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Original Research Paper

The Admissibility of the *P*-value for the Testing of Parameters in the Pareto Distribution

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Abstract. In this paper the problem of hypothesis testing is considered as an estimation problem within a decision-theoretic framework for estimating the accuracy of the test. The usual p-value is an admissible estimator for the one-sided testing of the scale parameter under the squared error loss function in the Pareto distribution. In the presence of nuisance parameter for model, the generalized p-value is inadmissible. Even though the usual p-value and the generalized p-value are inadmissible estimators for the one-sided testing of the shape parameter, it is difficult to exhibit a better estimator than the usual p-value. For the two-sided testing, although the usual p-value is generally inadmissible, it is remained as an estimator for the two-sided testing of the shape parameter.

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1 Introduction

The hypothesis testing usually involves data based decision making between two or more statistical hypotheses. In the significance test, the p-value is very common used as a measure of evidence against the null hypothesis. The p-value quantifies the consistency in the data with a statistical hypothesis. The investigation of the p-value has been noticed in many researches during the last decads. Although in these researches there are many Bayesian criticisms leveled at the p-value (see [3, 4, 5, 14]), some good properties of the p-value have been proved by other authors (see also [7, 11, 17]). These criticisms are all based on the fact that, the p-value may be much smaller than Bayesian criterion (the posterior probability of H_0) in the two-sided testing problem. In the one-sided testing problem, however, these criticisms do not appear and the p-value is a limit of Bayes rules.

A different view of hypothesis testing which is examined in this paper, the hypothesis testing is as a decision-theoretic problem. Consider the hypothesis $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$, based on observing $\mathbf{X} = \mathbf{x}$ with density function $f(\mathbf{x}|\theta)$, where Θ_0 and Θ_1 are two disjoint subsets of the natural parameter space Θ and $\Theta = \Theta_0 \cup \Theta_1$. The viability of the set specified by H_0 is considered with estimating the indicator function $I_{\Theta_0}(\theta)$ by an estimator, say $\delta(\mathbf{x})$. To measure the accuracy of the test, the performance of estimator $\delta(\mathbf{x})$ in terms of admissibility is evaluated with a loss function. Among the loss functions, the squared error loss function

$$L(\theta, \delta(\mathbf{x})) = (I_{\Theta_0}(\theta) - \delta(\mathbf{x}))^2, \tag{1}$$

is more favorable because it is common and proper, i.e, a Bayesian's best strategy is to tell the truth (see [15]). The estimator $\delta(\mathbf{x})$ has the interpretation that its large values confirm H_0 and small values confirm H_1, similar to a p-value and a posterior probability of H_0. Therefore, it can be applied by researcher in a same way. An estimator $\delta(\mathbf{x})$ is admissible for estimating $I_{\Theta_0}(\theta)$ if there does not exist an estimator $\delta'(\mathbf{x})$ which dominates $\delta(\mathbf{x})$, that is, such that,

$$\begin{aligned} &(i) \quad R(\theta, \delta(\mathbf{x})) \le R(\theta, \delta'(\mathbf{x})) \quad \text{for all } \ \theta \in \Theta, \\ &(ii) \quad R(\theta, \delta(\mathbf{x})) < R(\theta, \delta'(\mathbf{x})) \quad \text{for some } \ \theta \in \Theta, \end{aligned}$$

where $R(\theta,\delta(\mathbf{x}))=E[L(\theta,\delta(\mathbf{x}))]=\int_{\chi}(I_{\Theta_0}(\theta)-\delta(\mathbf{x}))^2 f(\mathbf{x}|\theta)\,\mathrm{d}\mathbf{x}$ is the risk function of $\delta(\mathbf{x})$ for estimating $I_{\Theta_0}(\theta)$. The Bayes estimator $\phi^{\pi}(\mathbf{x})$ associated with a (proper or improper) prior distribution $\pi(\theta)$ and a strictly convex loss function, is admissible if the Bayes risk

$$r(\pi,\phi^{\pi}(\mathbf{x}))=\int_{\Theta}R(\theta,\phi^{\pi}(\mathbf{x}))\pi(\theta)\,\mathrm{d}\theta,$$

is finite (see [16]). The investigation of the usual p-value behavior using loss functions was originated by Gutmann [10] and Schaafsma et al. [17]. Thereafter, Hwang et al. [11] considered estimating the accuracy of hypothesis testing. They showed that the usual p-value for the two-sided testing $H_0:\theta\in[\theta_0,\theta_1]$ against $H_1:\theta\notin[\theta_0,\theta_1]$ under the squared error loss function is generally inadmissible. For the one-sided testing, Chou [8] showed that the usual p-value is admissible under the squared error loss function in the Negative Binomial distribution. Wang [19] investigated for the simple hypothesis against simple alternative hypothesis testing. He demonstrated that the usual p-value derived by the Neyman-Pearson approach is inadmissible under the squared error loss function and also provided admissible estimators which dominate the usual p-value. In many partical appilications, the parameter space is restricted and the bounds are known. In this case, Woodroofe and Wang [21] and Wang [20] proposed a modified p-value based on the bounds of the parameter space. They showed that the modified p-value for the one-sided testing under the squared error loss function is admissible in the Poisson and Normal distributions.

In the present paper the admissibility of the p-value is discussed for the testing of parameters in the Pareto distribution

$$f(x|\alpha,\beta)=\frac{\alpha\beta^{\alpha}}{x^{\alpha+1}},\quad x\geq\beta,\,0<\alpha,\beta<\infty, \tag{2}$$

where α and β are the shape and scale parameters, respectively. The motivation for choosing this distribution is twofold. The Pareto distribution is a heavy-tailed distibution and belong to two important families, the Exponential family (when β is fixed) and the Nonregular family (when α is fixed). Also in many research areas such as economy, sociology, geophysical and other types of observable phenomena, the Pareto distribution is employed to fit the observed data. The paper is organized in five Sections. In Section 2, we consider the one-sided testing of the scale parameter, in two cases for the shape parameter, known and unknown. In latter case, we use the generalized p-value rather than the usual p-value. A similar approach is incorporated to the shape parameter in Section 3. Further, the one-point hypothesis against two-sided alternative hypothesis testing of the shape parameter which is more attractive and arguable than the one-sided testing is examined. Section 4, is devoted to simulation study carried out for the results of Section 3. Finally, in Section 5 the results obtained in the previous Sections are discussed as well as the conclusion. Throughout the paper we consider the squared error loss function.

