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On the Multiplication Operators and Multipliers on Weighted Spaces of Holomorphic Functions on the Upperhalfplane

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Abstract. In this paper, we find necessary and sufficient conditions such that under these conditions a self map multiplication operator between weighted spaces of holomorphic functions is Fredholm or closed range operator. We also obtain a characterization of multipliers for certain type of weights between weighted spaces of holomorphic functions on the upper halfplane. Our results will remain valid for any simply connected domain in the complex plane instead of the upper halfplane.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\}$ and $\mathbb{G} = \{\omega \in \mathbb{C} : Im\, \omega > 0\}$ be the open unit disc and upper halfplane respectively. Also, let O be an open subset of

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$\mathbb{C}$. By a weight we mean a continuous function $\upsilon : O \longrightarrow (0, \infty)$. For a holomorphic function $f : O \longrightarrow \mathbb{C}$, we define the weighted sup-norm

$$\|f\|_\upsilon = \sup_{z \in O} \mid f(z) \mid \upsilon(z)$$

and the weighted spaces

$$H_\upsilon(O) = \{f : O \longrightarrow \mathbb{C} : f \text{ is holomorphic}, \|f\|_\upsilon < \infty\},$$

$$H_{\upsilon_0}(O) = \{f \in H_\upsilon(O) :\mid f(z) \mid \upsilon(z) \text{ vanishes at infinity}\}.$$

Throughout this paper, we deal with the cases $O = \mathbb{D}$ or $O = \mathbb{G}$. In the case $H_{\upsilon_0}(\mathbb{G})$, $\mid f(z) \mid \upsilon(z)$ vanishes at infinity if for any $\epsilon > 0$ there is a compact subset K of $\mathbb{G}$ such that $\mid f(z) \mid \upsilon(z) < \epsilon$ for all $z \in \mathbb{G} \setminus K$. In $H_{\upsilon_0}(\mathbb{D})$, $\mid f(z) \mid \upsilon(z)$ vanishes at infinity is equivalent to $\lim_{|z| \to 1} \mid f(z) \mid \upsilon(z) = 0$ (uniform limit). It is wellknown that $H_\upsilon(O)$ and $H_{\upsilon_0}(O)$ are Banach spaces.

A weight $\upsilon : \mathbb{G} \longrightarrow (0, \infty)$ is called a standard weight on $\mathbb{G}$ if $\lim_{r \to 0} \upsilon(ir) = 0$ and $\upsilon(\omega_1) \leq \upsilon(\omega_2)$ whenever $Im\ \omega_1 \leq Im\ \omega_2$. We say a standard weight υ on $\mathbb{G}$ satisfies condition $(*)$ if

$$\sup_{k \in \mathbb{Z}} \frac{\upsilon(2^{k+1} i)}{\upsilon(2^k i)} < \infty.$$

A standard weight υ satisfies $(**)$ if

$$\inf_{n \in \mathbb{N}} \sup_{k \in \mathbb{Z}} \frac{\upsilon(2^k i)}{\upsilon(2^{k+n} i)} < 1.$$

Condition $(*)$ is equivalent to $\frac{\upsilon(ti)}{\upsilon(si)} \leq C(\frac{t}{s})^\beta$ whenever $0 < s \leq t$ for some constants $C > 0$ and $\beta > 0$ and condition $(**)$ is equivalent to $\frac{\upsilon(ti)}{\upsilon(si)} \geq d(\frac{t}{s})^\gamma$ whenever $0 < s \leq t$ for some constants $d, \gamma > 0$. (see Lemma 1.6 of [5]).

A weight $\upsilon : \mathbb{D} \longrightarrow (0, \infty)$ is called a standard weight on $\mathbb{D}$ if υ is radial (i.e $\upsilon(z) = \upsilon(\mid z \mid)$) and $\lim_{|z| \to 1^-} \upsilon(z) = 0$. We say a standard weight υ on $\mathbb{D}$ satisfies condition $(*)'$ if

$$\inf_{n \in \mathbb{N}} \frac{\upsilon(1 - 2^{-n-1})}{\upsilon(1 - 2^{-n})} > 0.$$

A standard weight υ on $\mathbb{D}$ satisfies condition $(**)'$ if

$$\inf_{k\in\mathbb{N}}\limsup_{n\to\infty}\frac{\upsilon(1-2^{-n-k})}{\upsilon(1-2^{-n})}<1.$$

