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A Note on LS-Category and Topological Complexity of Real Grassmannian Manifolds

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Abstract. Let $G_{k,n}$ be the Grassmann manifold of k-planes in $\mathbb{R}^{n+k}$. The Lusternik-Schnirelmann category and topological complexity are important invariants of topological spaces. In this note we calculate the Lusternik-Schnirelmann category and topological complexity of certain products of Grassmannian manifolds by using cup and zero-cup length. Also we will find the lower and upper bounds of the topological complexity of some Grassmannian manifolds by the same method.

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1 Introduction

In 1934, L. Lusternik and L. Schnirelmann described a new invariant of a manifold called category. Their purpose in creating this concept was to obtain a lower bound on the number of critical points for each

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smooth function on the manifold. This category examines the important concepts of geometry and dynamical systems. The topological complexity is a numerical homotopy invariant, introduced by M. Farber in 2001, and in [5], [6], [7] he examined the topological complexity of the robotics. Topological complexity has close relationship to classical invariant, Lusternik-Schnirelmann category. In [1] we have studied the product of projective spaces by here we are going to study real grassmannian manifolds.

In Section 2 we calculate by known results the category of products of $G_2(\mathbb{R}^{2^p+1})$ and $G_2(\mathbb{R}^{2^p+2})$. In Section 3 first we calculate the topological complexity of $G_2(\mathbb{R}^3)$ and $G_2(\mathbb{R}^4)$ by different method in [11] following the products of them. In Section 4 we give upper and lower bounds for topological complexity of certain Grasmanian manifolds. Specially we show that $10 \leq TC(G_2(\mathbb{R}^5)) \leq 11$ and $12 \leq TC(G_2(\mathbb{R}^6)) \leq 13$.

Definition 1.1. The Lusternik-Schnirelmann category of a space X is the least integer n such that there exists an open covering $U_1, \cdots, U_{n+1}$ of X with each U_i contractible to a point in the space X. We denote this by $cat(X) = n$ and we call such a covering U_i categorical. If no such integer exists, we write $cat(X) = \infty$.

In [5], Michael Farber, defined a numerical invariant $TC(X)$. We may out lined as follows: Let PX denote the space of all continuous paths $\gamma : [0,1] \longrightarrow X$ in X and $\pi : PX \longrightarrow X \times X$ denotes the map associating to any path $\gamma \in PX$ the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space PX with the compact-open topology.

Definition 1.2. The topological complexity of a path-connected space X, denoted by $TC(X)$, is the least integer n such that the Cartesian product $X \times X$ can be covered with n open subsets U_i, $X \times X = U_1 \cup U_2 \cup \cdots \cup U_n$ such that for any $i = 1, 2, \cdots, n$ there exists a continuous local section $s_i : U_i \longrightarrow PX$ of π, that is, $\pi \circ s_i = id$ over U_i. If no such m exists we will set $TC(X) = \infty$.

Theorem 1.3. *Let $G_{k,n}$ denote the Grassmann manifold of k-planes in $\mathbb{R}^{n+k}$. Then $H^*(G_{k,n}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k] / I_{k,n}$ where $I_{k,n}$ is the ideal generated by the dual Stiefel-Whitney classes $\bar{w}_{n+1}, \ldots, \bar{w}_{n+k}$.*

Proof. See [3] for a proof. □

Remark 1.4. The set $\{w_1^a w_2^b : a+b \leq n\}$ is vector space basis for the cohomology ring $H^*(G_{2,n};\mathbb{Z}_2)$.

2 LS-category of the products of $G_2(\mathbb{R}^{2^p+1})$, $G_2(\mathbb{R}^{2^p+2})$

This section is devoted to calculate LS-category of certain products of real Grassmannian manifolds by using cup-length.

Definition 2.1. Let R be a commutative ring and X be a space. The cup-length of X with coefficients in R is the least integer k (*or* ∞) such that all $(k+1)$ -fold cup products vanish in the reduced cohomology $\widetilde{H}^*(X;R)$; we denote this integer by $cup_R(X)$.