2 The *P*-value as an Estimator for the Test of the Scale Parameter

Let $X_1, X_2, \ldots, X_n$ be a random sample from (2). Suppose $X_{(i)}$ and $x_{(i)}, i = 1, 2, \ldots, n$ are the ith ordinal random variable and its observed value, respectively.

In this Section, we investigate the problem of estimating the accuracy of the one-sided testing for the scale parameter, i.e, the hypothesis testing

$$H_0 : \beta \leq \beta_0 \quad \text{against} \quad H_1 : \beta > \beta_0, \tag{3}$$

where β_0 is a specified value.

2.1 The Usual *P*-value

Suppose that in the density function (2), α is known and β is an unknown parameter. For the one-sided testing (3), the statistic $T(\mathbf{X}) = X_{(1)}$ is a sufficient statistic of the parameter β which has the Pareto distribution $\text{Pa}(n\alpha, \beta)$ and $t(\mathbf{x}) = x_{(1)}$ is the observed value of it. The usual p-value is given by

$$\begin{aligned} p(t) &= P_{H_0}(T(\mathbf{X}) \geq t(\mathbf{x})) = P_{H_0}(X_{(1)} \geq t) \\ &= \int_t^{+\infty} \frac{n\alpha\beta_0^{n\alpha}}{y^{n\alpha+1}} \, \mathrm{d}y = (\frac{\beta_0}{t})^{n\alpha}, \quad \beta_0 < t. \end{aligned}$$

The theorem 1 shows that for the one-sided testing (3) with the loss function (1), the usual p-value is a generalized Bayes estimator with the finite Bayes risk and therefore an admissible estimator for $I_{(0,\beta_0]}(\beta)$.

Theorem 2.1. *For n observations of random variable X with the density function (2), the usual p-value for the one-sided testing (3) and the loss function (1) is admissible.*

Proof. Choose the non-informative prior $\pi(\beta)=\frac{1}{\beta}$. Clearly, it is an improper prior distribution and one cannot choose another prior distribution because the usual p-value is not a Bayes estimator. The posterior density is as follows

$$\pi(\beta|t)=\frac{f(t|\beta)\pi(\beta)}{\int_0^t f(t|\beta)\pi(\beta)\,\mathrm{d}\beta}=\frac{n\alpha\beta^{n\alpha-1}}{t^{n\alpha}},$$

and form the Bayes estimator of $I_{(0,\beta_0]}(\beta)$ for this problem is

$$\begin{aligned}\phi^\pi(t) &= E(I_{(0,\beta_0]}(\beta)|T=t)\\ &= P(\beta\le\beta_0|T=t)\\ &= \int_0^{\beta_0}\frac{n\alpha\beta^{n\alpha-1}}{t^{n\alpha}}\,\mathrm{d}\beta=(\frac{\beta_0}{t})^{n\alpha},\quad \beta_0<t,\end{aligned}$$

which is $p(t)$. Then the usual p-value is a generalized Bayes estimator and the theorem is proved if the Bayes risk is finite. Therefore, the Bayes risk of the usual p-value is given by

$$\begin{aligned}r(\pi,p(t)) &= E[R(\beta,p(t))]\\ &= \int_0^{\beta_0}\int_{\beta_0}^{+\infty}[1-(\frac{\beta_0}{t})^{n\alpha}]^2\frac{n\alpha\beta^{n\alpha-1}}{t^{n\alpha+1}}\,\mathrm{d}t\,\mathrm{d}\beta\\ &+ \int_{\beta_0}^{+\infty}\int_{\beta}^{+\infty}[\frac{\beta_0}{t}]^{2n\alpha}\frac{n\alpha\beta^{n\alpha-1}}{t^{n\alpha+1}}\,\mathrm{d}t\,\mathrm{d}\beta\\ &= \int_0^{\beta_0}\frac{\beta^{n\alpha-1}}{3\beta_0^{n\alpha}}\,\mathrm{d}\beta+\int_{\beta_0}^{+\infty}\frac{\beta_0^{2n\alpha}}{3\beta^{2n\alpha+1}}\,\mathrm{d}\beta\\ &= \frac{1}{3n\alpha}+\frac{1}{6n\alpha}=\frac{1}{2n\alpha},\end{aligned}$$

which is finite as long as n is finite. □

The important question is that, whether the unknown of the parameter α influences on the obtained result in this Section? We follow this question in the next Section.

2.2 The Generalized P-value

In the hypothesis testing problem, the presence of nuisance parameters in model is very common in practice. In this case, the classical p-value is typically not available. Tsui and Weerahandi [18] introduced the definition of the generalized p-value, called gp-value, but they could not give a systematic approach to obtain it. Li et al. [13] provided a general method to obtain the generalized p-value by fiducial inference. In this paper, following Li et al. [13], the generalized p-value is formulated by this fiducial method. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from the two-parameter Pareto distribution (2). For the testing (3), it is known that $\mathbf{T} = (T_1, T_2) = (X_{(1)}, \sum_{i=1}^{n} \ln(\frac{X_i}{X_{(1)}}))$ is the joint sufficient statistic for $\mathbf{\Theta} = (\alpha, \beta)$ and $E_1 = 2n\alpha \ln(\frac{X_{(1)}}{\beta})$ and $E_2 = 2\alpha \sum_{i=1}^{n} \ln(\frac{X_i}{X_{(1)}})$ are independently distributed with χ^2_2 and χ^2_{2n-2}, respectively, where χ^2_ν is the Chi-square distribution with ν degree of freedom. We have $\mathbf{T} = (T_1, T_2) = (\beta e^{\frac{E_1}{2n\alpha}}, \frac{E_2}{2\alpha})$. Given an observation $\mathbf{t} = (t_1, t_2)$ of $\mathbf{T} = (T_1, T_2)$ and $\mathbf{e} = (e_1, e_2)$ of $\mathbf{E} = (E_1, E_2)$. The equation $\mathbf{t} = (t_1, t_2) = (\beta e^{\frac{e_1}{2n\alpha}}, \frac{e_2}{2\alpha})$ has a unique solution

$$(\alpha, \beta) = (\frac{e_2}{2t_2}, t_1 e^{\frac{-t_2 e_1}{n e_2}}), \tag{4}$$

and the fiducial distribution for β is

$$F_X(\beta) = P(t_1 e^{\frac{-t_2 E_1}{n E_2}} \leq \beta).$$

Therefore, the generalized p-value, $gp(\mathbf{t})$, with using the technique of changing variable in the integration is given by

$$\begin{aligned} gp(\mathbf{t}) &= F_X(\beta_0) = P(t_1 e^{\frac{-t_2 E_1}{n E_2}} \leq \beta_0) \\ &= P(\frac{E_1}{E_2} \geq \frac{n}{t_2} \ln(\frac{t_1}{\beta_0})) \\ &= \int_{\frac{n(n-1)}{t_2} \ln(\frac{t_1}{\beta_0})}^{+\infty} \frac{1}{(1+\frac{1}{n-1}x)^n} \, \mathrm{d}x \\ &= [\frac{t_2 + n\ln(\frac{t_1}{\beta_0})}{t_2}]^{1-n}, \end{aligned}$$

where $\frac{(n-1)E_1}{E_2}$ has the F-distribution with $(2, 2n-2)$ degree freedom.

