$ is a group. □

Proposition 5.2. *Let $a, b > 0$ be two real numbers, $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multiset on underlying group G and $\mathrm{Aut}(A)$. Define a map $\tau : \mathrm{Aut}(A) \longrightarrow [0, 1]$ as follows*

$$\tau(f) = \begin{cases} \frac{a}{b+a} & f \text{ is an even permutation} \\ 0 & f \text{ is an odd permutation,} \end{cases} \tag{1}$$

for all $f \in \mathrm{Aut}(A)$. Then τ is a fuzzy subgroup on $\mathrm{Aut}(A)$.

Proof. We prove this result in two following cases:

Case 1) Suppose that $f, g \in \mathrm{Aut}(A)$ both are even or odd permutation. Hence $f \circ g$ becomes an even permutation. If f and g both are even permutation, then

$$\begin{aligned} \tau(f \circ g) &= \frac{a}{b+a} \\ &\geq \min\left\{\frac{a}{b+a}, \frac{a}{b+a}\right\} \\ &= \min\{\tau(f), \tau(g)\}. \end{aligned}$$

If f and g both are odd permutation, then

$$\begin{aligned} \tau(f \circ g) &= \frac{a}{b+a} \\ &\geq \min\{0, 0\} \\ &= \min\{\tau(f), \tau(g)\}. \end{aligned}$$

Case 2) Suppose that $f, g \in \mathrm{Aut}(A)$ such that one of them is even permutation and the other one is odd permutation. Thus $f \circ g$ becomes an odd permutation. Thus

$$0 = \tau(f \circ g) = \min\left\{0, \frac{a}{b+a}\right\} = \min\{\tau(f), \tau(g)\}.$$

Moreover, for all $f \in \mathrm{Aut}(A)$, f and f^{-1} both are even permutation or both are odd permutation. This follows that $\tau(f^{-1}) = \tau(f)$, for all $f \in \mathrm{Aut}(A)$. Therefore, τ is a fuzzy subgroup on $\mathrm{Aut}(A)$. $\square$

Corollary 5.3. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup of G. Then $\mathrm{Aut}(A)_\tau$ is a fuzzy multigroup of $\mathrm{Aut}(A)$.*

Proof. Apply Proposition 5.2. $\square$

Corollary 5.4. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on underlying group Then*

(i) $(\mathrm{Aut}(A)_\tau)^k_\tau$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

(ii) $(\mathrm{Aut}(A)_\tau)^k_{\tau_i}$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

Proof. Apply Theorems 3.7, 3.8 and Corollary 5.3. □

Definition 5.5. Let $CM_A = (\mu_1, \mu_2, \dots, \mu_n)$ be a fuzzy multiset on group G. For every $x \in G$, define

$$\mu^k(x) = \sup\{\mu_1(u_1) \wedge \mu_2(u_2) \wedge \dots \wedge \mu_k(u_k) : x = u_1 u_2 \dots u_k, u_1, \dots, u_k \in G\},$$

and

$$\mu^\infty(x) = \sup\{\mu^k(x) : x \in G, k \in \mathbb{N}\}.$$

Lemma 5.6. *Let* $CM_A = (\mu_1, \mu_2, \dots, \mu_n)$ *be a fuzzy multigroup on underlying group* G. *If* $\phi \in \mathrm{Aut}(A)$, *then* $\mu^k(\phi(x)) = \mu^k(x)$.

Proof. Let $x = u_1 u_2 \dots u_k$ be an arbitrary products of k elements of G. Thus $\phi(x) = \phi(u_1)\phi(u_2)\dots\phi(u_k)$ is product of k elements of G. Since ϕ is a homomorphism, we have

$$\begin{array}{rcl} \mu^k(\phi(x) & \geq & \mu_1(\phi(u_1)) \wedge \mu_2(\phi(u_2)) \wedge \dots \wedge \mu_k(\phi(u_k)) \\ & \geq & \mu_1(u_1) \wedge \mu_2(u_2) \wedge \dots \wedge \mu_k(u_k). \end{array}$$