Proposition 2.2. *The R-cuplength of a space is less than or equal to the category of the space for all coefficients R. In notation, we write $cup_R(X) \leq cat(X)$.*

Proof. See Proposition 1.5 in [4]. □

Theorem 2.3. *For a path-connected locally contractible paracompact space, $cat(X) \leq dim(X)$.*

Proof. See Theorem 1.7 in [4]. □

Example 2.4. Since $H^*(\mathbb{R}P^n;\mathbb{Z}_2) = \mathbb{Z}_2[a]/\langle a^{n+1}\rangle$ with $deg(a) = 1$. Since $a^n \neq 0$, then $cup(\mathbb{R}P^n) = n \leq cat(\mathbb{R}P^n) \leq dim(\mathbb{R}P^n) = n$. Thus, $cat(\mathbb{R}P^n) = n$.

Theorem 2.5. *Suppose X and Y are path-connected spaces such that $X \times Y$ is completely normal. Then $cat(X \times Y) \leq cat(X) + cat(Y)$.*

Proof. See Theorem 1.37 in [4]. □

Theorem 2.6. *If X is a closed, connected n-manifold with $\pi_1(X) \approx \mathbb{Z}_2$, then $cat(X) = dim(X)$ iff $w^{dim(X)} \neq 0$, where w is the nonzero element of $H^1(X;\mathbb{Z}_2)$.*

Proof. See a proof [2, 10]. □ From Theorem 2.6 we have the following corollary.

Corollary 2.7. *$w^{dim(X)} = 0$ if and only if $cat(X) < dim(X)$.*

Theorem 2.8. *For any positive integers $p \geq 1$, we have:*

$$cat(G_2(\mathbb{R}^{2^p+1})) = 2^{p+1} - 2.$$

Proof. See [2] for a proof. □

Theorem 2.9. *For any positive integer $p_i \geq 1, m \geq 1$, we have:*

$$\begin{aligned} cat(G_2(\mathbb{R}^{2^{p_1}+1}) \times G_2(\mathbb{R}^{2^{p_2}+1}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+1})) &= \\ 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m \end{aligned}$$

Proof. Since $H^*(G_{k,n}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k] \big/ I_{k,n}$, by Künneth formulas

$$\begin{aligned} H^*(G_2(\mathbb{R}^{2^{p_1}+1}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+1})) &= \\ H^*(G_2(\mathbb{R}^{2^{p_1}+1})) \otimes ... \otimes H^*(G_2(\mathbb{R}^{2^{p_m}+1})) &= \mathbb{Z}_2[w_1, w_2]/\langle \bar{w}_{2^{p_1}}, \bar{w}_{2^{p_1}+1} \rangle \otimes \\ \cdots \otimes \mathbb{Z}_2[w_1, w_2]/\langle \bar{w}_{2^{p_m}}, \bar{w}_{2^{p_m}+1} \rangle, \end{aligned}$$

Since $cat(G_2(\mathbb{R}^{2^p+1})) = dim(G_2(\mathbb{R}^{2^p+1}))$, then by Theorem 2.6; $w_1^{dim} \neq 0$ and $w_1^{dim+1} = 0$. Set,

$$\begin{aligned} \alpha_1 &= w_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \\ \alpha_2 &= 1 \otimes w_1 \otimes 1 \otimes \cdots \otimes 1 \\ &\vdots \\ \alpha_m &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w_1. \end{aligned}$$

Thus

$$\begin{aligned} \alpha_1^{2^{p_1+1}-2} &= w_1^{2^{p_1+1}-2} \otimes \cdots \otimes 1 \\ \alpha_2^{2^{p_2+1}-2} &= 1 \otimes w_1^{2^{p_2+1}-2} \otimes \cdots \otimes 1 \\ &\vdots \\ \alpha_m^{2^{p_m+1}-2} &= 1 \otimes 1 \otimes \cdots \otimes w_1^{2^{p_m+1}-2}. \end{aligned}$$

Therefore

$\alpha_1^{2^{p_1+1}-2}\alpha_2^{2^{p_2+1}-2}\cdots\alpha_m^{2^{p_m+1}-2} = w_1^{2^{p_1+1}-2}\otimes w_1^{2^{p_2+1}-2}\otimes\cdots\otimes w_1^{2^{p_m+1}-2} \neq 0.$

