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Original Research Paper

Counting Fuzzy Subgroups in p-groups

Dedicated to my father on the occasion of his 74^{th} birthday

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Abstract. We give formulas for the number of fuzzy subgroups of all small p-groups of order at most p^4 as polynomials in prime p.

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Introduction

The theory of fuzzy groups began after few years the theory of fuzzy sets is introduced by Lotfi Asker Zadeh in 1965. In 1971, Rosenfeld [16] defined the notion of fuzzy groups and their fuzzy subgroups, which was the starting point to study algebraic structures from a fuzzy point of view. Study of fuzzy subgroups and their classification is one of the classical problems in the theory of fuzzy groups. Most results classifying fuzzy subgroups of fuzzy groups are devoted to finite abelian groups. In [9], Laszlo studies small fuzzy groups of order at most 6 and classified all their fuzzy subgroups. In [22], Zhang and Zou determine the number of fuzzy subgroups of cyclic groups of order p^n, where p is a prime number. Murali and Makamba in [12, 13] extend the results of Zhang and Zou

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and compute the number of fuzzy subgroups of all cyclic groups of order $p^m q^n$, where p and q are distinct primes. The number of fuzzy subgroups of finite cyclic groups is obtained by Tărnăuceanu and Bentea in [20] by establishing a recurrence relation for the number of fuzzy subgroups. They also give another recursive formula for the number of fuzzy subgroups of finite elementary abelian p-groups. Later, Tărnăuceanu in [18] gives a direct formula for the number of fuzzy subgroups of finite elementary abelian p-groups. The next step is two classify fuzzy subgroups of non-cyclic abelian groups. However, this case is too complicated to be solved in general. In [14], Ngcibi, Murali and Makamba compute the number of fuzzy subgroups of abelian p-groups $C_{p^m} \times C_{p^n}$ of rank two when $n \leq 3$. This result is generalized by Oh in [15] to all abelian p-groups of rank two. For abelian p-groups of rank three that are not elementary abelian, the only result we can state belongs to Appiah and Makamba in [1]. They obtain the number of fuzzy subgroups of abelian p-groups $C_{p^n} \times C_p \times C_p$ of rank three for all $n \geq 1$.

Tărnăuceanu in [17] provides a general techniques to count the number of distinct fuzzy subgroups of non-abelian finite groups and obtains the number of fuzzy subgroups of dihedral groups in some special cases. His results was extended by Darabi, Saeedi and Farrokhi in [3] to all dihedral groups and some other families of non-abelian finite groups including generalized quaternion groups, quasi-dihedral groups, and modular p-groups. Davvaz and Kamali Ardekani [4] compute the number of fuzzy subgroups of small p-groups of orders p^3 and 2^4 in the same year. Also, Tărnăuceanu in [19] gives formulas for the number of fuzzy subgroups of finite symmetric groups. In this paper, we will consider p-groups of orders at most p^4 and compute the number of their fuzzy subgroups. For this we analyze all subgroups of these groups and use a recursive formula to count their fuzzy subgroups.

1 Preliminaries

In this section, we provide all the required background and results we need in our investigation. Let $\mu : G \to [0, 1]$ be a fuzzy subset of a group G. Then μ is called a fuzzy subgroup of G if the following two conditions hold:

(a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$,

(b) $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$.

Note that condition (b) is equivalent to say that $\mu(x^{-1}) = \mu(x)$ for all $x \in G$. Also, condition (a) implies that $\mu(1) = \max \mu(G)$.

The main tool in studying fuzzy subgroups is the notion of level subsets. If $\alpha \in [0, 1]$, then the level subset corresponding to α is defined as

$$\mu_\alpha = \{x \in G : \mu(x) \geq \alpha\}.$$

These subsets characterize fuzzy subgroups of fuzzy group because a fuzzy subset μ of G is a fuzzy subgroup of G if its level subsets are all subgroups of G.

The above observation shows that many fuzzy subsets determine the same fuzzy subgroups of the groups. To compare these fuzzy subsets we need to introduce an equivalence relation on fuzzy subsets. Two fuzzy subsets μ and η of G are called naturally equivalent if

$$\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y)$$

for all $x, y \in G$. The above equivalence relation determines fuzzy subgroups of G up to equivalence classes. Indeed, that two fuzzy subgroups μ and η of G are distinct if $\mu \not\sim \eta$.

Now, let G be a finite group and $\mu : G \to [0, 1]$ be a fuzzy subgroup of G. Also, let $\mu(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Then we obtain the following chain of subgroups of G including G associated to μ

$$\mu_{\alpha_1} \subseteq \mu_{\alpha_2} \subseteq \cdots \subseteq \mu_{\alpha_n} = G. \tag{1}$$

Note that

$$\mu(x) = \alpha_i \Longleftrightarrow x \in \mu_{\alpha_i} \setminus \mu_{\alpha_{i-1}},$$

for all $x \in G$ and $i = 1, 2, \ldots, n$, where $\mu_{\alpha_0} = \emptyset$. In [21], Volf shows that two fuzzy subgroups μ, η of G are naturally equivalent if and only if μ and η have the same sets of level subgroups or equivalently they have the same chain of subgroups as shown in (1). This gives a bijection between equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G ending in G. For some further equivalence relations on fuzzy subgroups we refer to [6, 7]. See [8, 10, 11, 16] for more details on fuzzy groups and fuzzy algebras.

In the following we state the results we need to use in this paper. The most intricate and important result is the number of fuzzy subgroups of cyclic groups obtained by Tărnăuceanu and Bentea in 2008.

Proposition 1.1 ([20]). *Let G be a finite cyclic group of order $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots {p_k}^{\alpha_k}$. Then the number of all distinct fuzzy subgroups of G is equal to*

$$F(G)=2^{\sum\limits_{r=1}^{k}\alpha_r}\sum_{i_2=0}^{\alpha_2}\sum_{i_3=0}^{\alpha_3}\cdots\sum_{i_k=0}^{\alpha_k}\left(-\frac{1}{2}\right)^{\sum\limits_{r=2}^{k}i_r}\prod_{r=2}^{k}\binom{\alpha_r}{i_r}\binom{\alpha_1+\sum\limits_{m=2}^{r}(\alpha_m-i_m)}{\alpha_r},$$

if $k>1$, and $F(G)=2^{\alpha_1}$ if $k=1$.