Theorem 2.2. *Let $X_1, X_2, \ldots, X_n$ be a random sample with the density function (2) with α and β both are unknown. For the problem of testing (3) under the loss function (1), $gp(\mathbf{t})$ is inadmissible.*

Proof. Taking $\pi(\alpha, \beta) = \frac{1}{\alpha\beta}$. Then the joint posterior probability of α and β is

$$\pi(\alpha, \beta | \mathbf{X} = \mathbf{x}) = \frac{\alpha^{n-1} \beta^{n\alpha - 1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i}}{\int_0^{t_1} \int_0^{+\infty} \alpha^{n-1} \beta^{n\alpha-1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i} \, \mathrm{d}\alpha \, \mathrm{d}\beta},$$

hence, the marginal posterior probability of β is

$$\pi(\beta | \mathbf{X} = \mathbf{x}) = \frac{\int_0^{+\infty} \alpha^{n-1} \beta^{n\alpha - 1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i} \, \mathrm{d}\alpha}{\int_0^{t_1} \int_0^{+\infty} \alpha^{n-1} \beta^{n\alpha-1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i} \, \mathrm{d}\alpha \, \mathrm{d}\beta}.$$

Finally, the Bayes estimator under the loss function (1) is

$$\begin{aligned} \phi^{\pi}(\mathbf{x}) &= P(\beta \leq \beta_0 | \mathbf{X} = \mathbf{x}) \\ &= \frac{\int_0^{\beta_0} \int_0^{+\infty} \alpha^{n-1} \beta^{n\alpha - 1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i} \, \mathrm{d}\alpha \, \mathrm{d}\beta}{\int_0^{t_1} \int_0^{+\infty} \alpha^{n-1} \beta^{n\alpha-1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i} \, \mathrm{d}\alpha \, \mathrm{d}\beta} \\ &= \frac{n(\sum_{i=1}^n \ln(\frac{x_i}{t_1}))^{n-1}}{\Gamma(n-1)} \int_0^{\beta_0} \int_0^{+\infty} \alpha^{n-1} \beta^{n\alpha-1} e^{-\alpha \sum_{i=1}^n \ln x_i} \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ &= [\frac{t_2}{\sum_{i=1}^n \ln(\frac{x_i}{\beta_0})}]^{n-1} = [\frac{t_2 + n\ln(\frac{t_1}{\beta_0})}{t_2}]^{1-n} \\ &= gp(\mathbf{t}). \end{aligned}$$

Then $gp(\mathbf{t})$ is a generalized Bayes estimator. The random variables $T_1 \sim Pa(n\alpha, \beta)$ and $T_2 \sim \Gamma(n-1, \alpha) \equiv Gamma(n-1, \alpha)$ are independent and the Bayes risk of $gp(t)$ is given by

$$\begin{aligned} r(\pi, gp(\mathbf{t})) &= E[R(\beta, gp(\mathbf{t}))] \\ &= \int_0^{+\infty}\int_0^{\beta_0}\int_0^{+\infty}\int_{\beta_0}^{+\infty} [1-gp(\mathbf{t})]^2 \frac{n\alpha^{n-1}\beta^{n\alpha-1}}{\Gamma(n-1)t_1^{n\alpha+1}} t_2^{n-2} e^{-\alpha t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}\beta\,\mathrm{d}\alpha \\ &+ \int_0^{+\infty}\int_{\beta_0}^{+\infty}\int_0^{+\infty}\int_{\beta}^{+\infty} [gp(\mathbf{t})]^2 \frac{n\alpha^{n-1}\beta^{n\alpha-1}}{\Gamma(n-1)t_1^{n\alpha+1}} t_2^{n-2} e^{-\alpha t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}\beta\,\mathrm{d}\alpha, \end{aligned}$$

which is infinite. To show this, the first integral is divided into three integrals. One of them is

$$\int_0^{+\infty}\int_0^{\beta_0}\int_0^{+\infty}\int_{\beta_0}^{+\infty} \frac{n\alpha^{n-1}\beta^{n\alpha-1}}{\Gamma(n-1)t_1^{n\alpha+1}} t_2^{n-2} e^{-\alpha t_2}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}\beta\,\mathrm{d}\alpha = \int_0^{+\infty} \frac{1}{n\alpha^2}\,\mathrm{d}\alpha = +\infty.$$

Then the Bayes risk is infinite and the theorem is proved. □

Remark 2.3. *Using the similar argument, it can be shown that the obtained results are valid for the dual hypothesis of (3), i.e.,* $H_0 : \beta \geq \beta_0$ *against* $H_1 : \beta < \beta_0$.

3 The *P*-value as an Estimator for the Test of the Shape Parameter

In this Section, the admissibility of the p-value as an estimator for the testing of the shape parameter is examined.

3.1 The Estimation of the Accuracy in the One-sided Test

3.1.1 The Usual *P*-value

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a population with the density function (2), where α is unknown parameter and β is known. Consider the problem of estimating the accuracy of the one-sided testing

$$H_0 : \alpha \leq \alpha_0 \quad \text{against} \quad H_1 : \alpha > \alpha_0, \tag{5}$$

where α_0 is a specified value. The statistic $T(\mathbf{X}) = \sum_{i=1}^{n} \ln(\frac{X_i}{\beta})$ is the sufficient for the parameter α which has the Gamma distribution $\Gamma(n, \alpha)$

and $t(\mathbf{x}) = \sum_{i=1}^{n} \ln(\frac{x_i}{\beta})$ is the its observed value. The usual p-value with using the technique of the integrating by parts is given by

$$\begin{aligned} p(t) &= P_{H_0}(T(\mathbf{X}) \leq t(\mathbf{x})) \\ &= \int_0^t \frac{\alpha_0^n}{\Gamma(n)} y^{n-1} e^{-\alpha_0 y}\, \mathrm{d}y \\ &= \int_0^{\alpha_0} \frac{t^n}{\Gamma(n)} \alpha^{n-1} e^{-\alpha t}\, \mathrm{d}\alpha \\ &= 1 - \sum_{j=0}^{n-1} \frac{e^{-\alpha_0 t}(\alpha_0 t)^j}{j!}. \end{aligned} \tag{6}$$