From which,
$cup_{\mathbb{Z}_2}(G_2(\mathbb{R}^{2^{p_1}+1}) \times G_2(\mathbb{R}^{2^{p_2}+1}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+1})) \geq (2^{p_1+1}-2) + (2^{p_2+1}-2) + \cdots + (2^{p_m+1}-2)) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m.$

On the other hand, By Theorem 2.5,
$cat(G_2(\mathbb{R}^{2^{p_1}+1}) \times G_2(\mathbb{R}^{2^{p_2}+1}) \times \cdots \times G_2(R^{2^{p_m}+1})) \leq 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m.$

Now by Proposition 2.2,

$cat(G_2(\mathbb{R}^{2^{p_1}+1}) \times G_2(\mathbb{R}^{2^{p_2}+1}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+1})) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m.$ □

Corollary 2.10. *For any positive integer p, we have;*

$$cat\,\underbrace{(G_2(\mathbb{R}^{2^p+1}) \times G_2(\mathbb{R}^{2^p+1}) \times \cdots \times G_2(\mathbb{R}^{2^p+1}))}_{m-times} = m(2^{p+1}-2).$$

Theorem 2.11. *For any positive integer p, $cat(G_2(\mathbb{R}^{2^p+2})) = 2^{p+1}-1$.*

Proof. See [10] for a proof. □

Theorem 2.12. *For any positive integer $p_i \geq 1$, $m \geq 1$, we have:*

$$cat\,\underbrace{(G_2(\mathbb{R}^{2^{p_1}+2}) \times G_2(\mathbb{R}^{2^{p_2}+2}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+2}))}_{m-times} = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - m.$$

Proof. Since $H^*(G_{k,n};\mathbb{Z}_2) = \mathbb{Z}_2[w_1,\ldots,w_k]/I_{k,n}$, by Künneth formulas

$$\begin{aligned} H^*(G_2(\mathbb{R}^{2^{p_1}+2}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+2})) &= \\ H^*(G_2(\mathbb{R}^{2^{p_1}+2})) \otimes \ldots \otimes H^*(G_2(\mathbb{R}^{2^{p_m}+2})) &= \mathbb{Z}_2[w_1,w_2]/\langle \bar{w}_{2^{p_1}+1}, \bar{w}_{2^{p_1}+2}\rangle \otimes \\ \cdots \otimes \mathbb{Z}_2[w_1,w_2]/\langle \bar{w}_{2^{p_m}+1}, \bar{w}_{2^{p_m}+2}\rangle. \end{aligned}$$

Where

$$\bar{w}_{2^{p_i}+1} = w_1^{2^{p_i}+1} + \cdots + w_1 w_2^{2^{p_i-1}};$$

$$\bar{w}_{2^{p_i}+2} = w_1^{2^{p_i}+2} + w_1^{2^{p_i}} w_2 + \cdots + w_2^{2^{p_i-1}+1}.$$

Since $cat(G_2(\mathbb{R}^{2^{p_i}+2}) < dim(G_2(\mathbb{R}^{2^{p_i}+2})$, then by Corollary 2.7, $w_1^{2^{p_i+1}-1} = 0$ but $w_1^{2^{p_i+1}-2} \neq 0$. Set:

$$\begin{aligned}
\alpha_1 &= w_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \\
\alpha_2 &= 1 \otimes w_1 \otimes 1 \otimes \cdots \otimes 1 \\
&\vdots \\
\alpha_m &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w_1.
\end{aligned}$$

Thus

$$\begin{aligned}
\alpha_1^{2^{p_i+1}-2} &= w_1^{2^{p_i+1}-2} \otimes \cdots \otimes 1 \\
\alpha_2^{2^{p_i+1}-2} &= 1 \otimes w_1^{2^{p_i+1}-2} \otimes \cdots \otimes 1 \\
&\vdots \\
\alpha_m^{2^{p_i+1}-2} &= 1 \otimes 1 \otimes \cdots \otimes w_1^{2^{p_i+1}-2}.
\end{aligned}$$

Also let,

$$\begin{aligned}
\beta_1 &= w_2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \\
\beta_2 &= 1 \otimes w_2 \otimes 1 \otimes \cdots \otimes 1 \\
&\vdots \\
\beta_m &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w_2.
\end{aligned}$$