Remark 1.1. *Note that the space $H_\upsilon(\mathbb{G})$ (and therefore $H_{\upsilon_0}(\mathbb{G})$) will make sense whenever $H_\upsilon(\mathbb{G})\neq\{0\}$. There is a result of Stanev [14] which states that $H_\upsilon(\mathbb{G})\neq\{0\}$ if and only if there exist $a,b>0$ such that $\upsilon(it)\leq ae^{bt}$, $t>0$. Therefore, throughout this paper we always assume our weights satisfy Stanev condition. For example weights which satisfy condition $(*)$, satisfy Stanev condition.*

In this paper, we intend to obtain some results concerning to the self-map multiplication operator $M_\varphi : H_\upsilon(\mathbb{G})\longrightarrow H_\upsilon(\mathbb{G})$ defined by $M_\varphi(f)=f\varphi$ for each $f\in H_\upsilon(\mathbb{G})$ where $\varphi:\mathbb{G}\longrightarrow\mathbb{C}$ is a nonconstant holomorphic function. For this objective, we use wellknown results of [7], conformal map $\alpha:\mathbb{D}\longrightarrow\mathbb{G}$ defined by $\alpha(z)=\frac{1+z}{1-z}i$, Some Lemmas and proper arguments in order to transfer the results to the case of upper halfplane.

2 Preliminaries

In this section, we recall some definitions, notations and theorems which are necessary in the rest of this paper. For more details, we will refer the reader to the suitable references. We denote the space of all bounded holomorphic functions on $\mathbb{G}$ by $H^\infty(\mathbb{G})$. A sequence (ω_n) in $\mathbb{G}$ is called an interpolating sequence if for any bounded sequence (β_n) there is an $f\in H^\infty(\mathbb{G})$ for which $f(\omega_n)=\beta_n$ for all $n\in\mathbb{N}$. Equivalently, (ω_n) is an interpolating sequence if the bounded linear operator $T:H^\infty(\mathbb{G})\longrightarrow\ell^\infty$ defined by $(Tf)(z_j)=f(z_j)$ is an onto operator.
The maximal ideal space of $H^\infty(\mathbb{G})$ which is denoted by $M(H^\infty(\mathbb{G}))$ is the collection of all nonzero homeomorphisms of $H^\infty(\mathbb{G})\longrightarrow\mathbb{C}$, whenever $H^\infty(\mathbb{G})$ is endowed with weak* topology as a subset of $H^\infty(\mathbb{G})^*$.
The pseudohyperboilc distance between two points m and n in $M(H^\infty(\mathbb{G}))$ is defined by

$$\rho(m,n)=\sup\{|\hat{f}(n)|: f\in H^\infty(\mathbb{G}),\hat{f}(m)=0,\|f\|_\infty\leq 1\}$$

where $\|f\|_\infty = \sup\{\mid f(\omega) \mid : \omega \in \mathbb{G}\}$ and $\hat{f}$ is the Gelfand transform of f . Also For $\alpha, \omega \in \mathbb{G}, \rho(\omega, \alpha) = \mid \varphi_\alpha(\omega) \mid$ where $\varphi_\alpha(\omega) = \frac{\alpha-\omega}{\alpha-\overline{\omega}}$.
For the sake of simplicity, we denote the Gelfand transform of f by f itself futher on. Let F be a family of complex valued functions on a set X, a subset Γ of X is called a boundary for F if for each $f \in F$, there is an $x \in \Gamma$ such that $\mid f(x) \mid = \sup\{\mid f(y) \mid : y \in X\}$. Shilov proved that if A is a function algebra on a locally compact space X, then there exists a unique minimal(intersection of all boundaries of A) closed boundary for A. This minimal boundary is called the **Shilov boundary**.
The Gleason part of $m \in M(H^\infty(\mathbb{G}))$ is defined by $P(m) = \{n \in M(H^\infty(\mathbb{G})) : \rho(m,n) < 1\}$. The set of trivial Gleason parts $\{m \in M(H^\infty(\mathbb{G})) : P(m) = \{m\}\}$ is a closed subset of $M(H^\infty(\mathbb{G}))$ that contains properly the shilov boundary $\Gamma(H^\infty(\mathbb{G}))$ of $H^\infty(\mathbb{G})$. See [10].
A function F of the following form

$$F(\omega) = e^{i\gamma} \exp(\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1+t\omega)\log g(t)}{(t-\omega)(1+t^2)} dt)$$

is an **outer function** in $H^\infty(\mathbb{G})$. Here γ is a real number, $g(t) \geq 0$ is a measurable essentially bounded function in $\mathbb{R}$ ($g \in L^\infty(\mathbb{R})$) and

$$\int_{-\infty}^{\infty} \frac{\log g(t)}{1+t^2} dt > -\infty.$$

A **compactification** of a space X is an ordered pair (K, h) where K is a compact Hausdorff space and h is an embedding of X as a dense subset of K.
For a weight υ the function

$$\tilde{\upsilon}(z) = \frac{1}{\sup\{\mid h(z) \mid : h \in H_\upsilon(O), \|h\|_\upsilon \leq 1\}}$$

is called the **associated weight**. It is wellknown that (see [5]):
- $\|f\|_\upsilon = \|f\|_{\tilde{\upsilon}}$ for each $f \in H_\upsilon(O)$.
- For any $z \in O$, there is an $h \in H_\upsilon(O)$ with $\|h\|_\upsilon \leq 1$ such that $\tilde{\upsilon}(z) = \frac{1}{|h(z)|}$.
- $\tilde{\upsilon}(z) \leq \upsilon(z)$ for all $z \in O$.