Since $x = u_1 u_2 \dots u_k$,

$$\mu^k(\phi(x)) \geq \mu^k(x). \tag{2}$$

Now, assume that $\phi(x) = u'_1 u'_2 \dots u'_k$ is an arbitrary a products of k elements of G. Since $\phi \in \mathrm{Aut}(A)$, there are $u_1, u_2, \dots, u_k \in G$ such that $\phi(u_i) = u'_i$, for all $i \in \mathbb{N}_k$. Then

$$\mu^k(x) \geq \mu_1(u_1) \wedge \mu_2(u_2) \wedge \dots \wedge \mu_k(u_k).$$

On the other hand, we have $\mu(u_i) = \mu(\phi(u_i)) > 0$, thus

$$\mu^k(x) \;\; \geq \;\; \mu_1(\phi(u_1) \wedge \phi_2(u_2)) \wedge \dots \wedge \mu_k(\phi(u_k)).$$

This shows that

$$\mu^k(\phi(x)) \leq \mu^k(x). \tag{3}$$

Then (2) and (3) imply that $\mu^k(\phi(x)) = \mu^k(x)$. □

Proposition 5.7. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on underlying group G. Define a map $\lambda : \mathrm{Aut(A)} \longrightarrow [0,1]$ as follows*

$$\lambda(\phi) = \sup\{\mu^k(\phi(x)) : x \in G, k \in \mathbb{N}\}, \tag{4}$$

for all $\phi \in \mathrm{Aut(A)}$. Then λ is a fuzzy subgroup on underlying group $\mathrm{Aut(A)}$.

Proof. Let $\phi \in \mathrm{Aut(A)}$. Then by $\mu^k(\phi(x)) = \mu^k(x)$, we have

$$\lambda(\phi) = \sup\{\mu^k(x) : x \in G, k \in \mathbb{N}\} \tag{5}$$

Then for any $\phi, \psi \in \mathrm{Aut(A)}$, (5) implies that

$$\begin{aligned}\lambda(\phi \circ \psi) &= \sup\{\mu^k((\phi \circ \psi)(x) : x \in G, k \in \mathbb{N}\} \\ &= \sup\{\mu^k((\phi(\psi(x)) : x \in G, k \in \mathbb{N}\} \\ &\geq \mu^k(\psi(x)),\end{aligned}$$

for all $x \in G$. Thus

$$\lambda(\phi \circ \psi) \geq \lambda(\psi) \geq \lambda(\phi) \wedge \lambda(\psi).$$

Also,

$$\lambda(\phi^{-1}) = \sup\{\mu^k(\phi^{-1}(x)) : x \in G, k \in \mathbb{N}\} \geq \mu^k(x),$$

for all $x \in G$. Thus $\lambda(\phi^{-1}) \geq \lambda(\phi)$. This implies that λ is a fuzzy subgroup on underlying group $\mathrm{Aut(A)}$. $\square$

Corollary 5.8. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup of G. Then $\mathrm{Aut(A)}_\lambda$ is a fuzzy multigroup of $\mathrm{Aut(A)}$.*

Proof. Apply Proposition 5.7. $\square$

Corollary 5.9. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on underlying group. Then*

(i) $(\mathrm{Aut(A)}_\lambda)^k_\lambda$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

(ii) $(\mathrm{Aut(A)}_\lambda)^k_{\lambda_i}$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

Proof. By applying Theorems 3.7, 3.8 and Corollary 5.8, the Corollary holds. $\square$

Proposition 5.10. *Let $CM_A = (\mu_1, \mu_2, \dots, \mu_n)$ be a fuzzy multiset on underlying group G and $\mathrm{Aut}(A)$. Define a map $\psi : \mathrm{Aut}(A) \longrightarrow [0,1]$ as follows*

$$\psi(f) = \begin{cases} \min\left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : f(i) \neq i\right\}, & f \neq \mathrm{id}; \\ 1, & f = \mathrm{id}, \end{cases} \tag{6}$$

for all $f \in \mathrm{Aut}(A)$. Then ψ is a fuzzy subgroup on $\mathrm{Aut}(A)$.

Proof. Clearly, if $f, g \in \mathrm{Aut}(A)$ and $f \circ g = \mathrm{id}$, then $\psi(f \circ g) \geq \min\{\psi(f), \psi(g)\}$. Suppose that $f \circ g \neq \mathrm{id}$. If

$$\psi(f \circ g) = \min\left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : f \circ g(i) \neq i\right\},$$

then there exists $r \in G$ such that $f \circ g(r) \neq r$ and

$$\psi(f \circ g) = \min\left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : f \circ g(i) \neq i\right\} = \frac{\mu^{\infty}(r)}{1+\mu^{\infty}(r)}.$$