Therefore for $i = 1, \cdots, m$

$$\alpha_1^{2^{p_i+1}-2} \cdots \alpha_m^{2^{p_i+1}-2} \beta_1 \cdots \beta_m = w_1^{2^{p_i+1}-2} w_2 \otimes w_1^{2^{p_i+1}-2} w_2 \otimes \cdots \otimes w_1^{2^{p_i+1}-2} w_2 \neq 0.$$

From which,

$cup_{\mathbb{Z}_2}(G_2(\mathbb{R}^{2^{p_1}+2}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+2})) \geq (2^{p_1+1}-1) + \cdots + (2^{p_m+1}-1) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - m.$

Now by Theorem 2.5 and Proposition 2.2 we have

$$cat(G_2(\mathbb{R}^{2^{p_1}+2}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+2})) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - m.$$

□

Corollary 2.13. *For any positive integer p, we have;*

$$cat\,\underbrace{(G_2(\mathbb{R}^{2^p+2}) \times G_2(\mathbb{R}^{2^p+2}) \times \cdots \times G_2(\mathbb{R}^{2^p+2}))}_{m-times} = m(2^{p+1}-1).$$

3 Topological complexity of products of $G_2(\mathbb{R}^3)$, $G_2(\mathbb{R}^4)$

In this section we will calculate the topological complexity of $G_2(\mathbb{R}^3)$, $G_2(\mathbb{R}^4)$ following the product of them. We briefly recall a result from [3] giving a lower bound on $TC(X)$. It is quite useful since it allows us an effective computation of $TC(X)$ in many examples. A lower bound for topological complexity is obtained by using the zero-divisor-cup-length of X.

Definition 3.1. Let k be a field. The kernel of homomorphism

$$\cup : H^*(X;k) \otimes H^*(X;k) \longrightarrow H^*(X;k)$$

is called the ideal of the zero-divisors of $H^*(X;k)$. The zero-divisors-cup-length of $H^*(X;k)$ is the length of the longest nontrivial product in the ideal of the zero-divisors of $H^*(X;k)$. This number will be denoted by $zcl(X)$.

Theorem 3.2. *The number TC(X) is greater than the zero-divisors-cup-length of $H^*(X;K)$.*

Proof. See Theorem 7 in [6]. □

Theorem 3.3. *If X is path-connected and paracompact then*

$$cat(X) \leq TC(X) \leq 2.cat(X) - 1.$$

Proof. See Theorem 5 in [6]. □

Theorem 3.4. *For any path-connected metric spaces X and Y,*

$$TC(X \times Y) \leq TC(X) + TC(Y) - 1.$$

Proof. See Theorem 11 in [6]. □

Lemma 3.5. $TC(G_2(\mathbb{R}^3)) = 4$.

Proof. Since $G_2(\mathbb{R}^3)$ is infact $\mathbb{R}P^2$, so by [5], $TC(\mathbb{R}P^2) = 4 = TC(G_2(\mathbb{R}^3))$. We may give another proof with the method of zero divisior cup length. Since $H^*((G_2(\mathbb{R}^3)); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/\langle \bar{w}_2, \bar{w}_3\rangle$ and $\bar{w}_2 = w_1^2 + w_2$, $\bar{w}_3 = w_1^3$, we have $H^*((G_2(\mathbb{R}^3)); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/\langle w_1^2 + w_2, w_1^3\rangle$. Now define $\alpha, \beta \in H^*(G_2(\mathbb{R}^3) \otimes H^*(G_2(\mathbb{R}^3)$, by: $\alpha = (w_1 \otimes 1) + (1 \otimes w_1)$, $\beta = (w_2 \otimes 1) + (1 \otimes w_2)$.
Since $\alpha^2 = (w_1^2 \otimes 1) + (1 \otimes w_1^2)$, $\alpha^3 = (w_1^2 \otimes w_1) + (w_1 \otimes w_1^2)$, $\beta^2 = (w_2^2 \otimes 1) + (1 \otimes w_2^2) = 0$, but $\alpha^3\beta = 0$ on the other hand $\alpha^2\beta = (w_1^2 \otimes w_2) + (w_2 \otimes w_1^2) \neq 0$ consequently $zcl(G_2(\mathbb{R}^3)) \geq 3$, by Theorem 3.3, $3 < TC(G_2(\mathbb{R}^3)) \leq 4$, as a result $TC(G_2(\mathbb{R}^3)) = 4$. □