In the following theorem we prove that the usual p-value for the one-sided testing (5) and the loss function (1) is inadmissible.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_n$ be a random sample with the density function (2). The usual p-value for the problem of estimating the accuracy of the testing (5) under the loss function (1) is inadmissible.*

Proof. Choose an improper prior π on the parameter space $\alpha = (0, +\infty)$ with $\pi(\alpha) = \frac{1}{\alpha} I_{(0,+\infty)}(\alpha)$. Then the posterior density is

$$\pi(\alpha|t) = \frac{f(t|\alpha)\pi(\alpha)}{\int_0^{+\infty} f(t|\alpha)\pi(\alpha)\, \mathrm{d}\alpha} = \frac{t^n}{\Gamma(n)} \alpha^{n-1} e^{-\alpha t},$$

and the Bayes estimator of $I_{(0,\alpha_0]}(\alpha)$ is

$$\begin{aligned} \phi^\pi(t) &= E(I_{(0,\alpha_0]}(\alpha)|T = t) \\ &= P(\alpha \leq \alpha_0|T = t) \\ &= \int_0^{\alpha_0} \frac{t^n}{\Gamma(n)} \alpha^{n-1} e^{-\alpha t}\, \mathrm{d}\alpha \\ &= 1 - \sum_{j=0}^{n-1} \frac{e^{-\alpha_0 t}(\alpha_0 t)^j}{j!} = p(t). \end{aligned}$$

Therefore, the usual p-value is a generalized Bayes estimator. Using (6), the Bayes risk of the usual p-value is given by

$$\begin{aligned}
r(\pi, p(t)) &= E[R(\alpha, p(t))] \\
&= \int_0^{+\infty}\int_0^{+\infty} [I_{(0,\alpha_0]}(\alpha) - p(t)]^2 f(t|\alpha)\pi(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}t \\
&= \int_0^{+\infty}\int_0^{\alpha_0} [1-p(t)]^2 \frac{\alpha^{n-1}}{\Gamma(n)} t^{n-1} e^{-\alpha t}\,\mathrm{d}\alpha\,\mathrm{d}t \\
&+ \int_0^{+\infty}\int_{\alpha_0}^{+\infty} [p(t)]^2 \frac{\alpha^{n-1}}{\Gamma(n)} t^{n-1} e^{-\alpha t}\,\mathrm{d}\alpha\,\mathrm{d}t \\
&= \int_0^{+\infty} \frac{p(t)[1-p(t)]}{t}\,\mathrm{d}t \\
&= \int_0^{+\infty} \frac{[1-\sum_{j=0}^{n-1}\frac{e^{-\alpha_0 t}(\alpha_0 t)^j}{j!}][\sum_{j=0}^{n-1}\frac{e^{-\alpha_0 t}(\alpha_0 t)^j}{j!}]}{t}\,\mathrm{d}t \\
&= \int_0^{+\infty} [1-\cdots-\frac{e^{-\alpha_0 t}(\alpha_0 t)^{n-1}}{(n-1)!}][\frac{e^{-\alpha_0 t}}{t}+\cdots+\frac{e^{-\alpha_0 t}(\alpha_0 t)^{n-1}}{(n-1)!t}]\,\mathrm{d}t \\
&= \int_0^{+\infty} \frac{e^{-\alpha_0 t}}{t}\,\mathrm{d}t + \cdots - \int_0^{+\infty} \frac{e^{-2\alpha_0 t}(\alpha_0 t)^{2n-2}}{t[(n-1)!]^2}\,\mathrm{d}t.
\end{aligned}$$

Some of these integrals, say $\int_0^{+\infty} \frac{e^{-\alpha_0 t}}{t}\,\mathrm{d}t$, are infinite. Because

$$\int_0^{+\infty} \frac{e^{-\alpha_0 t}}{t}\,\mathrm{d}t = \int_0^{1} \frac{e^{-\alpha_0 t}}{t}\,\mathrm{d}t + \int_1^{+\infty} \frac{e^{-\alpha_0 t}}{t}\,\mathrm{d}t,$$

where the first integral $\int_0^1 \frac{e^{-\alpha_0 t}}{t}\,\mathrm{d}t = \int_0^1 \frac{1}{t}\,\mathrm{d}t + \sum_{i=1}^{+\infty}\frac{(-1)^i\alpha_0^i}{ii!}$ is infinite (see [9]). Then the risk Bayes of the usual p-value is infinite and the usual p-value is an inadmissible estimator for $I_{(0,\alpha_0]}(\alpha)$ and the theorem is proved. □

Remark 3.2. *It should be noted that the above obtained results are valid for the loss function (1) and different results may be obtained with other loss functions. In fact for the testing problem (5), suppose that d_0 and d_1 are accepting and rejecting H_0, respectively, for any loss function $L(\alpha, d_i)$, $i = 0, 1$ that satisfies*

$$\begin{aligned}
(i)\ L(\alpha, d_1) - L(\alpha, d_0) \geq 0 &\quad \text{as } \alpha \leq \alpha_0, \\
(ii)\ L(\alpha, d_1) - L(\alpha, d_0) \leq 0 &\quad \text{as } \alpha > \alpha_0,
\end{aligned} \tag{7}$$

the usual p-value is admissible for $I_{\Theta_0}(\alpha)$.
To prove this, note that $f(x|\alpha)$ has monotone likelihood ratio with $-\ln X$

and $\{x : f(x|\alpha) > 0\}$ is independent of α. Two loss functions that satisfy in (7) are the 0-1 and absolute loss functions (see [2, 16] for more detail).

Although the usual p-value has shown to be inadmissible for the testing problem (5), the following theorem shows that there is a prior distribution under which the Bayes estimator is admissible.