We have the following two cases:
Case 1) If $f \circ g(r) \neq g(r)$, then

$$\begin{aligned} \psi(f \circ g) &= \frac{\mu^{\infty}(r)}{1+\mu^{\infty}(r)} = \frac{\mu^{\infty}(g(r))}{1+\mu^{\infty}(g(r))} \\ &\geq \min\left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : f(i) \neq i\right\} = \psi(f). \end{aligned}$$

In the similar way we have
Case 2) Assume that $f \circ g(r) = g(r)$. Thus $g(r) \neq r$. Then we have

$$\begin{aligned} \psi(f \circ g) &= \frac{\mu^{\infty}(r)}{1+\mu^{\infty}(r)} \\ &\geq \min\left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : g(i) \neq i\right\} = \psi(g) \end{aligned}$$

So the Cases 1 and 2 imply that

$$\psi(f \circ g) \geq \min\{\psi(f), \psi(g)\}.$$

Moreover, $\left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : f(i) \neq i\right\} = \left\{\frac{\mu^{\infty}(i)}{1+\mu^{\infty}(i)} : f^{-1}(i) \neq i\right\}$, therefore $\psi(f^{-1}) = \psi(f)$. This completes the proof. $\square$

Corollary 5.11. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup of G. Then $\mathrm{Aut}(A)_\psi$ is a fuzzy multigroup of $\mathrm{Aut}(A)$.*

Proof. The proof is fulfilled by the Proposition 5.10. $\square$

Corollary 5.12. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on underlying group G. Then*

(i) $(\mathrm{Aut}(A)_\psi)^k_\psi$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

(ii) $(\mathrm{Aut}(A)_\psi)^k_{\psi_i}$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

Proof. By Theorems 3.7, 3.8 and Corollary 5.11 the proof is fulfilled. $\square$

Proposition 5.13. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on underlying group G. Define a map $\Upsilon : \mathrm{Aut}(A) \longrightarrow [0,1]$ as follows*

$$\Upsilon(f) = \begin{cases} \min\left\{\frac{C_A(i)}{1+C_A(i)} : f(i) \neq i\right\}, & f \neq \mathrm{id}; \\ 1, & f = \mathrm{id}. \end{cases} \tag{7}$$

Then Υ is a fuzzy subgroup on $\mathrm{Aut}(A)$.

Proof. The proof is similar to the proof of Proposition 5.10. $\square$

Corollary 5.14. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup of G. Then $\mathrm{Aut}(A)_\Upsilon$ is a fuzzy multigroup of $\mathrm{Aut}(A)$.*

Corollary 5.15. *Let $CM_A = (\mu_1, \mu_2, \ldots, \mu_n)$ be a fuzzy multigroup on underlying group G. Then*

(i) $(\mathrm{Aut}(A)_\Upsilon)^k_\Upsilon$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

(ii) $(\mathrm{Aut}(A)_\Upsilon)^k_{\Upsilon_i}$ *is a fuzzy multigroup on underlying group* $\mathrm{Aut}(A)$.

6 Code Generation by Fuzzy Multigroup Automorphisms

In this section, according to the obtained results in the previous sections, we give an application of fuzzy multigroups. Nowadays, one-time passwords (OTP) are used in e-banking and social media authentication. Here, we generate an OTP according to the time requested by a person to receive the OTP, and we can generate the OTP based on additional components in addition to time such as ID card number. First, we give the following algorithm and so by giving an example, we finish this section.

Algorithm:

1. Enter the date, logging time and put $m_1 = hmsab$, where h=Hour, m=Minutes, s=Seconds, a=Month, b=Day.
2. Let $CM_A = (\mu_1, \ldots, \mu_n)$ be a fuzzy multigroup on group $G = \mathbb{Z}_n$.
3. For all $2 \le i \le n$, $m_i = m_1 + n(i-1)$, when $m = \sum_{i=1}^{n} m_i$.
4. Choose the small number $\varepsilon = \frac{1-\frac{m_n}{m}}{2}$.
5. For every $1 \le i \le 8$, set $\sigma_i^\bullet = \frac{m_i}{m} + \frac{\varepsilon}{2^{n-i}}$, where $\sigma_i^\bullet$'s are distinct.
6. Set $\alpha = \sum_{i=1}^{n} \sigma_i^\bullet + \frac{|\mathrm{Aut(A)}|}{|\mathrm{Aut(A)}+2|}$.
7. For every $1 \le i \le n$, $p_i = \frac{\sigma_i^\bullet}{\alpha}$.
8. Set $sd = \frac{1}{|\mathrm{Aut(A)}|+3}\left(\frac{1}{n}\sum_{f\in \mathrm{Aut(A)}} \frac{C_\Upsilon(f)}{\alpha_{C_\Upsilon}} + \varepsilon\right)$.
9. Choose a number from 1 to n and call it k then set password$=10^{10}(sd+p_k)$