Lemma 3.6. *For any positive integer m, we have:*

$$zcl\,\underbrace{(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times ... \times G_2(\mathbb{R}^3))}_{m-times} \geq 3m.$$

Proof. Remember by Theorem 2.6, $w_1^2 \neq 0$. Let $\alpha_i, \beta_i \in H^*(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3)) \otimes H^*(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3))$, for $i = 1, 2, \cdots, m$, defined by:

$$\begin{aligned}
\alpha_1 &= (w_1 \otimes 1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (w_1 \otimes 1 \otimes \cdots \otimes 1),\\
\alpha_2 &= (1 \otimes w_1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes w_1 \otimes \cdots \otimes 1),\\
&\vdots\\
\alpha_m &= (1 \otimes 1 \otimes \cdots \otimes w_1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes 1 \otimes \cdots \otimes w_1)
\end{aligned}$$

and

$$\begin{aligned}
\beta_1 &= (w_2 \otimes 1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (w_2 \otimes 1 \otimes \cdots \otimes 1),\\
\beta_2 &= (1 \otimes w_2 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes w_2 \otimes \cdots \otimes 1),\\
&\vdots\\
\beta_m &= (1 \otimes 1 \otimes \cdots \otimes w_2) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes 1 \otimes \cdots \otimes w_2).
\end{aligned}$$

We may show by easy calculation that $\alpha_i s$ and $\beta_i s$ are in the kernel of $\cup : H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$. Clearly $\alpha_i^2 \neq 0$ and calculation shows that

$$\alpha_1^2\alpha_2^2 \cdots \alpha_m^2\beta_1 \cdots \beta_m = w_1^2w_2 \otimes w_1^2w_2 \otimes \cdots \otimes w_1^2w_2 \neq 0.$$

Consequently,

$$zcl(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3)) \geq 2m + m = 3m.$$

□

Theorem 3.7. *For any positive integer $m \geq 1$, we have:*

$$TC\underbrace{(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3))}_{m-times} = 3m + 1.$$

Proof. This proof follows by Theorems 3.4 and Lemmas 3.5, 3.6. □

Lemma 3.8. $TC(G_2(\mathbb{R}^4)) = 5$.

Proof. First, we calculate the zero divisior cup length of $G_2(\mathbb{R}^4)$. Since $H^*((G_2(\mathbb{R}^4)); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/\langle \bar{w}_3, \bar{w}_4 \rangle$ and $\bar{w}_3 = w_1^3$, $\bar{w}_4 = w_1^4 + w_1^2w_2 + w_2^2$, we have $H^*((G_2(\mathbb{R}^4)); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/\langle w_1^3, w_1^2w_2 + w_2^2 \rangle$. Now let $\alpha, \beta \in H^*(G_2(\mathbb{R}^4) \otimes H^*(G_2(\mathbb{R}^4)$, defined by:

$$\alpha = (w_1 \otimes 1) + (1 \otimes w_1), \qquad \beta = (w_2 \otimes 1) + (1 \otimes w_2).$$

By an easy calculation we see that

$$\alpha^3\beta = (w_1^2w_2 \otimes w_1) + (w_1^2 \otimes w_1w_2) + (w_1w_2 \otimes w_1^2) + (w_1 \otimes w_1^2w_2) \neq 0.$$

Consequently $zcl(G_2(\mathbb{R}^4)) \geq 4$, on the other hand by Theorem 3.3, $4 < TC(G_2(\mathbb{R}^4)) \leq 5$, as a result $TC(G_2(\mathbb{R}^4)) = 5$. □

K. J. Pearson and Tan Zhang in [11] used the equality $TC(X) = cat(X \times X)$, to compute the topological complexity of $G_2(\mathbb{R}^4)$, which is not true in general. In fact we have $TC(X) \leq cat(X \times X)$. See the following example.