A weight υ is called an **essential weight** if there exists a constant $C > 0$ such that $\tilde{\upsilon}(z) \leq C\upsilon(z)$ for all $z \in O$.

Let υ be a standard weight on $\mathbb{G}$. In [4], it has been shown that $\tilde{\upsilon}(\omega) = \tilde{\upsilon}(iIm\ \omega)$ and $\tilde{\upsilon}(it) \geq \tilde{\upsilon}(is)$ whenever $t \geq s > 0$.
We conclude this section by recalling the following definitions.
A Bounded linear operator $T : X \longrightarrow Y(X$ and Y are normed spaces) is called a Fredholm operator if it has closed range and dim ker(T) and dim $\frac{Y}{Im\ T} < \infty$. The Spectrum of a bounded linear operator $T : X \longrightarrow X$ which is denoted by $\sigma(T)$ is defined as follows.

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

The essential spectrum of T which is denoted by $\sigma_e(T)$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not a Fredholm operator}\}.$$

3 Main results

In this section we recall theorems on the boundedness and invertiblity of multiplication operators M_φ between weighted spaces of holomorphic functions on the upper halfplane. Then we characterize closed range and Fredholm self map multiplication operators between weighted spaces of holomorphic functions on the upper halfplane. Before that we prove some lemmas which have a major role in the proof of the main results of this paper. We recall that $\partial^\infty\mathbb{G} = \partial\mathbb{G} \cup \{\infty\}$.

Lemma 3.1. *Let (ω_n) be a sequence in $\mathbb{G}$, which has no cluster point in $\mathbb{G}$. Then (ω_n) has an interpolating subsequence in $\mathbb{G}$.*

Proof. Let (ω_n) be a sequence in $\mathbb{G}$ which has no cluster point in $\mathbb{G}$, then $(z_n = \alpha^{-1}(\omega_n))$ is a sequence in $\mathbb{D}$ such that $\mid z_n \mid\rightarrow 1$. Now Proposition 2.4 of [7] implies that (z_n) has an interpolating subsequence (z_{n_k}) in $\mathbb{D}$. Since (z_{n_k}) is an interpolating sequence in $\mathbb{D}$ so for any sequence $(a_k) \in \ell^\infty$, there exists $f \in H^\infty(\mathbb{D})$ such that $f(z_{n_k}) = a_k$. Obviously $\tilde{f} = f \circ \alpha^{-1} \in H^\infty(\mathbb{G})$ and $\tilde{f}(\omega_{n_k}) = a_k$. Therefore, (ω_{n_k}) is an interpolating subsequence of (ω_n). □

Lemma 3.2. *i) $\overline{\mathbb{D}}$ is a compactification of $\mathbb{G}$.*
ii) $H^\infty(\mathbb{G})$ is isometric isomorphic to $H^\infty(\mathbb{D})$.

Proof. i) Evidently, $\alpha^{-1} : \mathbb{G} \longrightarrow \mathbb{D}$ defined by $\alpha^{-1}(\omega) = \frac{\omega - i}{\omega + i} = z$ is a homeomorphism from $\mathbb{G}$ onto $\mathbb{D}$. So $\overline{\alpha^{-1}(\mathbb{G})} = \overline{\mathbb{D}}$ which is a compact set. Hence, the pair $(\alpha^{-1}, \overline{\mathbb{D}})$ is a compactification of $\mathbb{G}$.
ii) It is enough to define $T : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{G})$ by $T(f) = f \circ \alpha^{-1}$. Since α^{-1} is an onto map so

$$\|T(f)\|_\infty = \sup\{\mid (f \circ \alpha^{-1}(\omega) \mid : \ \omega \in \mathbb{G}\} = \sup\{\mid f(z) \mid : z \in \mathbb{D}\} = \|f\|_\infty.$$

□

Lemma 3.2 implies immediately the following two corollaries.

Corollary 3.3. $\overline{\mathbb{G}} = \mathbb{G} \cup \partial^\infty \mathbb{G}$ *is homeomorphic with* $\overline{\mathbb{D}}$.