Theorem 3.3. *Let $X_1, X_2, \ldots, X_n$ be a random sample with the density function (2). For the one-sided testing (5), the Bayes estimator with respect to the proper prior distribution $\pi(\alpha) = e^{-\alpha} I_{(0,+\infty)}(\alpha)$ and the loss function (1) is admissible.*

Proof. For the proper prior distribution $\pi(\alpha) = e^{-\alpha} I_{(0,+\infty)}(\alpha)$ and the sufficient statistic $T(\mathbf{X}) = \sum_{i=1}^{n} \ln(\frac{X_i}{\beta})$, the posterior density and the Bayes esimator with using the technique of the integrating by parts are, respectively,

$$\begin{aligned}
\pi(\alpha|t) &= \frac{\frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha(t+1)}}{\int_0^{+\infty} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha(t+1)}\, \mathrm{d}\alpha} \\
&= \frac{(t+1)^{n+1}}{\Gamma(n+1)} \alpha^n e^{-\alpha(t+1)},
\end{aligned}$$

and

$$\begin{aligned}
\phi^{\pi}(t) &= E(I_{(0,\alpha_0]}(\alpha)|T = t) \\
&= P(\alpha \leq \alpha_0 | T = t) \\
&= \int_0^{\alpha_0} \frac{(t+1)^{n+1}}{\Gamma(n+1)} \alpha^n e^{-\alpha(t+1)}\, \mathrm{d}\alpha \\
&= 1 - \sum_{j=0}^{n} \frac{e^{-\alpha_0(t+1)} [\alpha_0(t+1)]^j}{j!}. \qquad (8)
\end{aligned}$$

The Bayes risk with using (8) and the technique of changing variable in the integration is given by

$$
\begin{aligned}
r(\pi,\phi^{\pi}(t)) &= E[R(\alpha,\phi^{\pi}(t))] \\
&= \int_0^{+\infty}\int_0^{\alpha_0}[1-\phi^{\pi}(t)]^2\frac{\alpha^n}{\Gamma(n)}t^{n-1}e^{-\alpha(t+1)}\,\mathrm{d}\alpha\,\mathrm{d}t \\
&+ \int_0^{+\infty}\int_{\alpha_0}^{+\infty}[\phi^{\pi}(t)]^2\frac{\alpha^n}{\Gamma(n)}t^{n-1}e^{-\alpha(t+1)}\,\mathrm{d}\alpha\,\mathrm{d}t \\
&= \int_0^{+\infty}\frac{t^{n-1}\Gamma(n+1)}{(t+1)^{n+1}\Gamma(n)}[1-\phi^{\pi}(t)]\phi^{\pi}(t)\,\mathrm{d}t \\
&\leq \int_0^{+\infty}\frac{t^{n-1}\Gamma(n+1)}{(t+1)^{n+1}\Gamma(n)}\,\mathrm{d}t \\
&= \int_1^{+\infty}\frac{n(t-1)^{n-1}}{t^{n+1}}\,\mathrm{d}t < \int_1^{+\infty}\frac{nt^{n-1}}{t^{n+1}}\,\mathrm{d}t \\
&= \int_1^{+\infty}\frac{n}{t^2}\,\mathrm{d}t = n < +\infty.
\end{aligned}
$$

Therefore, the Bayes estimator (8) when considered as an estimator of $I_{(0,\alpha_0]}(\alpha)$ is admissible. □

The risk functions of the usual p-value (6) and the Bayes estimator (8) are compared in figures 1 and 2 for $\alpha_0 = 1$ and $\alpha_0 = 2$. For simplicity $n = 1$ has been considered. The interesting point is that the Bayes estimator (8) is admissible, but cannot dominate the usual p-value (6). Because, if the null hypothesis is true, $R(\alpha, p(t))$ will be bigger than $R(\alpha, \phi^{\pi}(t))$, otherwise $R(\alpha, p(t))$ will be smaller than $R(\alpha, \phi^{\pi}(t))$.

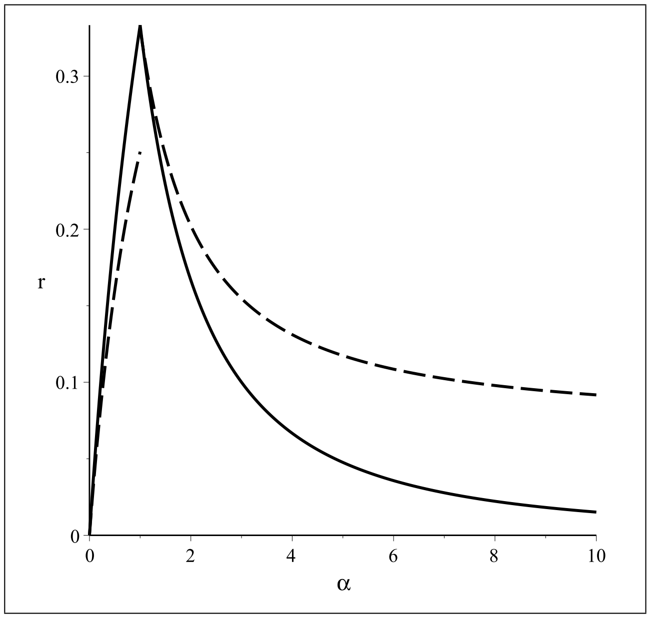

Figure 1: The risk functions of $p(t)$ (solid line) and $\phi^{\pi}(t)$ (dashed line) for $n = 1$ and $\alpha_0 = 1$.

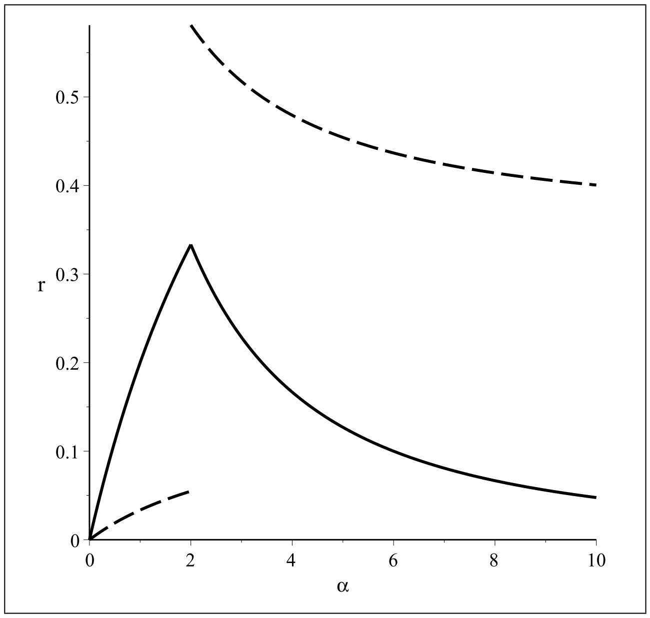

Figure 2: The risk functions of $p(t)$ (solid line) and $\phi^{\pi}(t)$ (dashed line) for $n=1$ and $\alpha_0=2$.