The next results provide the number of fuzzy subgroups of non-cyclic abelian finite p-groups in some special cases including all finite abelian p-groups of rank two.

Theorem 1.2 (Oh [15]). *Let $G = C_{p^m}\times C_{p^n}$ be an abelian p-group of rank two, where $m \geq n \geq 1$. Then the number of fuzzy subgroups of G is equal to*

$$F(G) = 2^{m+n} + 2^m\sum_{k=1}^{n}2^{n-k}\left[\sum_{r=1}^{k}T_{r,k}\binom{m+n-k-r-1}{k-r+1}\right]p^k,$$

where $T_{r,k}$ are the Schröder's numbers defined by

$$T_{r,k} = \begin{cases} 0, & k<r,\\ T_{r,k-1}+T_{r-1,k}+T_{r-1,k-1}, & k\geq r.\end{cases}$$

Theorem 1.3 (Tărnăuceanu [18]). *Let $G = C_p^n$ be an elementary abelian p-group of order p^n, where $n \geq 1$. Then the number of fuzzy subgroups of G is equal to*

$$F(G) = 2 + 2f(n)\sum_{k=1}^{n-1}\sum_{1\leq i_1<\cdots<i_k<n}\frac{1}{f(n-i_k)f(i_k)\cdots f(i_1)},$$

where $f : \mathbb{N} \longrightarrow \mathbb{N}$ is the function defined by $f(m) = (p-1)(p^2-1)\cdots(p^m-1)$.

Theorem 1.4 (Appiah and Makamba [1]). *Let $G = C_{p^n} \times C_p \times C_p$ be an abelian p-group of rank three, where $n \geq 1$. Then the number of fuzzy subgroups of G is equal to*

$$F(G) = 2^{n-1}n(n+1)p^3 + 2^{n-1}n(n+7)p^2 + 2^{n+1}(n+1)p + 2^{n+2}.$$

2 p-groups of Small Orders

In order to compute the number of fuzzy subgroups of small p-groups, we should recall some basic results from the theory of p-groups.

Lemma 2.1. *Let G be a group and $x, y \in G$ such that $[x, y]$ commutes with x and y. Then*

(1) $(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}$, *and*

(2) $[x, y^n] = [x^n, y] = [x, y]^n$

for all integers n

The intersection of all maximal subgroups of a group G is denoted by $\Phi(G)$ and called the Frattini subgroup of G. The Frattini subgroups of p-groups play a crucial rule in the study of p-groups.

Theorem 2.2. *Let G be a finite p-group. Then $\Phi(G) = G'G^p$, where G' is the derived subgroup of G and G^p is the subgroup of G generated by all elements g^p for arbitrary elements $g \in G$.*

Note that cyclic p-groups of order p^n have 2^n fuzzy subgroups. Also, the group $C_p \times C_p$ has $2(p+2)$ fuzzy subgroup. In what follows we shall use these results frequently.

The p-groups of orders p^3 have been studied in [4]. To be self-contained, we give a short proof of the following theorem. Notice that the only non-abelian 2-groups of order 8 are D_8 and Q_8 with 32 and 16 fuzzy subgroups, respectively. Our computation of the fuzzy subgroups is based on the following recursive formula

$$F(G) = 1 + \sum_{H \lneqq G} F(H) \tag{2}$$

for all finite groups G.

Theorem 2.3. *Let G be p-group of odd order p^3. Then*

(1) $F(G) = 8$ *if* $G = C_{p^3}$*;*

(2) $F(G) = 8(p+1)$ *if* $G = C_{p^2} \times C_p$*;*

(3) $F(G) = 2(p^3 + 4p^2 + 4p + 4)$ *if* $G = C_p \times C_p \times C_p$*;*

(4) $F(G) = 4(p^2 + 2p + 2)$ *if* $\exp(G) = p$*;*

(5) $F(G) = 8(p+1)$ *if* $\exp(G) = p^2$*.*

Proof. Part (1) is obvious. Part (2) can be obtained from Theorem 1.2. Also, part (3) can be obtained from Theorems 1.3 and 1.4.

(4) It is evident that G has $(p^3 - 1)/(p - 1) = p^2 + p + 1$ subgroups of order p. Let H be a subgroup of G of order p^2. Then H is abelian containing $Z(G)$ otherwise $G = \langle H, Z(G)\rangle$ would be abelian, which is a contradiction. Hence, the subgroups H are in one-to-one correspondence to subgroups of $G/Z(G)$ of order p, which implies that G has $p+1$ subgroups of order p^2. Therefore, the number of fuzzy subgroups of G equals

$$\begin{aligned} F(G) &= 1 + (p+1)F(C_p \times C_p) + (p^2 + p + 1)F(C_p) + F(1) \\ &= 1 + 2(p+1)(p+2) + 2(p^2 + p + 1) + 1 \\ &= 4(p^2 + 2p + 2). \end{aligned}$$

(5) Let $G = \langle x, y \mid x^{p^2} = y^p = 1, x^y = x^{p+1}\rangle$ be the non-abelian p-group of order p^3 and exponent p^2. Then every element of G has the form $x^i y^j$ for some $0 \le i < p^2$ and $0 \le j < p$. A simple computation shows that

$$o(x^i y^j) = \begin{cases} p^2, & p \nmid i, \\ 1, & i = j = 0, \\ p, & \text{otherwise}. \end{cases}$$

As a result, G has $(p^3 - p\varphi(p^2) - 1)/(p-1) = p+1$ subgroups of order p and p cyclic subgroups of order p^2. Clearly, G has a non-cyclic subgroup of order p^2, namely $\langle x^p, y\rangle$. If H is any non-cyclic subgroup of G of order p^2, then H has $p+1$ cyclic subgroups of order p so that it contains all cyclic subgroups of G. Hence, G has a unique non-cyclic subgroup of order p^2. Therefore, the number of fuzzy subgroups of G equals

$$\begin{aligned} F(G) &= 1 + F(C_p \times C_p) + pF(C_{p^2}) + (p+1)F(C_p) + 1 \\ &= 8(p+1). \end{aligned}$$

□

Corollary 2.4. *The number of fuzzy subgroups of a p-group of order p^3 and exponent p^2 equals $8(p+1)$.*

The groups of order p^4 are classified by Hölder and Young and presented in Burnside's book [2]. Here we give all these groups by their presentation and use them to compute their fuzzy subgroups.