Corollary 3.4. $M(H^\infty(\mathbb{G}))$ *is homeomorphic to* $M(H^\infty(\mathbb{D}))$. $L^\infty(\partial \mathbb{D})$ *is isomorphic to* $L^\infty(\partial^\infty \mathbb{G})$. *This is true for Shilov boundary, trivial(nontrivial) Gleason parts of* $H^\infty(\mathbb{G})$ *and* $H^\infty(\mathbb{D})$

Also, the disc algebra $A(\mathbb{D})$ is isomorphic to the $A(\mathbb{G})$, the algebra of all holomorphic functions on $\mathbb{G}$ and continuous on $\overline{\mathbb{G}}$.

Lemma 3.5. *The Shilov boundary of* $H^\infty(\mathbb{G})$ *is the maximal ideal space of* $L^\infty(\partial^\infty \mathbb{G})$.

Proof. We know that $\mathbb{G}$ is homeomorphic to $\mathbb{D}$. The map $\alpha : \mathbb{D} \longrightarrow \mathbb{G}$ maps $\partial \mathbb{D}$ onto $\partial^\infty \mathbb{G}$ homeomorphically. Also, $H^\infty(\mathbb{G})$ and $H^\infty(\mathbb{D})$ are isomorphic spaces and for a function $f \in H^\infty(\mathbb{G})$, $\mid f \circ \alpha(z) \mid$ attains its supremum through the points $z \in \mathbb{D}$, tending to a point $z_0 \in \partial \mathbb{D}$ if and only if $\mid f(\omega) \mid$ attains its supremum thourgh the points $\omega = \alpha(z) \in \mathbb{G}$ tending to a point $\omega_0 = \alpha(z_0) \in \partial^\infty \mathbb{G}$ and vice versa. So the shilov boundaries of $H^\infty(\mathbb{G})$ and $H^\infty(\mathbb{D})$ are homeomorphic. By Corollary 3.4 $L^\infty(\partial \mathbb{D})$ and $L^\infty(\partial^\infty \mathbb{G})$ are isomorphic. From these facts and the result which states that the Shilov boundary of $H^\infty(\mathbb{D})$ is the maximal ideal space of $L^\infty(\partial \mathbb{D})$ (see V.1.7 of [10] and page 169 of [11]) the lemma is proved. □

Lemma 3.6. *Let* X *be the maximal ideal space of* $L^\infty(\partial^\infty \mathbb{G})$. *As* E *varies over the measurable subsets of the* $\partial^\infty \mathbb{G}$ *the open closed sets* $\{\phi \in X : \hat{\chi}_E(\phi) = 0\}$ *give a basis for the topology of* X, *where* $\hat{\chi}$ *is the Gelfand transform of characteristic function. In particular* X *is totally disconnected.*

Proof. By a similar lemma on $L^\infty(\partial\mathbb{D})$ (see page 169 of [11]) and arguments similar to what has been done in Lemma 3.5 we have done. □

Lemma 3.7. *A function $\varphi \in H^\infty(\mathbb{G})$ does not vanish at any point of the Shilov boundary of $H^\infty(\mathbb{G})$ if and only if $\mid \varphi \mid$ is essentially bounded away from zero on $\partial^\infty\mathbb{G}$.*

Proof. $\mid \varphi \mid$ is essentially bounded away from zero means that for some $\epsilon > 0$, $\mid \varphi(\omega) \mid \geq \epsilon$ a.e on $\partial^\infty\mathbb{G}$. Now if φ is zero on any point of the shilov boundary of $H^\infty(\mathbb{G})$ by Lemma 3.6 it vanishes on a totally disconnected open-closed set of positive Lebesgue measure. □

Remark 3.8. *Indeed the maximal ideal space of $L^\infty(\partial^\infty\mathbb{G})$, X is a closed subset of the maximal ideal space of $H^\infty(\mathbb{G})$, $M(H^\infty(\mathbb{G}))$ and because of the injection $H^\infty(\mathbb{G}) \hookrightarrow L^\infty(\partial^\infty\mathbb{G})$, X is a Shilov boundary of $H^\infty(\mathbb{G})$. For more details see page 184 of [10] .*

Now, we continue by recalling some results from [3] . For the sake of completeness, here, we state a modified proof for Theorem 3.9 .

Theorem 3.9. *Let υ be a weight on $\mathbb{G}$. The following statements are equivalent.*
(a) $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ is bounded.
(b) $\varphi \in H^\infty(\mathbb{G})$.
If M_φ is bounded then $\|M_\varphi\| = \|\varphi\|_\infty$. Besides, theorem is also true for $H_{\upsilon_0}(\mathbb{G})$ instead of $H_\upsilon(\mathbb{G})$.