3.1.2 The Generalized P-value

Let $X_1, X_2, \ldots, X_n$ be a random sample from a variable with the density function (2), where α and β are both unknown parameters. For the problem of the one-sided testing (5), $gp(t)$ using (4) and the fiducial distribution for α, $F_X(\alpha)=P(\frac{E_2}{2t_2}\leq\alpha)$, is given by

$$\begin{aligned}
gp(\mathbf{t}) &= F_X(\alpha_0)\\
&= P(E_2 \leq 2\alpha_0 t_2)\\
&= \int_0^{2\alpha_0 t_2} \frac{1}{2^{n-1}\Gamma(n-1)} y^{n-2} e^{-\frac{y}{2}}\,\mathrm{d}y\\
&= \frac{t_2^{n-1}}{\Gamma(n-1)} \int_0^{\alpha_0} y^{n-2} e^{-t_2 y}\,\mathrm{d}y\\
&= 1-\sum_{j=0}^{n-2} \frac{e^{-\alpha_0 t_2}(\alpha_0 t_2)^j}{j!}.
\end{aligned}$$

Theorem 3.4. *Let $X_1, X_2, \ldots, X_n$ be a random sample with the density function (2) where α and β both are unknown. $gp(\mathbf{t})$ for the problem of estimating the accuracy of the testing (5) under the loss function (1) is inadmissible.*

Proof. If $\pi(\alpha, \beta) = \frac{1}{\alpha\beta}$, then the Bayes estimator with using the loss function (1) is

$$\begin{aligned}
\phi^{\pi}(x) &= P(\alpha \leq \alpha_0 | \mathbf{X} = \mathbf{x}) \\
&= \int_0^{\alpha_0} \pi(\alpha | \mathbf{X} = \mathbf{x})\, \mathrm{d}\alpha \\
&= \int_0^{\alpha_0} \int_0^{t_1} \pi(\alpha, \beta | \mathbf{X} = \mathbf{x})\, \mathrm{d}\beta\, \mathrm{d}\alpha \\
&= \frac{\int_0^{\alpha_0} \int_0^{t_1} \alpha^{n-1} \beta^{n\alpha - 1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i}\, \mathrm{d}\beta\, \mathrm{d}\alpha}{\int_0^{+\infty} \int_0^{t_1} \alpha^{n-1} \beta^{n\alpha - 1} e^{-(\alpha+1)\sum_{i=1}^n \ln x_i}\, \mathrm{d}\beta\, \mathrm{d}\alpha} \\
&= \frac{t_2^{n-1}}{\Gamma(n-1)} \int_0^{\alpha_0} \alpha^{n-2} e^{-\alpha t_2}\, \mathrm{d}\alpha \\
&= \frac{\alpha_0^{n-1}}{\Gamma(n-1)} \int_0^{t_2} y^{n-2} e^{-\alpha_0 y}\, \mathrm{d}y \\
&= 1 - \sum_{j=0}^{n-2} \frac{e^{-\alpha_0 t_2} (\alpha_0 t_2)^j}{j!} = gp(\mathbf{t}),
\end{aligned} \tag{9}$$

i.e, $gp(\mathbf{t})$ is a generalized Bayes estimator. Since $T_1 \sim Pa(n\alpha, \beta)$ and $T_2 \sim \Gamma(n-1, \alpha)$ are independent random variables. Using (9) the Bayes risk of $gp(\mathbf{t})$ is given by

$$\begin{aligned}
r(\pi, gp(\mathbf{t})) &= E[R(\alpha, gp(\mathbf{t}))] \\
&= \int_0^{+\infty} \int_{\beta}^{+\infty} \int_0^{\alpha_0} \int_0^{t_1} [1 - gp(\mathbf{t})]^2 \frac{n\alpha^{n-1}\beta^{n\alpha-1}}{\Gamma(n-1) t_1^{n\alpha+1}} t_2^{n-2} e^{-\alpha t_2}\, \mathrm{d}\beta\, \mathrm{d}\alpha\, \mathrm{d}t_1\, \mathrm{d}t_2 \\
&+ \int_0^{+\infty} \int_{\beta}^{+\infty} \int_{\alpha_0}^{+\infty} \int_0^{t_1} [gp(\mathbf{t})]^2 \frac{n\alpha^{n-1}\beta^{n\alpha-1}}{\Gamma(n-1) t_1^{n\alpha+1}} t_2^{n-2} e^{-\alpha t_2}\, \mathrm{d}\beta\, \mathrm{d}\alpha\, \mathrm{d}t_1\, \mathrm{d}t_2 \\
&= \int_0^{+\infty} \int_{\beta}^{+\infty} \frac{gp(\mathbf{t})[1 - gp(\mathbf{t})]}{t_1 t_2}\, \mathrm{d}t_1\, \mathrm{d}t_2 \\
&= \int_0^{+\infty} \int_{\beta}^{+\infty} \frac{e^{-\alpha_0 t_2}}{t_1 t_2}\, \mathrm{d}t_1\, \mathrm{d}t_2 + \cdots - \int_0^{+\infty} \int_{\beta}^{+\infty} \frac{e^{-2\alpha_0 t_2} (\alpha_0 t_2)^{2n-4}}{t_1 t_2 [(n-1)!]^2}\, \mathrm{d}t_1\, \mathrm{d}t_2.
\end{aligned}$$

Since some of these integrals are infinite, $gp(\mathbf{t})$ is an inadmissible estimator for $I_{(0,\alpha_0]}(\alpha)$ and the theorem is proved. □

Remark 3.5. By a similar argument, it can be shown that the obtained results for the hypothesis testing $H_0 : \alpha \geq \alpha_0$ against $H_1 : \alpha < \alpha_0$ are valid.

3.2 The Estimation of the Accuracy in the Two-sided Test

The usual p-value is most routinely used as the standard measure of uncertainly in the statistical testing in the one-sided and two-sided hypotheses. However, in the one-point hypothesis against two-sided alternative hypothesis testing, the usual p-value can be very misleading impression as the validity of H_0 (see [4]). To investigate the admissibility of the usual p-value in this case, we examine

$$H_0 : \alpha = \alpha_0 \quad \text{against} \quad H_1 : \alpha \neq \alpha_0, \tag{10}$$

in the Pareto distribution when β is considered to have a known value. For this test, the statistic $T(\mathbf{X}) = \sum_{i=1}^{n} \ln(\frac{X_i}{\beta})$ is a appropriate statistic which has the Gamma distribution $\Gamma(n, \alpha)$. This statistic is a sufficient statistic for α and its extreme values are against H_0. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is an observed vector of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, the likelihood ratio test with size p, using equal tails, rejects H_0 when $T(\mathbf{x}) < \frac{1}{2\alpha_0}\chi^2_{(2n)}(1 - \frac{p}{2})$ and $T(\mathbf{x}) > \frac{1}{2\alpha_0}\chi^2_{(2n)}(\frac{p}{2})$ where $\chi^2_{(\nu)}(q)$ denotes the qth fractile of a Chi-square distribution with ν degree of freedom. Under H_0,

$$2\alpha_0 T(\mathbf{X}) = 2\alpha_0 \sum_{i=1}^{n} \ln(\frac{X_i}{\beta}) \sim \chi^2_{2n}.$$