Theorem 2.5 ([2, Pages 87–88])**.** *Let G be an abelian p-group of odd order p^4. Then G is isomorphic to one of the following groups*

(1) $G_1(p) = C_{p^4}$;

(2) $G_2(p) = C_{p^3} \times C_p$;

(3) $G_3(p) = C_{p^2} \times C_{p^2}$;

(4) $G_4(p) = C_{p^2} \times C_p \times C_p$;

(5) $G_5(p) = C_p \times C_p \times C_p \times C_p$;

(6) $G_6(p) = \langle a, b \mid a^{p^3} = b^p = 1, a^b = a^{p^2+1} \rangle$;

(7) $G_7(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = [a, b] = [a, c] = 1, b^c = ba^p \rangle$;

(8) $G_8(p) = \langle a, b \mid a^{p^2} = b^{p^2} = 1, a^b = a^{p+1} \rangle$;

(9) $G_9(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = [a, b] = [b, c] = 1, a^c = a^{p+1} \rangle$;

(10) $G_{10}(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = [a, b] = [b, c] = 1, a^c = ab \rangle$;

(11) $G_{11}(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, a^b = a^{p+1}, a^c = ab, b^c = b \rangle$;

(12) $G_{12}(p) = \langle a, b, c \mid a^9 = b^3 = [b, c] = 1, c^3 = a^3, a^b = a^4, a^c = ab^{-1} \rangle$ *for* $p = 3$ *and* $G_{12}(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, a^b = a^{p+1}, a^c = ab, b^c = a^p b \rangle$ *for* $p > 3$;

(13) $G_{13}(p) = \langle a, b, c \mid a^9 = b^3 = [b, c] = 1, c^3 = a^{-3}, a^b = a^4, a^c = ab^{-1}\rangle$ *for* $p = 3$ *and* $G_{13}(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, a^b = a^{p+1}, a^c = ab, b^c = a^{p\lambda}b\rangle$ *for* $p > 3$ *and non-residue* λ *modulo* p*;*

(14) $G_{14}(p) = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = [a, b] = [a, c] = [a, d] = [b, c] = [b, d] = 1, c^d = ac\rangle$*;*

(15) $G_{15}(p) = \langle a, b, c \mid a^9 = b^3 = c^3 = [a, b] = 1, a^c = ab, b^c = a^{-3}b\rangle$ *for* $p = 3$ *and* $G_{15}(p) = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = [a, b] = [a, c] = [a, d] = [b, c] = 1, b^d = ab, c^d = bc\rangle$ *for* $p > 3$.

There are only 14 non-abelian 2-groups of order 2^4. These groups and the number of their fuzzy subgroups are described in the following theorem (by using computations with GAP [5] in conjunction with the fact that the number of fuzzy subgroups of a group can be computed inductively via equality (2)).

Theorem 2.6. *Let* G *be a* 2*-group of order* 2^4*. Then*

(1) $F(G) = 16$ *if* $G = C_{2^4}$*;*

(2) $F(G) = 64$ *if* $G = C_{2^3} \times C_2$*;*

(3) $F(G) = 112$ *if* $G = C_{2^2} \times C_{2^2}$*;*

(4) $F(G) = 304$ *if* $G = C_{2^2} \times C_2 \times C_2$*;*

(5) $F(G) = 1392$ *if* $G = C_2 \times C_2 \times C_2 \times C_2$*;*

(6) $F(G) = 64$ *if* $G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^5\rangle$*;*

(7) $F(G) = 240$ *if* $G = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, b] = [a, c] = 1, b^c = ba^2\rangle$*;*

(8) $F(G) = 112$ *if* $G = \langle a, b \mid a^4 = b^4, a^b = a^{-1}\rangle$*;*

(9) $F(G) = 432$ *if* $G = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, b] = [b, c] = 1, a^c = a^{-1}\rangle$*;*

(10) $F(G) = 208$ *if* $G = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, b] = [b, c] = 1, a^c = ab\rangle$*;*

(11) *$F(G) = 72$ if $G = \langle a, b, c \mid a^4 = b^4 = c^2 = [a, c] = [b, c] = 1, a^2 = b^2, a^b = a^{-1}, b^c = b^{-1}\rangle$;*

(12) *$F(G) = 128$ if $G = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1}\rangle$;*

(13) *$F(G) = 96$ if $G = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3\rangle$;*

(14) *$F(G) = 64$ if $G = \langle a, b \mid a^8 = b^4 = 1, a^4 = b^2, a^b = a^{-1}\rangle$.*

Now, we are ready to prove our main theorem. We will consider every group in Theorem 2.5 and construct all its subgroups, which enables us to compute all fuzzy subgroups of the given group.

Theorem 2.7. *Let G be p-group of odd order p^4. Then*

(1) *$F(G) = 16$ if $G = G_1(p)$;*

(2) *$F(G) = 8(3p + 2)$ if $G = G_2(p)$;*

(3) *$F(G) = 4(3p^2 + 6p + 4)$ if $G = G_3(p)$;*

(4) *$F(G) = 4(2p^3 + 9p^2 + 6p + 4)$ if $G = G_4(p)$;*

(5) *$F(G) = 2(p^6 + 6p^5 + 12p^4 + 18p^3 + 18p^2 + 12p + 8)$ if $G = G_5(p)$;*

(6) *$F(G) = 8(3p + 2)$ if $G = G_6(p)$;*

(7) *$F(G) = 4(2p^3 + 7p^2 + 6p + 4)$ if $G = G_7(p)$;*

(8) *$F(G) = 4(3p^2 + 6p + 4)$ if $G = G_8(p)$;*

(9) *$F(G) = 4(3p^3 + 9p^2 + 6p + 4)$ if $G = G_9(p)$;*

(10) *$F(G) = 4(p^3 + 7p^2 + 6p + 4)$ if $G = G_{10}(p)$;*

(11) *$F(G) = 4(p^3 + 7p^2 + 6p + 4)$ if $G = G_{11}(p)$ and $p > 3$, and $F(G) = 520$ if $G = G_{11}(3)$;*

(12) *$F(G) = 4(5p^2 + 6p + 4)$ if $G = G_{12}(p)$ and $p > 3$, and $F(G) = 196$ if $G = G_{12}(3)$;*

(13) *$F(G) = 4(5p^2 + 6p + 4)$ if $G = G_{13}(p)$ and $p > 3$, and $F(G) = 268$ if $G = G_{13}(3)$;*

(14) $F(G) = 4(2p^4 + 7p^3 + 9p^2 + 6p + 4)$ *if* $G = G_{14}(p)$*;*

(15) $F(G) = 4(3p^3 + 7p^2 + 6p + 4)$ *if* $G = G_{15}(p)$*.*

Proof. (1) It is obvious.