Proof. (b)$\Rightarrow$ (a): It is obvious, since for arbitrary $f \in H_\upsilon(\mathbb{G})$ we have

$$\begin{aligned}\|M_\varphi(f)\|_\upsilon &= \sup\{\mid f(\omega) \mid\mid \varphi(\omega) \mid \upsilon(\omega) : \omega \in \mathbb{G}\} \\ &\leq \sup\{\mid f(\omega) \mid \upsilon(\omega) : \omega \in \mathbb{G}\} \sup\{\mid \varphi(\omega) \mid : \omega \in \mathbb{G}\} \\ &\leq \|f\|_\upsilon \|\varphi\|_\infty.\end{aligned}$$

Hence, $\|M_\varphi\| \leq \|\varphi\|_\infty$.
(a)$\Rightarrow$ (b): $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ is bounded. So the adjoint map $M^*_\varphi : H_\upsilon(\mathbb{G})^* \longrightarrow H_\upsilon(\mathbb{G})^*$ is bounded too and $\|M_\varphi\|_\upsilon = \|M^*_\varphi\|_\upsilon$. Now let

$\omega \in \mathbb{G}$ be given. Evidently, the evaluational function $\delta_\omega : H_\upsilon(\mathbb{G}) \longrightarrow \mathbb{C}$ defined by $\delta_\omega(f) = f(\omega)$ belongs to the dual space $H_\upsilon(\mathbb{G})^*$. Note that

$$| \varphi(\omega) | = \frac{\|M_\varphi^*(\delta_\omega)\|}{\|\delta_\omega\|} \leq \|M_\varphi^*\| = \|M_\varphi\| < \infty.$$

Therefore,

$$\sup\{| \varphi(\omega) | : \omega \in \mathbb{G}\} = \|\varphi\|_\infty \leq \|M_\varphi\| < \infty.$$

Where by $\|M_\varphi\|$ and $\|M_\varphi^*\|$ we mean the operator norm of linear operators M_φ and M_φ^*. □

Remark 3.10. *From now on we always assume $\varphi \in H^\infty(\mathbb{G})$. Because this assumption is equivalent to that M_φ is welldefined.*

Theorem 3.11. (See [3]) *Let υ be a weight on $\mathbb{G}$. The following statements are equivalent.*
(a) $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ is invertible.
(b) $\frac{1}{\varphi} \in H^\infty(\mathbb{G})$ (or, equivalently, there exists $\epsilon > 0$ such that $| \varphi(\omega) | \geq \epsilon$ for all $\omega \in \mathbb{G}$).

Remark 3.12. *Since $\varphi \in H^\infty(\mathbb{G})$ then $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ is a bounded linear operator. So $\sigma(M_\varphi)$ is defined.*

Theorem 3.13. $\sigma(M_\varphi) = \overline{\varphi(\mathbb{G})} = \varphi(M(H^\infty(\mathbb{G})))$.

Proof. Proof of the first equality can be found in [3]. For proving the last equality note that $\overline{\tilde{\varphi}(\mathbb{D})} = \tilde{\varphi}(M(H^\infty(\mathbb{D})))$ (see pages 159-162 of [11]). Here $\tilde{\varphi} = \varphi \circ \alpha$. Now using Corollary 3.3 and Corollary 3.4 we conclude that
$\overline{\varphi(\mathbb{G})} = \overline{\tilde{\varphi}(\mathbb{D})} = \tilde{\varphi}(M(H^\infty(\mathbb{D}))) = \varphi(M(H^\infty(\mathbb{G})))$. □

Corollary 3.14. (See [3]) *M_φ is not a compact operator.*

In the next Theorem we state a necessary and sufficient condition such that M_φ be a Fredholm operator. Since proof is very similar to the proof in [7], we do not repeat it again. we only explain necessary changes in order to transfer the proof to the upper halfplane case. Note that condition on φ in theorem 3.15 is completely different from the case of unit disc since compact subsets of $\mathbb{G}$ have wide variety in comparison with compact subsets of $\mathbb{D}$.