Example 3.9. Let $X = G_2(\mathbb{R}^4)$ by Lemma 3.11 $TC(X) = 5$ and by Corollary 2.13 $cat(X \times X) = 6$. This shows that the equality $TC(X) = cat(X \times X)$ is not true in general.

Lemma 3.10. *For any positive integer m, we have:*

$$zcl\,\underbrace{\left(G_2(\mathbb{R}^4)\times G_2(\mathbb{R}^4)\times ...\times G_2(\mathbb{R}^4)\right)}_{m-times}\geq 4m.$$

Proof. Let $\alpha_i, \beta_i \in H^*(G_2(\mathbb{R}^4)\times\cdots\times G_2(\mathbb{R}^4))\otimes H^*(G_2(\mathbb{R}^4)\times\cdots\times G_2(\mathbb{R}^4))$, for $i=1,2,\cdots,m$, defined by:

$$\begin{aligned}
\alpha_1 &= (w_1\otimes 1\otimes\cdots\otimes 1)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(w_1\otimes 1\otimes\cdots\otimes 1)\\
\alpha_2 &= (1\otimes w_1\otimes\cdots\otimes 1)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(1\otimes w_1\otimes\cdots\otimes 1)\\
&\vdots\\
\alpha_m &= (1\otimes 1\otimes\cdots\otimes w_1)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(1\otimes 1\otimes\cdots\otimes w_1)
\end{aligned}$$

and

$$\begin{aligned}
\beta_1 &= (w_2\otimes 1\otimes\cdots\otimes 1)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(w_2\otimes 1\otimes\cdots\otimes 1)\\
\beta_2 &= (1\otimes w_2\otimes\cdots\otimes 1)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(1\otimes w_2\otimes\cdots\otimes 1)\\
&\vdots\\
\beta_m &= (1\otimes 1\otimes\cdots\otimes w_2)\otimes(1\otimes\cdots\otimes 1)+(1\otimes\cdots\otimes 1)\otimes(1\otimes 1\otimes\cdots\otimes w_2)
\end{aligned}$$

We may show by an easy calculation that $\alpha_i s$ and $\beta_i s$ are in the kernel of $\cup : H^*(X)\otimes H^*(X)\longrightarrow H^*(X)$. Since $w_1^2\neq 0$ and $w_2\neq 0$, then calculation shows that

$$\alpha_1^3\alpha_2^3\cdots\alpha_m^3\beta_1\cdots\beta_m = w_1^3w_2\otimes w_1^3w_2\otimes\cdots\otimes w_1^3w_2\neq 0.$$

Consequently,

$$zcl(G_2(\mathbb{R}^4)\times G_2(\mathbb{R}^4)\times\cdots\times G_2(\mathbb{R}^4))\geq 3m+m=4m.$$

□

Corollary 3.11. *For any positive integer $m\geq 1$, we have:*

$$TC\,\underbrace{\left(G_2(\mathbb{R}^4)\times G_2(\mathbb{R}^4)\times\cdots\times G_2(\mathbb{R}^4)\right)}_{m-times}=4m+1.$$

Proof. The proof follows by Theorems 3.4 and Lemmas 3.8 and 3.10.
□

4 Lower and upper bounds on Topological complexity of certain real Grassmannian manifolds

In this section we calculate lower and upper bounds of Topological complexity of $G_2(\mathbb{R}^{2^p+1})$ and $G_2(\mathbb{R}^{2^p+2})$.

Theorem 4.1. *For any positive integer $p \geq 2$ we have:*

$$zcl(G_2(\mathbb{R}^{2^p+1})) \geq 3(2^p - 1).$$

Proof. Let $w_1, w_2 \in H^*(G_2(\mathbb{R}^{2^p+1}); \mathbb{Z}_2)$ be generators.Then $w_1^{2^{p+1}-2} \neq 0$, but $w_1^{2^{p+1}-1} = 0$ and $w_2^{2^p-1} \neq 0$ but $w_2^{2^p} = 0$. Let $\alpha, \beta \in H^*(G_2(\mathbb{R}^{2^p+1}) \otimes H^*(G_2(\mathbb{R}^{2^p+1})$ defined by:

$$\alpha = (w_1 \otimes 1) + (1 \otimes w_1), \qquad \beta = (w_2 \otimes 1) + (1 \otimes w_2).$$