Therefore, the distribution of $2\alpha_0 T(\mathbf{X})$ under H_0 is not symmetric and the usual p-value is given by

$$\begin{aligned} p(t) &= 2\min\{P_{\alpha=\alpha_0}(T(\mathbf{X}) \leq t(\mathbf{x})), P_{\alpha=\alpha_0}(T(\mathbf{X}) \geq t(\mathbf{x}))\} \\ &= \begin{cases} 2P_{H_0}(\chi^2_{2n} \geq 2\alpha_0 t(\mathbf{x})) & t(\mathbf{x}) > \frac{n}{\alpha_0}, \\ 2P_{H_0}(\chi^2_{2n} \leq 2\alpha_0 t(\mathbf{x})) & t(\mathbf{x}) \leq \frac{n}{\alpha_0}, \end{cases} \end{aligned} \tag{11}$$

where $t(\mathbf{x})$ is the observed value of the test statistic. The Bayes estimators for this problem is

$$\phi^{\pi}(t) = P(H_0|T = t) = \frac{\pi_0 f(t|\alpha_0)}{\pi_0 f(t|\alpha_0) + \pi_1 \int_{\alpha \neq \alpha_0} f(t|\alpha)\pi(\alpha)\, \mathrm{d}\alpha}, \tag{12}$$

where $f(t|\alpha)$ has the density function Gamma with parameters n and α. Amounts π_0 and $\pi_1 = 1 - \pi_0$ are the primary probability of H_0 and H_1, respectively. π_0 is usually assumed to be $\frac{1}{2}$ in the calculation. This choice of π_0 is necessary since it reflects the assumption of equal believable of H_0 and H_1.

Theorem 3.6. *Let $X_1, X_2, \ldots, X_n$ be a random sample from the density function (2). For the testing problem (10), the usual p-value under the loss function (1) cannot be dominated by the Bayes estimators with $\pi_0 = \frac{1}{2}$.*

Proof. See Appendix. □

Thus, in the two-sided testing problem, the usual p-value remains as a frequently used estimator of the accuracy, although is inadmissible and is not within in the range of Bayesian's rule.

4 Simulation

In this Section, we conduct a simulation study to compare the usual p-value and the Bayes estimator for the testing of the shape parameter in the previous Section. This simulation is based on real-data, the Norwegian fire claims data set which is taken from Beirlant et al. [1]. It represents the total damage done by 142 fires in Norway for the year 1975, for claims above 500000 Norwegian krones. The losses are recorded in 1000's of Norwegian krones. Brazauskas and Serfling [6] fitted the Pareto distribution with $\alpha = 1.218$ and $\beta = 500000$ to the data.

First, we compare the usual p-value (6) and the Bayes estimator (8) as measures of evidence against the null hypothesis in the one-sided testing (5). Table 1 shows the ratios of the usual p-value (6) and the Bayes estimator (8) less than 0.05 and 0.01 based on 1000 samples $n = 10$ corresponding with values of α_0 ranging from 1.206 to 1.230 and $\beta = 500000$. The results of simulation in Table 1 show that when α_0 is larger than 1.218, which means $H_0 : \alpha \leq \alpha_0$ is true, the number of the Bayes estimator less than 0.05 and 0.01 are less than that of the usual p-value. When α_0 is less than 1.218, the number of the usual p-value less than 0.05 and 0.01 are more than that of the Bayes estimator. They reveal

Table 1: The ratios of the usual p-value and the Bayes estimator (the posterior probability of H_0) less than 0.05 and 0.01 for the testing (5) based on 1000 replicates.

α_0	H_0 true or false	$p(t) < 0.05$	$\phi^\pi(t) < 0.05$	$p(t) < 0.01$	$\phi^\pi(t) < 0.01$
1.206	false	0.067	0.044	0.020	0.008
1.209	false	0.044	0.023	0.003	0.002
1.212	false	0.055	0.038	0.018	0.005
1.215	false	0.051	0.030	0.017	0.009
1.217	false	0.059	0.028	0.012	0.004
1.219	true	0.052	0.031	0.010	0.004
1.221	true	0.038	0.022	0.008	0.005
1.224	true	0.051	0.034	0.010	0.001
1.227	true	0.044	0.026	0.010	0.004
1.230	true	0.053	0.028	0.008	0.002

that the usual p-value (6) is not dominated by the Bayes estimator (8) which is an admissible estimator for $I_{(0,\alpha_0]}(\alpha)$.

Table 2: The ratios of the usual p-value and the Bayes estimator (the posterior probability of H_0) less than 0.05 and 0.01 for the testing (10) based on 1000 replicates.

α_0	H_0 true or false	$p(t) < 0.05$	$\phi^\pi(t) < 0.05$	$p(t) < 0.01$	$\phi^\pi(t) < 0.01$
1.206	false	0.046	0.032	0.009	0.007
1.210	false	0.064	0.036	0.017	0.011
1.213	false	0.042	0.017	0.008	0.003
1.216	false	0.057	0.034	0.012	0.007
1.218	true	0.059	0.028	0.013	0.009
1.220	false	0.044	0.026	0.012	0.006
1.223	false	0.054	0.030	0.013	0.006
1.226	false	0.047	0.021	0.010	0.005
1.230	false	0.056	0.028	0.005	0.002

Second, we consider the usual p-value (11) and the Bayes estimator (12) with $\pi(\alpha) = 1$ and $\pi_0 = \frac{1}{2}$ for the two-sided testing (10). Table 2 shows the ratios of the usual p-value (11) and the Bayes estimator (12) less than 0.05 and 0.01 based on 1000 samples $n = 10$ corresponding with values of α_0 ranging from 1.206 to 1.230 and $\beta = 500000$. As can be seen from Table 2, when α_0 is equal to 1.218, which means $H_0 : \alpha = \alpha_0$ is true, the number of the Bayes estimator less than 0.05 and 0.01 are less than the number of the usual p-value less than 0.05 and 0.01. When

α_0 is not equal to 1.218, in most situations, the number of the usual p-value less than 0.05 and 0.01 are more than the number of the Bayes estimator less than 0.05 and 0.01. Therefore, the usual p-value (11) is not dominated by the Bayes estimator (12), although the usual p-value (11) is inadmissible.