Theorem 3.15. *Let υ be a weight on $\mathbb{G}$ and $\varphi \in H^\infty(\mathbb{G})$. The operator $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ is Fredholm if and only if there exist $\epsilon > 0$ and a compact subset $K \subset \mathbb{G}$ such that $\mid \varphi(\omega) \mid > \epsilon$ for all $\omega \in \mathbb{G} \setminus K$. Consequently $\sigma_e(M_\varphi) = \varphi(M(H^\infty(\mathbb{G}))) \setminus \mathbb{G}$. The same holds for $H_{\upsilon_0}(\mathbb{G})$ instead of $H_\upsilon(\mathbb{G})$ whenever $H_{\upsilon_0}(\mathbb{G}) \neq \{0\}$.*

Proof. Firstly, suppose that M_φ is Fredholm but there is a sequence $(\omega_n) \subset \mathbb{G}$ such that $\mid \varphi(\omega_n) \mid \to 0$ whenever $\omega_n \to \partial^\infty \mathbb{G}$. By Lemma 3.1 we can assume that (ω_n) is an interpolating sequence in $H^\infty(\mathbb{G})$. Rest of the proof is quite similar. Last assertion of the theorem follows from Corollary 3.4 and equality $\sigma_e(M_\varphi) = \varphi(M(H^\infty(\mathbb{D})) \setminus \mathbb{D})$ in [7]. ☐
With a similar proof as in [7] we have:

Theorem 3.16. *Let $\varphi \in H^\infty(\mathbb{G})$. If υ and w are two weights on $\mathbb{G}$ and $u := \frac{\upsilon}{w}$ is equivalent to an essential weight, then every closed range map $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ also has closed range as a map $M_\varphi : H_w(\mathbb{G}) \longrightarrow H_w(\mathbb{G})$. An analogous result holds for $H_{\upsilon_0}(\mathbb{G})$ and $H_{w_0}(\mathbb{G}) \neq \{0\}$.*

Theorem 3.17. *The map $M_\varphi : H^\infty(\mathbb{G}) \longrightarrow H^\infty(\mathbb{G})$ has closed range if and only if φ does not vanish at any point of the Shilov boundary $\Gamma(H^\infty(\mathbb{G}))$ of $H^\infty(\mathbb{G})$.*

Proof. Let φ^* be the a.e limit of φ on $\partial^\infty \mathbb{G}$. If $\mid \varphi^* \mid < \epsilon$ on a subset $A \subset \partial^\infty \mathbb{G}$ of positive measure and $\mid \varphi^* \mid \leq 1$ elsewhere, then we take the outer function

$$F(\omega) = \exp \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz}{(t - z)(t^2 + 1)} \log \mid F^*(t) \mid dt$$

where $\mid F^* \mid = 1$ on A and $\mid F^* \mid = \epsilon$ on $\partial^\infty \mathbb{G} \setminus A$ to get $\|M_\varphi F\|_\infty \leq \epsilon$. Thus if $\mid \varphi \mid$ is not essentially bounded from below then M_φ is not bounded from below and this is equivalent to M_φ does not have closed range. Conversely, suppose that φ does not vanish at any point of the Shilov boundary. Hence, for some $\epsilon > 0$, $\mid \varphi(\omega) \mid > \epsilon$ almost everywhere . Now if M_φ is not of closed range then we can consider an $f \in H^\infty(\mathbb{G})$ with $\|f\|_\infty = 1$ and $\|M_\varphi f\|_\infty \to 0$. Since $\mid \varphi(\omega) \mid > \epsilon$ almost everywhere so we must have $\mid f(\omega) \mid = 0$ a.e on $\partial^\infty \mathbb{G}$. But this is a contradiction since $f \in H^\infty(\mathbb{G}) \subset N$ (Nevalinna class) and a function in the Nevalinna class can not be zero on a subset of $\partial^\infty \mathbb{G}$ of positive Lebesgue measure

(see Theorem 2.2 of [9]).
□
Here we recall the following theorem.

Theorem 3.18. (See [7]) *For any weight υ on $\mathbb{D}$ (not necessarily radial) there is a closed set A_υ, $\Gamma(H^\infty(\mathbb{D})) \subset A_\upsilon \subset M(H^\infty(\mathbb{D})) \setminus \mathbb{D}$, such that $M_\varphi : H_\upsilon(\mathbb{D}) \longrightarrow H_\upsilon(\mathbb{D})$ has closed range if and only if φ does not vanish on A_υ.*

Now we Prove Theorem 3.18 for the upper halfplane.

