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Random Fixed Point Theorems in Fréchet Spaces with their Applications

H. R. Goudarzi
Yasouj University

Abstract. In this paper some random fixed point theorems Fréchet spaces have been introduced. Some of them will be the stochastic analogue of the well known fixed point theorems. Also as their applications we see some notable results of fixed points in Banach and linear topological spaces.

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1. Introduction

Random operator theory is an interesting subject needed to study of various classes of random equations. The initiated point in the study of random fixed point theorems belongs to Prague school of probabilistic in 1950s. The interest in this subject enhanced after publication of the survey paper by Bharucha Ried [1]. Random fixed point theory has received much attention in the recent years (see [3], [12], [13] and [15]). Some important stochastic analogue cases of well known fixed point theorems have been introduced by some authors (see [8], [10], [5], [14] and [17]). The main purpose of this article is to prove the stochastic analogue of some well known results such as Markov-kakutani and Krasnoselskii fixed point theorems. Moreover as their applications, we see some other interesting results.

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2. Preliminaries

Let $(X,\ \tau)$ be a Hausdorff locally convex space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms on X is called an associated family for τ if the family $\{\gamma U : \gamma > 0\}$ forms a base of neighborhoods of zero for τ, where $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$. A family $\{p_\alpha : \alpha \in I\}$ of seminorms on X is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family such that $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$, for any $\alpha, \beta \in I$. We will denote by $\mathcal{A}(\tau)$ and $\mathcal{A}^*(\tau)$, the associated and augmented associated semi norms $\{p_\alpha : \alpha \in I\}$, respectively. As a well known result, there always exists a family $\{p_\alpha : \alpha \in I\}$ of semi norms on X such that $\{p_\alpha : \alpha \in I\} = \mathcal{A}^*(\tau)$(see [6], P. 203). A subset M of X is τ-bounded in X if and only if each p_α is bounded on M.
For any τ-bounded subset M of X we can choose a number $\lambda_\alpha > 0$ such that $M \subseteq \lambda_\alpha U_\alpha$, where $U_\alpha = \{x : p_\alpha(x) \leqslant 1\}$ and $\alpha \in I$. It is easy to show that $B = \bigcap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains M. Also the linear span X_B of B in X is $\bigcup_{n=1}^{\infty} nB$ and the Minkowski linear functional is a norm which is denoted by $\|.\|_B$. This means that $(X_B,\ \|.\|_B)$ is a normed space with closed unit ball B and for each $x \in X_B$, $\|x\|_B = \sup_\alpha p_\alpha(\frac{x}{\lambda_\alpha})$. To see the details one can refer to [6] and [16].
Let A and B be two self maps on M. Then A is called:

i) $\mathcal{A}^*(\tau)$-non expansive if for all $x, y \in M$

$$p_\alpha(Ax - Ay) \leqslant p_\alpha(x - y),\ \forall p_\alpha \in \mathcal{A}^*(\tau).$$

ii) $\mathcal{A}^*(\tau)$-B-non expansive if for all $x, y \in M$

$$p_\alpha(Ax - Ay) \leqslant p_\alpha(Bx - By),\ \forall p_\alpha \in \mathcal{A}^*(\tau).$$

Throughout this paper for simplicity, we shall call $\mathcal{A}^*(\tau)$-non expansive ($\mathcal{A}^*(\tau)$-B-non expansive) maps by expansive (B-non expansive) maps.
Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is Cauchy if and only for each p_α, $p_\alpha(x_n - x_m) \longrightarrow 0$ as $m, n \longrightarrow \infty$.
Let C be a nonempty convex subset of X and $q \in C$. A self mapping T on C is said to be affine if

$$T(\lambda x + (1 - \lambda)y) = \lambda T(x) + (1 - \lambda)T(y),$$

for all $x, y \in C$ and $\lambda \in (0,1)$. Also T is said to be affine with respect to q if

$$T(\lambda x + (1-\lambda)q) = \lambda T(x) + (1-\lambda)T(q),$$

for all $x \in C$ and $\lambda \in (0,1)$. There is an example of an affine mapping with respect to a point which is not affine (see[18]).

Let X be a topological vector space. A subset $A \subseteq X$ is said to be star-shaped if there exists an element $x \in A$ such that $tx + (1-t)y \in A$, for all $t \in [0,1]$ and for all $y \in A$. Such an element is called a star-point of A. The set of all star-points of A is called the star-core of A.