Example 6.1. Let $G = \mathbb{Z}_8$ and $CM_A = (\mu_1, \mu_2, \ldots, \mu_8)$ be a fuzzy multigroup on the group G such that $\mu_i : G \longrightarrow [0, 1]$, for any $i \in \mathbb{N}$ is defined as follows:

$$\mu_i(x) = \begin{cases} 1, & x = 0; \\ \frac{1}{2^{\frac{i-1}{i}}}, & x \in \mathbb{Z}_8 - \{0\}. \end{cases}$$

By the above stated algorithm, we generate an OPT depending on the fuzzy multigroup $CM_A = (\mu_1, \mu_2, \ldots, \mu_8)$ for the time 12:20:30 PM, Day=17, Month=3. We have $\mathrm{Aut}(A) = \{f_1, f_3, f_5, f_7\}$, where for $r = 1, 3, 5, 7$, $f_r : \mathbb{Z}_8 \longrightarrow \mathbb{Z}_8$ is defined by $f_r(a) = ra$. Then by Proposition 5.13, $\Upsilon : \mathrm{Aut}(A) \longrightarrow \mathbb{W}$ and $C_\Upsilon : \mathbb{Z}_8 \longrightarrow \mathbb{W}$ are as follows:

$$\Upsilon(f) = \begin{cases} 1, & f = f_1; \\ \frac{8}{9}, & f \neq f_1. \end{cases} \quad \text{and} \quad C_\Upsilon(f) = \begin{cases} 4, & f = f_1; \\ 1, & f \neq f_1. \end{cases}$$

and $m_1 = 122030173, \varepsilon = \frac{1-\frac{m_8}{m}}{2} = \frac{1-\frac{122030229}{976241608}}{2} = 0.43749986$ Thus

$$sd = \frac{1}{7}\left(\frac{1}{8}\left(\frac{4+3\times 1}{4}\right) + \varepsilon\right) = 0.093749998$$

The Table 1 shows generated OTP's for the time 12:20:30.,17/3.

Table 1: Generated OTP's for the time 12:20:30.,17/3

No.	m_i	$\frac{m_i}{m}$	ε_i	$\frac{m_i}{m} + \varepsilon_i$	p_i	$sd + p_i$
1	122030173	0.124999971	0.003417969	0.12841794	0.04141514	0.135165138
2	122030181	0.12499998	0.006835937	0.131835917	0.042517447	0.136267445
3	122030189	0.124999988	0.013671875	0.138671862	0.044722058	0.138472056
4	122030197	0.124999996	0.027343749	0.152343745	0.049131278	0.142881276
5	122030205	0.125000004	0.054687498	0.179687502	0.057949715	0.151699713
6	122030213	0.125000012	0.109374996	0.234375009	0.075586587	0.169336584
7	122030221	0.12500002	0.218749993	0.343750013	0.118060327	0.204610325
8	122030229	0.125000029	0.437499986	0.562500014	0.181407806	0.275157804
	976241608		1.871582003			
			3.100748684			

Hence according to the Table 1, an OTP for $n = 5$ is 1516996950.

In the above example, if we notice the column $sd + p_i$ in the above table, then we can see the efficiency of the algorithm for generating the quasi random numbers. For example, the numbers 5138, 7445, 2038, 2056, 1276,9713, 6584 are the last four digits in the column $sd + p_i$, where all of them have applications in various fields of social media, banking, safe boxes and etc.

7 Conclusion

In this work we obtained some results in fuzzy multigroups. We presented some suitable chains of fuzzy subgroups of a fuzzy subgroup μ on group G and obtained fuzzy multigroups with underlying group G.

We introduced an equivalence relations on fuzzy multigroups and by the notion of automorphisms on multigroups, we described a fuzzy multigroup with an underlying group of permutations from a fuzzy multigroup with an underlying group G. Finally, we generated one-time passwords (OTP) by using fuzzy multigroups.

For future work, it will be interesting to introduce a t-norm over a fuzzy subgroup and obtain a t-norm fuzzy multigroup by a t-norm in a natural way. Also, In [1, 2, 6, 17] the authors studied a generalization of fuzzy subgroups and fuzzy multigroups. It will be interesting to introduce the idea of this paper on Multi-polygroups and Multi-hypergroups.

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