Theorem 3.19. *For any weight υ on $\mathbb{G}$ there is a closed set A_υ, $\Gamma(H^\infty(\mathbb{G})) \subset A_\upsilon \subset M(H^\infty(\mathbb{G}))\backslash\mathbb{G}$, such that $M_\varphi : H_\upsilon(\mathbb{G}) \longrightarrow H_\upsilon(\mathbb{G})$ has closed range if and only if φ does not vanish on A_υ.*

Proof. Let υ be a weight on $\mathbb{G}$. then $u = \upsilon \circ \alpha$ is a weight on $\mathbb{D}$ and $T : H_\upsilon(\mathbb{G}) \longrightarrow H_u(\mathbb{D})$ defined by $Tg = g \circ \alpha$ is an isometric isomorphism since

$$\|Tg\|_u = \sup_{z\in\mathbb{D}} \mid (g \circ \alpha)(z) \mid (\upsilon \circ \alpha)(z) = \sup_{\omega\in\mathbb{G}} \mid g(\omega) \mid \upsilon(\omega) = \|g\|_\upsilon$$

Now consider the following diagram.

$$\begin{array}{cccc} M_\varphi : & H_\upsilon(\mathbb{G}) & \longrightarrow & H_\upsilon(\mathbb{G}) \\ & \downarrow T & & \downarrow T \\ M_{\tilde{\varphi}} : & H_u(\mathbb{D}) & \longrightarrow & H_u(\mathbb{D}) \end{array}$$

where $\tilde{\varphi} = \varphi \circ \alpha$. For any $g \in H_\upsilon(\mathbb{G})$
$(T^{-1} \circ M_{\tilde{\varphi}} \circ T)(g) = (T^{-1} \circ M_{\tilde{\varphi}})(g \circ \alpha) = T^{-1}((\varphi \circ \alpha)(g \circ \alpha)) = [(\varphi \circ \alpha) \circ \alpha^{-1}][(g \circ \alpha) \circ \alpha^{-1}] = \varphi g = M_\varphi(g)$.
Thus the diagram is commutataive. Therefore M_φ has closed range on $H_\upsilon(\mathbb{G})$ if and only if $M_{\tilde{\varphi}}$ has closed range on $H_u(\mathbb{D})$. Now let M_φ be of closed range. Since $M_{\tilde{\varphi}}$ has closed range, Theorem 3.18 implies that $\tilde{\varphi}$ does not vanish on a closed subset $A_{u=\upsilon\circ\alpha}$ of $\overline{\mathbb{D}}$ and this implies that φ does not vanish on a closed subset A_υ of $\overline{\mathbb{G}}$ (an isomorhic map takes a closed set to a closed set). A_u satisfies $\Gamma(H^\infty(\mathbb{D})) \subset A_u \subset M(H^\infty(\mathbb{D})) \setminus \mathbb{D}$. Now Lemma 3.2 and Corollary 3.4 imply that A_υ satisfies $\Gamma(H^\infty(\mathbb{G})) \subset A_\upsilon \subset M(H^\infty(\mathbb{G})) \setminus \mathbb{G}$.
Conversely if φ does not vanish on a set A_υ in $\overline{\mathbb{G}}$, $\Gamma(H^\infty(\mathbb{G})) \subset A_\upsilon \subset$

$M(H^\infty(\mathbb{G})) \setminus \mathbb{G}$ then $\tilde{\varphi}$ does not vanish on a homeomorphic set $A_u = A_{\upsilon\circ\alpha}$ in $\overline{\mathbb{D}}$ and $\Gamma(H^\infty(\mathbb{D})) \subset A_u \subset M(H^\infty(\mathbb{D})) \setminus \mathbb{D}$. Again by Theorem 3.18 $M_{\tilde{\varphi}}$ has closed range and equivalently M_φ has closed range. □

Remark 3.20. *It is worth to be mentioned that all of our results hold in every simply connected domain in the complex plane and can be obtained by methods which we have used here, since all of simply connected domains are homeomorphic. (see Chap. 13 of [12]). For example the strip* $\{z \in \mathbb{C} : z = x + iy,\ -\frac{\pi}{2} < x < \frac{\pi}{2}\}$ *and* $\mathbb{C} \setminus \{(x, 0) :\ x \geq 0\}$*. Nevertheless the simply connected domain* $\mathbb{G}$ *is of special interest, since it is frequently easier to handle .*

4 Multipliers

In this section we characterize multipliers of $H_\upsilon(\mathbb{G})$ for certain type of weights on the upper half-plane. We recall that a sequence $\{\lambda_n\}$ is a multiplier of a sequence space A if $\{\lambda_n a_n\} \in A$ for each $\{a_n\} \in A$. Since an analytic function has a unique Talyor coefficients in its expansion , so any space of holomorphic functions can be regarded as a sequence space, particularly, $H_\upsilon(\mathbb{D})$. Shields and Williams proved the following characterization for multipliers of $H_\upsilon(\mathbb{D})$ whenever υ is a normal weight i.e υ is a standard weight on $\mathbb{D}$ such that there are $\epsilon, k > 0$ such that $0 < \epsilon < k$, $\frac{\upsilon(r)}{(1-r)^k} \nearrow \infty$ as $r \to 1^-$ and $\frac{\upsilon(r)}{(1-r)^\epsilon} \searrow 0$ as $r \to 1^-$.