Let $(\Omega,\ A)$ be a measurable space and X be a metric space. A mapping $f : \Omega \times X \longrightarrow X$ is called a random operator if for any $x \in X$, $f(.,x)$ is measurable. Some authors consider the random operator as $f : \Omega \times K \longrightarrow X$ (see [2]), where K is a non empty subset of X. A measurable mapping $\xi : \Omega \longrightarrow X$ is called a random fixed point of a random operator $f : \Omega \times X \longrightarrow X$, if for each $\omega \in \Omega$, $\xi(\omega) = f(\omega, \xi(\omega))$. Clearly $\xi_x(\omega) = x$ is a random fixed point for $f : \Omega \times X \longrightarrow X$ if and only if, $x \in X$, is a fixed point for $f(\omega, .)$, for any $\omega \in \Omega$. A random operator $f : \Omega \times X \longrightarrow X$ is continuous (respectively, non expansive, g-non expansive(for self mapping g of X)) if for each $\omega \in \Omega$, $f(\omega, .)$ is continuous (respectively, non expansive, g-non expansive), see [11]. Two maps $f, g : X \longrightarrow X$ are called commutative, if for each $x \in X$, $f(g(x)) = g(f(x))$. Random operators $f, g : \Omega \times X \longrightarrow X$ is said to be commutative, if $f(\omega, .)$ and $g(\omega, .)$ are commutative for each $\omega \in \Omega$. The random operator $T : \Omega \times X \longrightarrow X$ is called affine with respect to $p \in X$ (resp. demiclosed at p) if for each $\omega \in \Omega$, $T(\omega, .)$ is affine with respect p (resp. demiclosed at p). The other definitions such as linearity, invariant property, compactness, demicompactness, weakly compactness and closeness can be introduced similarly. Throughout this paper we denote by $RF(T)$, the set of random fixed points of a random operator $T : \Omega \times X \longrightarrow X$.

Theorem 2.1. (Markov-Kakutani) ([4]) *Let X be a nonempty, compact and convex subset of a Hausdorff locally convex space and let $\mathcal{A}$ be a commutating family of continuous affine mappings. Then there exists an element $\overline{x} \in X$ such that $f(\overline{x}) = \overline{x}$, for each $f \in \mathcal{A}$.*

Theorem 2.2. ([14]) *Let X be separable closed and convex subset of a Hilbert space H and $A, B : \Omega \times X \longrightarrow H$ be random operators such that A is contraction and B is compact and continuous. Then there exists a measurable map $\phi : \Omega \longrightarrow X$ such that for each $\omega \in \Omega$*

$$\|T(\omega, \phi(\omega)) - \phi(\omega)\| = \min\{\|T(\omega, \phi(\omega)) - x\| : x \in X\},$$

where $T = A + B$. If additionally $T(\Omega \times \partial X) \subseteq X$ then ϕ is a random fixed point.

For a Banach space E, suppose that $A \subseteq E$ be a subset of E and let $\delta(A)$ denotes the diameter of A. Let $\alpha(A) = \inf \epsilon$, where $\epsilon > 0$ and A can be covered by a finite number of subsets $A_1, A_2, ...A_n$ of E with $\delta(A_i) \leqslant \epsilon$, for each i. Then $\alpha(A)$ is called the Kuratowski's measure of noncompactness of A (see [9]). We recall that for arbitrary bounded subsets B, B_1, B_2 of E we have the following properties.
i)$\alpha(A) = 0$ if and only if B is relatively compact set.
ii)$\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}$.
iii)$\alpha(\overline{B}) = \alpha(B)$.
For a nonempty subset $X \subseteq E$, a random operator $T : \Omega \times X \longrightarrow E$ is called condensing if for each bounded subset $A \subseteq X$ with $\alpha(A) > 0$ and each fixed $\omega \in \Omega$ we have $\alpha(T(\omega, A)) \leqslant \alpha(A)$.