The simalution results in Tables 1 and 2 confirm the obtained results in the previous Section. Thus, the usual p-value can remain as an estimator of the accuracy in the one-sided and two-sided testing of the shape parameter, although it is an inadmissible estimator.

5 Conclusions

In this paper the problem of hypothesis testing is considered as an estimation in the decision-theoretic framework. The accuracy of the test for the parameters of the Pareto distribution is examined. The usual p-value for the one-sided testing of the scale parameter is admissible when the shape parameter is fixed. When the shape parameter considered as nuisance parameter, the generalized p-value is inadmissible.

The usual p-value and the generalized p-value are inadmissible for the one-sided testing of the shape parameter. It should be noted that finding an estimator which dominates the p-value for these cases is difficult. So, it can be a new task for further researches. In the case of one-point hypoyhesis against two-sided alternative hypothesis testing of the shape parameter, the usual p-value is inadmissible but it is not so bad to be dominated by the Bayes estimators. The loss function used in this paper is the squared loss function which is a proper loss function. Therefore, the results may be contradicted when the loss function is changed. For example, in the testing problem $H_0 : \alpha \leq \alpha_0$ against $H_1 : \alpha > \alpha_0$, or vice versa, the usual p-value is usually admissible for the loss functions under which the accepting $H_0(H_1)$ is not larger than the rejecting $H_0(H_1)$ for $\alpha \in \Theta_0(\Theta_1)$. Two important common used loss functions are the 0-1 loss and the absolute loss (see [2, 12] for more detail).

One restriction which is proved and is not mentioned in this paper, for saving in writing, is that the admissibility of the usual p-value is depend on whole parameter space for the testing of shape parameter. When the parameter space is restricted the results are changed. One way to get

rid of this problem is the modification of the usual p-value, similar to Wang [20] and Woodroofe and Wang [21].
The generalization of these results to other distributions or to the family of distributions, require a further study.

Appendix

The Proof of Theorem 3.6. Without loss of generality, assume that $\alpha_0 = 1$. For the proof, we show that as $\alpha \to +\infty$, $R(\alpha, p(t)) < R(\alpha, \phi^{\pi}(t))$. Consider two cases:
case 1: $t \le \frac{n}{\alpha_0}$. In this case for $0 < a < t \le \frac{n}{\alpha_0}$, is $\phi^{\pi}(t) > p(t)$. This follows from the fact that

$$\phi^{\pi}(t) \ge \frac{f(t|\alpha_0)}{f(t|\alpha_0) + f(t|\hat{\alpha})} > p(t), \tag{13}$$

where $\hat{\alpha} = \frac{n}{t(\mathbf{x})}$ is the maximum likelihood estimator of α. For $\alpha \ne 1$, the difference in risks two estimators is equal with

$$R(\alpha, \phi^{\pi}(t)) - R(\alpha, p(t)) = E_{\alpha}(\phi^{\pi}(T)^2 - p(T)^2).$$

From (13), by continuity, there exists an $\varepsilon > 0$ such that for all $a < t < a + \varepsilon$, is $\phi^{\pi}(T)^2 - p(T)^2 > \varepsilon$. Hence

$$\begin{aligned} E_{\alpha}(\phi^{\pi}(T)^2 - p(T)^2) &= \int_a^{a+\varepsilon} (\phi^{\pi}(t)^2 - p(t)^2)\, \mathrm{d}p \\ &+ \int_{a+\varepsilon}^{\frac{n}{\alpha_0}} (\phi^{\pi}(t)^2 - p(t)^2)\, \mathrm{d}p \\ &\ge \varepsilon P_{\alpha}(a < T < a+\varepsilon) - P_{\alpha}(a+\varepsilon < T < \frac{n}{\alpha_0}). \end{aligned}$$

This lower bound for large α is positive. Since with using L'Hopital's rule as $\alpha \to +\infty$,

$$\begin{aligned} \frac{\varepsilon P_{\alpha}(a < T < a+\varepsilon)}{P_{\alpha}(a+\varepsilon < T < \frac{n}{\alpha_0})} &= \frac{\varepsilon \int_a^{a+\varepsilon} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t}\, \mathrm{d}t}{\int_{a+\varepsilon}^{\frac{n}{\alpha_0}} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t}\, \mathrm{d}t} \\ &= \frac{\varepsilon \int_{\alpha a}^{\alpha(a+\varepsilon)} t^{n-1} e^{-t}\, \mathrm{d}t}{\int_{\alpha(a+\varepsilon)}^{\alpha(\frac{n}{\alpha_0})} t^{n-1} e^{-t}\, \mathrm{d}t} \to +\infty. \end{aligned}$$

case 2: $t > \frac{n}{\alpha_0}$. In this case for $t > a > \frac{n}{\alpha_0}$, similar to case 1, we have $\phi^\pi(t) > p(t)$. For $\alpha \neq 1$, the difference in risks two estimators is equal with $R(\alpha, \phi^\pi(t)) - R(\alpha, p(t)) = E_\alpha(\phi^\pi(T)^2 - p(T)^2)$. Then, by continuity, there exists an $\varepsilon > 0$ such that for all $a < t < a + \varepsilon$, is $\phi^\pi(T)^2 - p(T)^2 > \varepsilon$. Hence

$$E_\alpha(\phi^\pi(T)^2 - p(T)^2) \quad \geq \quad \varepsilon P_\alpha(a < T < a + \varepsilon) - P_\alpha(a + \varepsilon < T).$$

This lower bound is also positive. Since as $\alpha \to +\infty$,

$$\begin{aligned} \frac{\varepsilon P_\alpha(a < T < a + \varepsilon)}{P_\alpha(a + \varepsilon < T)} &= \frac{\varepsilon \int_a^{a+\varepsilon} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t}\, \mathrm{d}t}{\int_{a+\varepsilon}^{+\infty} \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t}\, \mathrm{d}t} \\ &= \frac{\varepsilon \int_{\alpha a}^{\alpha(a+\varepsilon)} t^{n-1} e^{-t}\, \mathrm{d}t}{\int_{\alpha(a+\varepsilon)}^{+\infty} t^{n-1} e^{-t}\, \mathrm{d}t} \to +\infty, \end{aligned}$$

by L'Hopital's rule. From cases 1 and 2, we deduce that the difference in risks for large α is strictly positive and $\phi^\pi(t)$ cannot dominate $p(t)$. □

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