(2) and (3) The result follows from Theorem 1.2.

(4) The result obtains from Theorem 1.4.

(5) It follows from Theorem 1.3.

In what follows we denote the non-abelian p-groups of order p^3 of exponent p and p^2 by E_p and E_{p^2}, respectively. Recall that $F(E_p) = 4(p^2 + 2p + 2)$ and $F(E_{p^2}) = 8(p + 1)$ by Theorem 2.3.

(6) Let $G = G_6(p) = \langle a, b \mid a^{p^3} = b^p = 1, a^b = a^{p^2+1}\rangle$. Since G is non-abelian and $(a^p)^b = a^{p(p^2+1)} = a^p$, we have $Z(G) = \langle a^p\rangle$ and so G is nilpotent of class 2 for $G/Z(G)$ is a p-group of order p^2. As $G = \langle a\rangle \rtimes \langle b\rangle$ every element of G can be written uniquely as a^ib^j for some $0 \leq i < p^3$ and $0 \leq j < p$. Then by Lemma 2.1,

$$(a^ib^j)^{p^k} = a^{ip^k}b^{jp^k}[b, a]^{ijp^k(p^k-1)/2} = a^{ip^k(1-ip^2(p^k-1)/2)}b^{jp^k}$$

for $k \geq 0$. Hence,

$$o(a^ib^j) = \begin{cases} p^3, & p \nmid i, \\ p^2, & p \mid i \text{ and } p^2 \nmid i, \\ p, & p^2 \mid i \text{ and } i \neq 0, \\ 1, & i = j = 0. \end{cases}$$

Therefore, G has $p^2 - 1$ elements of order p and so $(p^2 - 1)/\varphi(p) = p + 1$ subgroups of order p. Similarly, G has $(p^3 - p^2)/\varphi(p^2) = p$ cyclic subgroups of order p^2 and $(p^4 - p^3)/\varphi(p^3) = p$ cyclic subgroups of order p^3. Now, let H be a non-cyclic subgroup of G of order p^2. Then H has $p^2 - 1$ elements of order p so that it contains all elements of G of order p. Thus $H = \langle a^{p^2}, b\rangle$ is the unique non-cyclic subgroup of G of order p^2. We only need to find non-cyclic subgroups of order p^3. So, let H be a non-cyclic subgroup of G of order p^3. As H is a maximal subgroup of G, it contains $\Phi(G)$ by Theorem 2.2, hence $a^p \in H$. Since H is non-cyclic and $o(a^ib^j) = p^3$ when $\gcd(i, p) = 1$, H must have an element $a^{pi}b^j$ such that $j \neq 0$. Thus H contains b, which implies that

$H = \langle a^p, b\rangle \cong C_{p^2} \times C_p$ is the unique non-cyclic subgroup of G of order p^3. Therefore, the number of fuzzy subgroups of G is equal to

$$\begin{aligned} F(G) &= 1 + pF(C_{p^3}) + F(C_{p^2} \times C_p) + pF(C_{p^2}) + F(C_p \times C_p) \\ &\quad + (p+1)F(C_p) + 1 \\ &= 8(3p+2). \end{aligned}$$

(7) Let $G_7(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = [a, b] = [a, c] = 1, b^c = ba^p\rangle$. Clearly, $Z(G) = \langle a\rangle$ and $\Phi(G) = \langle a^p\rangle$ by Lemma 2.2. Also, every element of G can be written uniquely as $a^i b^j c^k$ for some $0 \leq i < p^2$ and $0 \leq j, k < p$. Since $[b, c]$ commutes with b and c, a simple calculation shows that

$$(a^i b^j c^k)^p = a^{(i-jk(p-1)/2)p}.$$

Thus,

$$o(a^i b^j c^k) = \begin{cases} p^2, & p \nmid i - jk(p-1)/2, \\ p, & p \mid i - jk(p-1)/2 \text{ and } (i, j, k) \neq (0, 0, 0), \\ 1, & i = j = k = 0. \end{cases}$$

Hence, G has $p^3 - 1$ elements of order p and consequently $(p^3 - 1)/\varphi(p) = p^2 + p + 1$ cyclic subgroups of order p. Since G has $p^4 - p^3$ elements of order p^2, it has $(p^4 - p^3)/\varphi(p^2) = p^2$ cyclic subgroups of order p^2. Let H be a non-cyclic subgroup of G of order p^2. Since G is non-abelian and $a \in Z(G)$ we should have $a^p \in H$ otherwise $H \cap Z(G) = 1$ and so $G = HZ(G) \cong H \times Z(G)$ would be abelian. Hence, H has p non-central cyclic subgroups of order p. Since these cyclic subgroups together with $\langle a^p\rangle$ all generate the same subgroup H, it follows that G has $(p^2 + p + 1 - 1)/p = p + 1$ non-cyclic subgroups of order p^2. Now, let H be a non-cyclic subgroup of order p^3. Clearly, $a^p \in H$ for H is a maximal subgroup of G and $a^p \in \Phi(G)$. We have three cases:

Case 1. $a \in H$. Then $H = \langle a, x\rangle$ for some $x\Phi(G) \in \langle b\Phi(G), c\Phi(G)\rangle$. Since $\langle b\Phi(G), c\Phi(G)\rangle \cong C_p \times C_p$ has $p + 1$ cyclic subgroups $\langle x\Phi(G)\rangle$ of order p and each of which gives a distinct subgroup $\langle a, x\rangle$, we obtain $p + 1$ non-cyclic subgroups of G isomorphic to $C_{p^2} \times C_p$.

Case 2. $a \notin H$ and $\exp(H) = p^2$. Let $x \in H$ be an element of order p^2. If H is abelian, then $H = C_G(x) \supseteq Z(G) = \langle a\rangle$ for $x \notin Z(G)$,

$H \subseteq C_G(x)$ and H is a maximal subgroup of G, a contradiction. Thus H is non-abelian and consequently $\langle x\rangle$ is the only cyclic subgroup of H of order p^2. Moreover, $H = N_G(\langle x\rangle) \cong E_{p^2}$ is uniquely determined by $\langle x\rangle$. Since G has p^2-1 non-central cyclic subgroups of order p^2, it has p^2-1 subgroup of order p^3 and exponent p^2 such that $a \notin H$.