Lemma 2.3. ([14])*Let X be a separable, closed and convex subset of a Banach space E and $T : \Omega \times X \longrightarrow X$ be a bounded, continuous and condensing random operator. Then T has a random fixed point.*

Theorem 2.4. (Krasnosel'skii) ([7]) *Suppose A is a closed bounded convex subset of a Banach space X. If $T : A \longrightarrow X$ is a contraction, $C : A \longrightarrow X$ is compact and $T(A) + C(A) = \{T(x) + C(y),\ x, y \in A\} \subseteq A$, then $T + C$ has a fixed point in A.*

Theorem 2.5. ([4]) *Let K be a compact and star-shaped subset of a topological linear space X. Then every decreasing chain of nonempty, compact and star-shaped subsets of K has a nonempty intersection that is compact and star-shaped.*

Lemma 2.6. ([4]) *Suppose that K is a compact star-shaped subset of* *a topological linear space X and A is the corresponding star-core of K. Then A is compact convex subset of A.*

3. Main Results

In pure and applied aspects of Fixed point Theory, there are useful results such as Markov-kakutani and Krasnoselski theorems. In this section, we try to prove the stochastic analogue of these theorems. Moreover as their applications, we see some other interesting results.

Theorem 3.1. *Let K be a compact convex subset of a Fréchet space E and $T : \Omega \times K \longrightarrow K$ be a continuous affine random operator. Then T has a random fixed point.*

Proof. Suppose that $\omega \in \Omega$ be given. If the theorem were false, the intersection of the diagonal $\triangle = \{(x, x) : x \in K\}$ of $K \times K$ with the graph of $T(\omega, .)$. i.e. $\Gamma = \{((\omega, x), T(\omega, x))\}$ would be empty. But for $\omega \in \Omega$, $\triangle$ and Γ are compact and convex subset of $E \times E$. By applying Han-Banach theorem there are continuous linear functionals L_1^ω and L_2^ω on E and scalars $\beta < \alpha$ such that

$$L_1^\omega(x) + L_2^\omega(x) \leqslant \alpha < \beta \leqslant L_1^\omega(y) + L_2^\omega(y),$$

for all $x, y \in K$. Hence for $x \in K$ we have

$$L_2^\omega(T(\omega, x)) - L_2^\omega(x) \geqslant \beta - \alpha.$$

By iterating this equality we can deduce that

$$L_2^\omega(T^n(\omega, x)) - L_2^\omega(x) \geqslant n(\beta - \alpha) \longrightarrow \infty,$$

for arbitrary $x \in K$ so that the sequence $\{L_2^\omega(T^n(\omega, x))\}_{n\in\mathbb{N}}$ is unbounded which contradicts the compactness of $L_2^\omega(K)$. $\square$

Corollary 3.2. *Let K be a compact convex subset of a Fréchet space E. Suppose that $\mathcal{A}$ be a family of commutative, continuous and affine random operators $T : \Omega \times K \longrightarrow K$. Then $\mathcal{A}$ has a random fixed point.*

Proof. Suppose that $\omega \in \Omega$ be given, $\mathcal{A} = \{T_i : i \in I\}$ and $K_i = RF(T_i)$. According to theorem 3.1, $K_i \neq \emptyset$ and also K_i is convex and compact. Since for each $i, j \in I$, T_i and T_j commute we conclude that $T_i(\omega, K_j) \subset K_j$. Hence $T_i(\omega, .)|_{K_j}$ has a fixed point by theorem 3.1 so that $K_i \cap K_j \neq \emptyset$. This means that the class $\{A_i : i \in I\}$ has the finite intersection property. So $\bigcap_{i\in I} K_i \neq \emptyset$. $\square$

If we regard the fixed points as the special cases of random fixed points, also we have the following application.

Corollary 3.3. *Let X be a nonempty, compact and convex subset of a Fréchet space and let $\mathcal{A}$ be a commutating family of continuous affine mappings. Then there exists an element $\overline{x} \in X$ such that $f(\overline{x}) = \overline{x}$, for each $f \in \mathcal{A}$.*

Now we wish to weakened the conditions of theorem 3.1.

Lemma 3.4. *Suppose that K is a star-shaped subset of a Fréchet space X and $T : \Omega \times K \longrightarrow K$ be a surjective random operator that is affine. Then the star-core of K is invariant under T.*

Proof. Let C be the star-core of K and $x_0 \in C$. We will show that for each $\omega \in \Omega$, $T(\omega, x_0) \in C$. Let $y \in K$ and $0 \leqslant \lambda \leqslant 1$ be arbitrary. since T is surjective for given $\omega \in \Omega$, there exists some $x \in K$ such that $T(\omega, x) = y$. Since x_0 is a star-point of K, we have $\lambda x_0 + (1-\lambda)x \in K$. But T is affine and K is invariant under T. So

$$\begin{aligned}\lambda T(\omega, x_o) + (1-\lambda)y &= \lambda T(\omega, x_0) + (1-\lambda)T(\omega, x)\\ &= T(\omega, \lambda x_0 + (1-\lambda)x)\\ &\in T(\omega, K) \subseteq K.\end{aligned}$$