Theorem 4.1. (See [13]) *Let* υ *be a normal weight on* $\mathbb{D}$*. The sequence* $\{\lambda_n\}$ *is a multiplier of* $H_\upsilon(\mathbb{D})$ *if and only if*
(i) The power series $h(z) = \Sigma_{n=1}^{\infty} \lambda_n z^n$ *converges for* $\mid z \mid < 1$ *and*
(ii) $M_1(h', r) = O(\frac{1}{1-r})$.
Where $M_1(h, r) = \frac{1}{2\pi}\int_0^\infty \mid h(re^{i\theta}) \mid d\theta$ *and (ii) means* $M_1(h', r)(1 - r)$ *is bounded.*

We intend to obtain a result similar to Theorem 4.1 for $H_\upsilon(\mathbb{G})$ for a special kind of weights on $\mathbb{G}$. We recall the Definition 1.2 of [2]. We say a weight υ on $\mathbb{G}$ is of type(II) if $\upsilon(\omega) = \upsilon_1(\omega)$ ($\mid \omega \mid \leq 1$ and υ_1 is a standard weight on $\mathbb{G}$) and there is a constant $C > 0$ such that $\frac{\upsilon(\omega)}{\upsilon(-\frac{1}{\omega})} \leq C$. An example of a type(II) weight is $\upsilon(\omega) = (\frac{Im\ \omega}{\max(\mid \omega \mid^2, 1)})^\beta$

for some $\beta > 0$. For more examples of type(II) weights we refer the reader to the Example 1.3 of [2].

Lemma 4.2. (See Lemma 3.1 of [2]) *Let υ be a type(II) weight satisfying $(*)$. Put $\tilde{\upsilon}(z) = \upsilon(\alpha(- \mid z \mid))$. Then $\tilde{\upsilon}(z)$ is a radial weight on $\mathbb{D}$. Moreover, the map T defined by $(Tf)(z) = f(\alpha(z))$ is an isomorphism from $H_\upsilon(\mathbb{G})$ on to $H_{\tilde{\upsilon}}(\mathbb{D})$.*

Note that the proof of 4.2 reveals that weights $\upsilon \circ \alpha$ and $\tilde{\upsilon}$ are equivalent. Before stating the main result of this section we need the following lemmas.

Lemma 4.3. *If $f : \mathbb{G} \longrightarrow \mathbb{C}$ is a holomorphic function, then there are $\alpha_k \in \mathbb{C}$ such that $f(\omega) = \Sigma_{k=0}^{\infty} \alpha_k (\frac{\omega - i}{\omega + i})^k$, where the series converges uniformly on compact subsets of $\mathbb{G}$.*

Proof. Clearly $f \circ \alpha : \mathbb{D} \longrightarrow \mathbb{C}$ is analytic. So $(f \circ \alpha)(z) = \Sigma_{k=0}^{\infty} \alpha_k z^k$ for some α_k and series converges uniformly on the compact subsets of $\mathbb{D}$. Put $\alpha(z) = \omega$ then $z = \alpha^{-1}(\omega) = \frac{\omega - i}{\omega + i}$. This completes the proof. □

Lemma 4.4. (See Theorem 2.2.3 of [1]) *Let υ be a type(II) weight on $\mathbb{G}$ satisfying $(*)$ and $(**)$. Then $\tilde{\upsilon}(\tilde{\upsilon}$ is as in Lemma 4.2) satisfies $(*)'$ and $(**)'$ respectively.*

Remark 4.5. *Lemma 4.3 enables us to regard $H_\upsilon(\mathbb{G})$ as a sequence space. Also it is easy to see that a weight υ is normal if and only if it satisfies $(*)'$ and $(**)'$. See also section three of [7].*

What we have obtained in this section can be summarized in the following theorem.

Theorem 4.6. *Let υ be a type (II) weight on $\mathbb{G}$ satisfying $(*)$ and $(**)$. Then (λ_n) is a multiplier on $H_\upsilon(\mathbb{G})$ if and only if*
(i) The power series $h(\omega) = \Sigma_{n=1}^{\infty} \lambda_n (\frac{\omega - i}{\omega + i})^n$ converges on $\mathbb{G}$ and
(ii) $M_1(\tilde{h}', r) = O(\frac{1}{1-r})$,
where $\tilde{h} = h \circ \alpha$.

Proof. Only note that since α is an onto map so $h(\omega) = h \circ \alpha(z) = \tilde{h}(z)$ $(\omega \in \mathbb{G},\ z \in \mathbb{D})$ again can be considered as an element of $H_{\tilde{\upsilon}}(\mathbb{D})$. □

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