Case 3. $a \notin H$ and $\exp(H) = p$. Then H has p^3-1 elements of order p, which implies that it contains all elements of order p. Thus $H = \langle a^p, b, c\rangle \cong E_p$ is unique.

Utilizing the above information, we obtain

$$\begin{aligned} F(G) &= 1 + (p^2-1)F(E_{p^2}) + F(E_p) + (p+1)F(C_{p^2}\times C_p) \\ &\quad + (p+1)F(C_p\times C_p) + p^2F(C_{p^2}) + (p^2+p+1)F(C_p) + 1 \\ &= 4(2p^3+7p^2+6p+4). \end{aligned}$$

(8) Let $G = G_8(p) = \langle a, b \mid a^{p^2} = b^{p^2} = 1, a^b = a^{p+1}\rangle$. Clearly, $G = \langle a\rangle \rtimes \langle b\rangle$ and so every element of G can be written uniquely as a^ib^j for some $0 \le i, j < p^2$. Since $(a^i)^{b^j} = a^{i(p+1)^j}$, we observe that $a^p, b^p \in Z(G)$. The fact that G is non-abelian shows that $Z(G) = \langle a^p, b^p\rangle$. Furthermore, $a^p, b^p \in \Phi(G)$, which implies that $\Phi(G) = \langle a^p, b^p\rangle$. A simple computation shows that

$$o(a^ib^j) = \begin{cases} p^2, & p\nmid i \text{ or } p\nmid j,\\ p, & p \mid i, j \text{ and } (i,j)\neq(0,0),\\ 1, & i=j=0.\end{cases}$$

Thus G has $(p^2-1)/\varphi(p) = p+1$ cyclic subgroups of order p and $(p^4-p^2)/\varphi(p^2) = p(p+1)$ cyclic subgroups of order p^2. Also, it is obvious that $\langle a^p, b^p\rangle$ is the only non-cyclic subgroup of G of order p^2. Let H be a subgroup of G of order p^3. Then $a^p, b^p \in H$ as H is a maximal subgroup of G. As $a^p, b^p \in Z(G)$ it follows that H is abelian. Now H contains $(p^3-p^2)/\varphi(p^2) = p$ cyclic subgroups of order p^2 each of which together with $\langle a^p, b^p\rangle$ generates H. Thus G has $p(p+1)/p = p+1$ subgroups of order p^3 all of which are abelian and isomorphic to $C_{p^2}\times C_p$. Note that $\langle a^p, b^p, x\rangle$ is an abelian group of order p^3 for all $x \in G\setminus\langle a^p, b^p\rangle$.

Therefore, the number of fuzzy subgroups of G is given by

$$\begin{aligned}F(G) &= 1+(p+1)F(C_{p^2}\times C_p)+F(C_p\times C_p)+p(p+1)F(C_{p^2})\\&\quad +(p+1)F(C_p)+1\\&= 4(3p^2+6p+4).\end{aligned}$$

(9) Let $G=G_9(p)=\langle a,b,c \mid a^{p^2}=b^p=c^p=[a,b]=[b,c]=1, a^c=a^{p+1}\rangle$. It is easy to see that $G=\langle a,c\rangle\times\langle b\rangle$, which implies that $Z(G)=\langle a^p,b\rangle$ and $\Phi(G)=\langle a^p\rangle$. Also, every element of G can be written as $a^ib^jc^k$ for some $0\leq i<p^2$ and $0\leq j,k<p$. Since G is nilpotent of class two, we observe that

$$o(a^ib^jc^k)=\begin{cases}p^2, & p\nmid i,\\ p, & p\mid i \text{ and } i\neq 0,\\ 1, & i=j=k=0.\end{cases}$$

Thus G has $(p^3-1)/\varphi(p)=p^2+p+1$ cyclic subgroups of order p and $(p^4-p^3)/\varphi(p^2)=p^2$ cyclic subgroups of order p^2. Let H be a non-central non-cyclic subgroup of G of order p^2. If $H\cap Z(G)=1$, then $G=Z(G)H\cong Z(G)\times H$ is abelian as $|Z(G)|=p^2$, a contradiction. Thus $Z(G)\cap H\neq 1$. Let $H\cap Z(G)=\langle z\rangle$. Then $H=\langle z,g\rangle$ for some non-central element g of order p. Thus H has p non-central cyclic subgroups each of which together with $\langle z\rangle$ generate H. Since G has $(p^2+p+1)-(p+1)=p^2$ non-central cyclic subgroups of order p, it follows that G has $(p^2/p)\times(p+1)=p^2+p$ non-central non-cyclic subgroups of order p^2. Thus G has p^2+p+1 non-cyclic subgroups of order p^2. Now, let H be a subgroup of G of order p^3. Then $a^p\in H$. If H has exponent p, then it contains p^3-1 elements of order p, which implies that $H=\langle a^p,b,c\rangle\cong C_p\times C_p\times C_p$ is the only subgroup of G of order p^3 and exponent p. So, we may assume that H has exponent p^2. A simple computation shows that the following facts hold in G:

(1) every element of G of order p^2 is a power of ab^jc^k for some $0\leq j,k<p$.

(2) $C_G(ab^jc^k)=\langle ac^k,b\rangle\cong C_{p^2}\times C_p$ depends only on k.

(3) $\langle ab^jc^k\rangle$ is a normal subgroup of G for all $0\leq j,k<p$.