Thus $T(\omega, x_0)$ is star-point of K and hence C is invariant under T. $\square$

Theorem 3.5. *Let X be a nonempty compact and star-shape subset of a Fréchet space V, C be the star-core of X and $\mathcal{A}$ be a commutating family of continuous random operators $T : \Omega \times X \longrightarrow X$ such that for each $T \in \mathcal{A}$ and $\omega \in \Omega$, $F(T(\omega, .)) \subseteq C$. Also suppose that T is p-affine, for each $p \in C$. Then $\bigcap_{T\in\mathcal{A}} RF(T) \neq \emptyset$.*

Proof. By applying Theorem 2.5 and Zorn's lemma we can obtain a set $M \subseteq X$ such that M is minimal with respect to being nonempty, compact, star-shaped and invariant under each $T \in \mathcal{A}$. We will show that for each $\omega \in \Omega$ and $T \in \mathcal{A}$, we have $T(\omega, M) = M$. Assume the contrary and suppose that $S \in \mathcal{A}$, be such that $S(\omega, M) \subset M$, for some $\omega \in \Omega$. Let $S(\omega, M) = N$. Since S is p-affine then N is star-shaped. Also N is nonempty and compact. Now suppose that $(\omega, x) \in N = S(\omega, M)$. Then there exists $y \in M$ such that $(\omega, x) = S(\omega, y)$. Since $\mathcal{A}$ is commutative, for given $\omega \in \Omega$ and $T \in \mathcal{A}$ we have

$$T(\omega, x) = T(S(\omega, y)) = S(T(\omega, y)) \in S(\omega, M).$$

Consequently, for each $T \in \mathcal{A}$ and $\omega \in \Omega$, $T(\omega, N) \subseteq N$. Thus $N \subset M$ is a nonempty, compact and star-shaped subset of K that is invariant under each $T \in \mathcal{A}$. But this is a contradiction to the minimality of M and hence $T(\omega, M) = M$, $\omega \in \Omega$ and $T \in \mathcal{A}$. Let C be the star-core of M. Since C is nonempty, convex and compact subset of X, by lemma 3.4, we deduce that for each $T \in \mathcal{A}$, C is invariant under C. Now by applying corollary 3.2, $\mathcal{A}$ has a random fixed point. $\square$

In the next step we wish to extend theorem 2.4. But first we extend the lemma 2.3.

Lemma 3.6. *Let $(X,\ \tau)$ be a separable Férechet space and M be a τ-bounded subset of X. Then we have*
i) *$(X_B,\ \|.\|_B)$ is complete.*
ii) *If $T : \Omega \times M \longrightarrow X$ is continuous, then it is continuous in $(X_B,\ \|.\|_B)$.*
iii) *If T is condensing and bounded with respect to τ then it is condensing and bounded in $(X_B,\ \|.\|_B)$.*

Proof. *i)* Let $\{x_m\}$ be a sequence in M such that $x_m \longrightarrow x$ and suppose that τ is compatible with the family of seminorms $\{p_n\}$. Then for each $\epsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$p_n(x_m - x) < \epsilon,\ \forall m \geqslant k_0,$$

for all $p_n \in \mathcal{A}^*(\tau)$. Hence

$$\sup_n p_n(\frac{x_m - x}{\lambda_n}) \leqslant \epsilon,\ \forall m \geqslant k_0,$$

and so

$$\parallel x_m - x \parallel_B \leqslant \epsilon,\ \forall m \geqslant k_0.$$

The other parts can be similarly proved. □

According to the previous lemma we have the following extension of Lemma 2.3.

Corollary 3.7. *Let X be a separable, closed and convex subset of a Férechet space E and $T : \Omega \times X \longrightarrow X$ be a bounded, continuous and condensing random operator. Then T has a random fixed point.*

Now we are ready to extend the second result of this article (Theorem 3.9).

Definition 3.8. *Let E be a Férechet space, $K \subseteq E$, compatible with the family of seminorms $\{p_n\}$ and $T : E \times K \longrightarrow E$ be an operator. The family $\{T(., y) : y \in K\}$ is said to be equicontractive if there exists a $k \in [0, 1)$ such that*

$$p_n(T(x_1, y) - T(x_2, y) \leqslant k p_n(x_1 - x_2),$$

for each p_n and for all $(x_1, y), (x_2, y)$ in the domain of T.