From (1) and (2) we observe that G has exactly p abelian subgroups H of order p^3 and exponent p^2. Hence, we may assume that H is non-abelian and contains the element ab^jc^k for some $0 \leq j,k < p$. Then $H = \langle ab^jc^k, g\rangle$ for some $g \in G \setminus C_G(ab^jc^k)$. As $|H| = p^3$ and $\exp(H) = p^2$ we must have $H \cong E_{p^2}$ and H contains elements of order p which does not belong to $\langle ab^jc^k\rangle$, that is, we can assume that g has order p. Moreover, all such elements together with ab^jc^k generate H. Now since $G \setminus C_G(ab^jc^k)$ and $H \setminus \langle ab^jc^k\rangle$ contain $(p^3-1)-(p^2-1) = p^3-p^2$ and $(p^2-1)-(p-1) = p^2-p$ elements of order p, respectively, it follows that G has $(p^3-p^2)/(p^2-p) = p$ non-abelian subgroups of order p^3 containing ab^jc^k. As $0 \leq j,k < p$ were arbitrary and every subgroup with the above properties has p cyclic subgroups of order p^2, G has $(p^2 \times p)/p = p^2$ non-abelian subgroups of order p^3 and exponent p^2, which are isomorphic to E_{p^2}. Therefore, the number of fuzzy subgroups of G is equal to

$$
\begin{aligned}
F(G) &= 1 + p^2F(E_{p^2}) + pF(C_{p^2} \times C_p) + F(C_p \times C_p \times C_p)\\
&\qquad + p^2F(C_{p^2}) + (p^2+p+1)F(C_p \times C_p) + (p^2+p+1)F(C_p) + 1\\
&= 4(3p^3 + 9p^2 + 6p + 4).
\end{aligned}
$$

(10) Let $G = G_{10}(p) = \langle a,b,c \mid a^{p^2} = b^p = c^p = [a,b] = [b,c] = 1, a^c = ab\rangle$. A simple verification shows that $Z(G) = \Phi(G) = \langle a^p, b\rangle$ and so G is nilpotent of class two. Also, every element of G can be written as $a^ib^jc^k$ for some $0 \leq i < p^2$ and $0 \leq j,k < p$. We have

$$
o(a^ib^jc^k) = \begin{cases} p^2, & p \nmid i,\\ p, & p \mid i \text{ and } i \neq 0,\\ 1, & i = j = k = 0. \end{cases}
$$

Thus, as in part (9), G has p^2+p+1 and p^2 cyclic subgroups of orders p and p^2, respectively. Also, the same argument as in (9) shows that G has p^2+p+1 non-cyclic subgroups of order p^2. Now, let H be a subgroup of G of order p^3. As H is a maximal subgroup of G and $Z(G) = \Phi(G) \subseteq H$, it follows that H is abelian. Since $\langle a^p, b, c\rangle$ contains all elements of order p in G, it is the only subgroup of G of order p^3 and exponent p. Hence, we may assume that $\exp(H) = p^2$. Clearly, $H \cong C_{p^2} \times C_p$ as H is non-cyclic. Thus H contains $(p^3-p^2)/\varphi(p^2) = p$ cyclic subgroups of order

p^2 each of which together with $Z(G)$ generate H. Since $\langle Z(G), g\rangle$ is a subgroup of order p^3 for all elements g of order p^2 and G contains p^2 cyclic subgroups of order p^2, G has $p^2/p = p$ abelian subgroups of order p^3 and exponent p^2. Therefore, the number of fuzzy subgroups of G is equal to

$$\begin{aligned} F(G) &= 1 + pF(C_{p^2} \times C_p) + F(C_p \times C_p \times C_p) \\ &\quad + p^2F(C_{p^2}) + (p^2+p+1)F(C_p \times C_p) + (p^2+p+1)F(C_p) + 1 \\ &= 4(p^3 + 7p^2 + 6p + 4). \end{aligned}$$

(11) Let $G = G_{11}(p) = \langle a, b, c \mid a^{p^2} = b^p = c^p = [b, c] = 1, a^b = a^{p+1}, a^c = ab\rangle$. If $p = 3$, then we apply GAP and obtain $F(G) = 520$. Hence, we assume that $p \geq 5$. Clearly, $\Phi(G) = \langle a^p, b\rangle$ as $a^p, b = [a, c] \in \Phi(G)$ and $|\Phi(G)| \leq p^2$. Since $[a, [a, c]] = [a, b] = a^p \neq 1$, the group G has class ≥ 3, which implies that G is nilpotent of class 3 and so $Z(G) = \langle a^p\rangle$ has order p. Also, as $G = Z_3(G)$, we must have $[a, c] = b \in Z_2(G)$, from which it follows that $Z_2(G) = \langle a^p, b\rangle \cong C_3 \times C_3$. Let $M := \langle a^p, b, c\rangle$. Since $a^p \in Z(G)$ and $[b, c] = 1$, we have $M \cong C_p \times C_p \times C_p$ and M is a maximal subgroup of G. Using induction, we can show that

$$(a^{pi}b^jc^k)^{a^t} = a^{p\left(i-tj+\binom{t}{2}k\right)}b^{j-kt}c^k$$

for all $0 \leq i, j, k < p$ and $t \geq 0$. Now, since

$$(a^tm)^s = (a^tm)\cdots(a^tm) = a^{ts}m^{a^{t(s-1)}}\cdots m^{a^t}m$$

for all $s, t \geq 0$ and $m \in M$, we obtain

$$\begin{aligned} (a^t(a^{pi}b^jc^k))^p &= a^{tp}\left(a^{p\left(i-t(p-1)j+\binom{t(p-1)}{2}k\right)}b^{j-t(p-1)k}c^k\right)\cdots(a^{pi}b^jc^k) \\ &= a^{tp+p^2i-p\binom{p}{2}tj+\frac{1}{2}p\binom{p}{2}tk\left(\frac{t(2p-1)}{3}-1\right)}b^{pj-\binom{p}{2}tk}c^{pk} \\ &= a^{tp}. \end{aligned}$$

Hence

$$o(a^ib^jc^k) = \begin{cases} p^2, & p \nmid i, \\ p, & p \mid i \text{ and } (i,j,k) \neq (0,0,0), \\ 1, & i = j = k = 0. \end{cases}$$

Therefore, G has $(p^3-1)/\varphi(p) = p^2+p+1$ cyclic subgroups of order p and $(p^4-p^3)/\varphi(p^2) = p^2$ cyclic subgroups of order p^2. Since M contains all elements of order p, it follows that G has $(p^3-1)/(p-1) = p^2+p+1$ non-cyclic subgroups of order p^2. Also, G has a unique subgroup M of exponent p. Now, let H be a subgroup of G of order p^3 and exponent p^2. As $a^p, b \in \Phi(G)$ and H is a maximal subgroup of G we have $a^p, b \in H$. Hence H contains an element of the form ac^k for some $0 \leq k < p$ and that H is defined uniquely by k. Thus G has p subgroups of order p^3 and exponent p^2. Also, all these subgroups H are non-abelian otherwise G has two abelian maximal subgroups, say H and M, which implies that $H \cap M = Z(G)$ has order p^2, a contradiction. Therefore, the number of fuzzy subgroups of G is equal to

$$\begin{aligned} F(G) &= 1 + pF(E_{p^2}) + F(C_p \times C_p \times C_p) \\ &\quad + (p^2+p+1)F(C_p \times C_p) + p^2F(C_{p^2}) + (p^2+p+1)F(C_p) + 1 \\ &= 4(p^3+7p^2+6p+4). \end{aligned}$$

(12) Let $G : G_{12}(p)$. For $p = 3$, we get $F(G) = 196$ by GAP. Hence assume that $p > 3$. Then $G = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, a^b = a^{p+1}, a^c = ab, b^c = a^pb \rangle$. Similar to part (11), we can show that $Z(G) = \langle a^p \rangle$ and $\Phi(G) = \langle a^p, b \rangle$. Also, G is nilpotent of class 3 with $Z_2(G) = \langle a^p, b \rangle$. Moreover, $M := \langle a^p, b, c \rangle \cong E_p$ is a maximal subgroup of G of exponent p. Similar to part (11), we can show that

$$(a^{pi}b^jc^k)^{a^t} = a^{p\left(i-jt+kt\binom{t}{2}+t\binom{k+1}{2}\right)}b^{j-kt}c^k$$

for all $0 \leq i, j, k < p$ and $t \geq 0$. Thus

$$\begin{aligned} &(a^t(a^{ip}b^jc^k))^p \\ &\quad = a^{pt}\left(a^{p\left(i-jt(p-1)+kt(p-1)\binom{t(p-1)}{2}+t(p-1)\binom{k+1}{2}\right)}b^{j-kt(p-1)}c^k\right)\cdots(a^{ip}b^jc^k) \\ &\quad = a^{pt+p^2i-p\binom{p}{2}jt+\frac{1}{2}t^4\binom{p}{2}^2+\frac{1}{2}t^2\binom{p}{2}-t^2\binom{p}{3}+pt\binom{p}{2}\binom{k+1}{2}} \\ &\quad = a^{pt}, \end{aligned}$$

which implies that

$$o(a^ib^jc^k) = \begin{cases} p^2, & p \nmid i, \\ p, & p \mid i \text{ and } (i,j,k) \neq (0,0,0), \\ 1, & i = j = k = 0. \end{cases}$$

Hence, as in part (11), G has $(p^3-1)/\varphi(p) = p^2+p+1$ cyclic subgroups of order p and $(p^4-p^3)/\varphi(p^2) = p^2$ cyclic subgroups of order p^2. Let H be a non-cyclic subgroup of G of order p^2. Since M is non-abelian and contains all elements of order p, it follows that $H \leq M$. Also, as H is abelian we should have $a^p \in H$ otherwise $H \cap \langle a^p \rangle = 1$ and consequently $M = H\langle a^p \rangle \cong H \times \langle a^p \rangle$ is abelian, which is a contradiction. Clearly, H generates with a^p and some element g of $M \setminus \langle a^p \rangle$ of order p. As $M \setminus \langle a^p \rangle$ contains $p^3 - p$ elements and $H \setminus \langle a^p \rangle$ contains $p^2 - p$ elements, G has $(p^3-p)/(p^2-p) = p+1$ non-cyclic subgroup H of order p^2. Now, let H be a subgroup of G of order p^3. Clearly, M is the unique subgroup of G of order p^3 and exponent p. Hence, we can assume that H has exponent p^2. Note that $b \in H$ for H is a maximal subgroup of G. The H contains an element of the form ac^k for some $0 \leq k < p$ and this element determines H uniquely. So, G has p subgroups of order p^3 and exponent p. However, as $[a,b] = [b,c] = a^p \in Z(G)$ we obtain $[ac^{-1}, b] = 1$ and so $\langle b, ac^{-1} \rangle \cong C_{p^2} \times C_p$ is abelian. Since G cannot have two abelian maximal subgroups, by the same reason as in part (11), G has $p-1$ non-abelian subgroups of order p^3 and exponent p^2. Therefore, the number of fuzzy subgroups of G is equal to

$$\begin{aligned} F(G) &= 1 + (p-1)F(E_{p^2}) + F(E_p) + F(C_{p^2} \times C_p) \\ &= (p+1)F(C_p \times C_p) + p^2 F(C_p) + (p^2+p+1)F(C_p) + 1 \\ &= 4(5p^2+6p+4). \end{aligned}$$

(13) Let $G = G_{13}(p)$. For $p = 3$ we obtain $F(G) = 268$ by GAP. So, we assume that $p > 3$. Then $G = \langle a, b, c \mid a^{p^2} = b^p = c^p = 1, a^b = a^{p+1}, a^c = ab, b^c = a^{p\lambda} b \rangle$ for some quadratic non-residue λ modulo p. The group G is similar to $G_{12}(p)$ with the only difference that $[b,c] = a^{p\lambda}$ in G but $[b,c] = a^p$ in $G_{12}(p)$. All computations and subgroups are similar, which implies that $F(G) = 4(5p^2+6p+4)$ in this case too.

(14) Let $G = G_{14}(p) = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = [a,b] = [a,c] = [a,d] = [b,c] = [b,d] = 1, c^d = ac \rangle$. Clearly, $Z(G) = \langle a, b \rangle$ and $\Phi(G) = \langle a \rangle$. Moreover, $G = \langle c, d \rangle \times \langle b \rangle \cong E_p \times C_p$. Then G is a group of exponent p and so G has $(p^4-1)/\varphi(p) = p^3+p^2+p+1$ cyclic subgroups of order p. If H is a subgroup of G of order p^2, then $H \cap Z(G) \neq 1$ otherwise $H \cap Z(G) = 1$ and $G = HZ(G) \cong H \times Z(G)$ is abelian, which is a contradiction. Suppose $H \neq Z(G)$. Then $H \cap Z(G) \cong C_p$. Also,