Theorem 3.9. *Suppose that A is a closed convex bounded subset of the Férechet space X and $C : \Omega \times A \longrightarrow X$ a compact random operator. Also suppose that $T : \Omega \times A \times C(A) \longrightarrow A$ is a random operator such that for each $\omega \in \Omega$, the family $\{T(\omega, ., y) : y \in C(A)\}$ is equicontractive and for each $x \in A$ and $\omega \in \Omega$, the map $T(\omega, x, .)$ is uniformly continuous. If $T(\omega, A, C(A)) \subseteq A$, the random operator $S(\omega, x) = T(\omega, x, C(x))$, for each $x \in A$, has a random fixed point.*

Proof. Consider a set $B \subseteq A$, $\omega \in \Omega$ and let $\epsilon > 0$ be given. The uniformly continuity guarantees that there exists a $\delta(\epsilon) > 0$ such that for all $x \in A$ and $y_1, y_2 \in C(A)$, $p_n(T(\omega, x, y_1) - T(\omega, x, y_2)) < \epsilon$, whenever we have $p_n(y_1, y_2) < \delta(\epsilon)$ and for each p_n. According to the definition of Kuratowski's measure of noncompactness, see [9], there exists a finite family of sets $B_1, ..., B_n$ such that $B = \bigcup_{i=1}^{n} B_i$ and $diam(B_i) \leqslant \alpha(B) + \epsilon,\ i = 1, 2, ..., n$.

Also $C(\omega, B)$ is precompact. Thus we have $\alpha(C(\omega, B)) = \alpha(C(\overline{\omega, B})) = 0$ and hence there exists a finite family of sets $V_1, ..., V_m$ such that $C(\omega, B) = \bigcup_{j=1}^{m} V_j$ and $diam(V_j) \leqslant \delta(\epsilon),\ j = 1, 2, ...m$. Hence $B \subseteq \bigcup_{j=1}^{m} C^{-1}(\omega, V_j)$ and therefore

$$B = (\bigcup_{i=1}^{n} B_i) \cap (\bigcup_{j=1}^{m} C^{-1}(V_j)) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (B_i \cap C^{-1}(V_j)).$$

This gives that

$$S(\omega, B) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} S(\omega, B_i \cap C^{-1}(V_j)).$$

Fix two indices i, j, take points $x_1, x_2 \in B_i \cap C^{-1}(V_j)$. Hence for each p_n, we have

$$p_n(x_1 - x_2) \leqslant diam(B_i) \leqslant \alpha(B) + \epsilon,$$

and

$$p_n(C(\omega, x_1) - C(\omega, x_1)) \leqslant diam(V_j) \leqslant \delta(\epsilon).$$

Hence

$$p_n(T(\omega, x_1, C(\omega, x_1)) - T(\omega, x_2, C(\omega, x_2))) \leqslant \epsilon,$$

and therefore we get

$$\begin{aligned} p_n(S(\omega, x_1) - S(\omega, x_2)) &\leqslant p_n(T(\omega, x_1, C(\omega, x_1)) - T(\omega, x_2, C(\omega, x_2))) \\ &+ p_n(T(\omega, x_2, C(\omega, x_1)) - T(\omega, x_2, C(\omega, x_2))) \\ &\leqslant k\alpha(B) + k\epsilon + \epsilon. \end{aligned}$$

But this means that $diam S(\omega, B_i \cap C^{-1}(V_j)) \leqslant k\alpha(B) + k\epsilon + \epsilon$ and so we have

$$\begin{aligned} \alpha(S(\omega, B)) &= \max\{diam S(\omega, B_i \cap C^{-1}(\omega, V_j)) : \\ i &= 1, 2, ..., n,\ j = 1, 2, ..., m\} \\ &\leqslant k\alpha(B) + k\epsilon + \epsilon. \end{aligned}$$

Since ϵ is arbitrary it implies that $\alpha(S(\omega, B)) \leqslant k\alpha(B)$. Now by Corollary 3.7 the proof is complete. $\square$

Corollary 3.10. *A simple observation shows that the Theorems 2.5 and the second part of Theorem 2.4 are the special cases of the above theorem.*

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Hamid Reza Goudarzi
Department of Mathematics
Faculty of Science
Assistant Professor of Mathematics
Yasouj University
Yasouj, Iran
E-mail: goudarzi@mail.yu.ac.ir