H generates with $H \cap Z(G)$ and some non-central cyclic subgroup of G. As $H \cap Z(G)$ can be any of the $p+1$ cyclic subgroups of $Z(G)$ and H contains p non-central cyclic subgroups of G, G has

$$\frac{(p^3+p^2+p+1)-(p+1)}{p} \times (p+1) = p(p+1)^2$$

non-central subgroups of order p^2. Thus G has $p(p+1)^2+1$ non-cyclic subgroups of order p^2. Now, let H be a subgroup of G of order p^3. Then $a \in H$ as H is a maximal subgroup of G and $a \in \Phi(G)$. If H is abelian, then H contains $Z(G)$ and so $H = \langle Z(G), g\rangle$ for any $g \in H \setminus Z(G)$. Now, since all nontrivial elements of G have order p and H contains p^3-p^2 non-central elements, G has $(p^4-p^2)/(p^3-p^2) = p+1$ abelian maximal subgroups. Finally, assume that H is non-abelian. Then H generates by two non-commuting elements. On the other hand, every two non-commuting elements of G generates a non-abelian subgroup of order p^3 as G is nilpotent of class 2. Now, since G has $(p^4-p^2)(p^4-p^3)$ non-commuting pairs and any non-abelian group of order p^3 has $(p^3-p)(p^3-p^2)$ non-commuting pairs, G has

$$\frac{(p^4-p^2)(p^4-p^3)}{(p^3-p)(p^3-p^2)} = p^2$$

non-abelian subgroup of order p^3. Therefore, the number of fuzzy subgroups of G is equal to

$$\begin{aligned} F(G) &= 1 + p^2F(E_p) + (p+1)F(C_p \times C_p \times C_p) \\ &\quad + (p(p+1)^2+1)F(C_p \times C_p) + (p^3+p^2+p+1)F(C_p) + 1 \\ &= 4(2p^4+7p^3+9p^2+6p+4). \end{aligned}$$

(15) Let $G = G_{15}(p)$. If $p = 3$, then $F(G) = 412$ by GAP. Hence, we assume that $p > 3$ and $G = \langle a, b, c, d \mid a^p = b^p = c^p = d^p = [a, b] = [a, c] = [a, d] = [b, c] = 1, b^d = ab, c^d = bc\rangle$. Clearly, $\Phi(G) = \langle a, b\rangle$ for $a = [b, d]$ and $b = [c, d]$ and $|\Phi(G)| \leq p^2$. Also, we can see that $G = \langle a, b, c\rangle \rtimes \langle d\rangle$, which implies that every element of G can be written as $a^ib^jc^kd^l$ for some $0 \leq i, j, k, l < p$. Clearly, $M := \langle a, b, c\rangle \cong C_p \times C_p \times C_p$ and so we must have $Z(G) \subseteq M$ otherwise $G = MZ(G)$ is abelian, which

is a contradiction. If $z \in Z(G)$, then $z = a^i b^j c^k$ for some $0 \leq i, j, k < p$. Now, we have

$$a^i b^j c^k = (a^i b^j c^k)^d = a^i (ab)^j (bc)^k = a^{i+j} b^{j+k} c^k,$$

from which it follows that $j = k = 0$. Hence $Z(G) = \langle a \rangle \cong C_p$. As a result, M is the only abelian maximal subgroup of G. Using induction on t, we can show that

$$(a^i b^j c^k)^{d^t} = a^{i+tj+\binom{t}{2}k} b^{j+tk} c^k$$

for all $t \geq 0$. Thus,

$$\begin{aligned}
\left(d^l (a^i b^j c^k)\right)^p &= d^{pl} (a^i b^j c^k)^{d^{l(p-1)}} \cdots (a^i b^j c^k) \\
&= d^{pl} \left(a^{i+(p-1)lj+\binom{l(p-1)}{2}k} b^{j+(p-1)lk} c^k\right) \cdots (a^i b^j c^k) \\
&= d^{pl} a^{pi+\binom{p}{2}lj+\frac{1}{2}lk\binom{p}{2}\left(\frac{2p-1}{3}l-1\right)} b^{pj+\binom{p}{2}lk} c^{pk} \\
&= 1
\end{aligned}$$

for all $0 \leq i, j, k, l < p$, which implies that G has exponent p. Then G has $(p^4-1)/\varphi(p) = p^3+p^2+p+1$ cyclic subgroups of order p. Let H be subgroup of G of order p^2. Since M is elementary abelian, it contains $(p^3-1)/(p-1) = p^2+p+1$ subgroups of order 2. Hence, we can assume that $H \not\subseteq M$. Then $|H \cap M| = p$. Since M is an abelian maximal subgroup of G and $|Z(G)| = p$, it follows that $C_G(g) = \langle Z(G), g \rangle$ has order p^2 for all $g \in G \setminus M$. Now, since H is abelian and contains an element of $G \setminus M$, we must $H \cap M = Z(G)$ and have $H = C_G(g)$ for all $g \in H \setminus Z(G)$. Thus G contains $(p^4 - p^3)/(p^2 - p) = p^2$ subgroups of order p^2 not containing in M. Hence G contains $2p^2 + p + 1$ subgroups of order p^2. Clearly, M is the unique abelian subgroup of G of order p^3. Now, let H be a non-abelian subgroup of G of order p^3. Since H contains $\Phi(G)$, $H/\Phi(G)$ is a cyclic subgroup of $G/\Phi(G) = \langle c\Phi(G), d\Phi(G) \rangle$, which implies that $H = \langle a, b, c^i d \rangle$ for some $0 \leq i < p$. Note that $H \neq M = \langle a, b, c \rangle$. Hence G has p non-abelian subgroups of order p^3. Therefore, the number of fuzzy subgroups of G is equal to

$$\begin{aligned}
F(G) &= 1 + pF(E_p) + F(C_p \times C_p \times C_p) \\
&\quad + (2p^2 + p + 1)F(C_p \times C_p) + (p^2 + p^2 + p + 1)F(C_p) + 1 \\
&= 4(3p^3 + 7p^2 + 6p + 4).
\end{aligned}$$

The proof is complete. $\square$

Theorem 2.7 has the following immediate corollary.

Corollary 2.8.

(1) The number of fuzzy subgroups of the p-groups $G_2(p)$ and $G_6(p)$ of order p^4 are similar and equal to $8(3p+2)$.

(2) The number of fuzzy subgroups of the p-groups $G_3(p)$ and $G_8(p)$ of order p^4 are similar and equal to $4(3p^2+6p+4)$.

(3) The number of fuzzy subgroups of the p-groups $G_{10}(p)$ and $G_{11}(p)$ of order p^4 $(p>3)$ are similar and equal to $4(p^3+7p^2+6p+4)$.

(4) The number of fuzzy subgroups of the p-groups $G_{12}(p)$ and $G_{13}(p)$ of order p^4 $(p>3)$ are similar and equal to $4(5p^2+6p+4)$.

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