By an easy calculation,

$$\alpha^{2^{p+1}-1} = (w_1^{2^{p+1}-2} \otimes w_1) + (w_1 \otimes w_1^{2^{p+1}-2}) + \cdots \neq 0,$$

$$\beta^{2^p-2} = (w_2^{2^p-2} \otimes 1) + (1 \otimes w_2^{2^p-2}) + (w_2^{2^p-4} \otimes w_2^2) + (w_2^2 \otimes w_2^{2^p-4}) + \cdots \neq 0,$$

$$\beta^{2^p-1} = (w_2^{2^p-1} \otimes 1) + (1 \otimes w_2^{2^p-1}) + (w_2^{2^p-2} \otimes w_2) + (w_2 \otimes w_2^{2^p-2}) + \cdots \neq 0.$$

Clearly α, β are in the kernel of $\cup : H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$. And the calculation shows that $\alpha^{2^{p+1}-1}\beta^{2^p-1} = 0$ but $\alpha^{2^{p+1}-1}\beta^{2^p-2} \neq 0$. Consequently,

$$zcl(G_2(\mathbb{R}^{2^p+1})) \geq (2^{p+1} - 1) + (2^p - 2) = 3(2^p - 1).$$

□

Corollary 4.2. *For any positive integer $p \geq 2$, we have:*

$$3(2^p) - 2 \leq TC(G_2(\mathbb{R}^{2^p+1})) \leq 2^{p+2} - 5.$$

Proof. It follows from Theorem 3.2, Theorem 3.3. □

Remark 4.3. If $p = 1$ then $TC(G_2(\mathbb{R}^3)) = 4$. Note that $G_2(\mathbb{R}^3)$ is infact $\mathbb{R}P^2$ wich is consistent with previous calculations. For $p = 2$, $10 \leq TC(G_2(\mathbb{R}^5)) \leq 11$. We see there is a gape between lower and upper bounds. For $p \geq 3$ we find a gape between lower and upper bounds by $2^p - 7$.

At the end we calculate topological complexity of $G_2(\mathbb{R}^{2^p+2})$ for $p \geq 2$ using the same method of Theorem 3.8, but we see there is a gape between lower and upper bounds.

Theorem 4.4. *For any positive integer, $p \geq 2$, we have:*

$$zcl(G_2(\mathbb{R}^{2^p+2})) \geq (2^{p+1} - 2) + (2^p + 1) = 2^{p+1} + 2^p - 1 = 3(2^p) - 1.$$

Proof. Let $w_1, w_2 \in H^*(G_2(\mathbb{R}^{2^p+2}); \mathbb{Z}_2)$ be generators. Clearly $w_1^{2^{p+1}-2} \neq 0$, $w_1^{2^{p+1}-1} = 0$ and $w_2^{2^p} \neq 0$, $w_2^{2^{p+1}} = 0$. Let $\alpha, \beta \in H^*(G_2(R^{2^p+2}) \otimes H^*(G_2(\mathbb{R}^{2^p+2})$, defined by:

$$\alpha = (w_1 \otimes 1) + (1 \otimes w_1), \qquad \beta = (w_2 \otimes 1) + (1 \otimes w_2).$$

By an easy calculation,

$$\alpha^{2^{p+1}-2} = (w_1^{2^{p+1}-2} \otimes 1) + (w_1^{2^{p+1}-4} \otimes w_1^2) + \cdots + (w_1^2 \otimes w_1^{2^{p+1}-4}) + (1 \otimes w_1^{2^{p+1}-2})$$

$$\alpha^{2^{p+1}-1} = (w_1^{2^{p+1}-1} \otimes 1) + (w_1^{2^{p+1}-2} \otimes w_1) + \cdots + (w_1 \otimes w_1^{2^{p+1}-2}) + (1 \otimes w_1^{2^{p+1}-1})$$

and also

$$\begin{aligned}
\beta^{2^p+1} &= (w_2^{2^p} \otimes w_2) + (w_2 \otimes w_2^{2^p}) \\
\beta^{2^p+2} &= (w_2^{2^p} \otimes w_2^2) + (w_2^2 \otimes w_2^{2^p}) \\
&\vdots \\
\beta^{2^{p+1}-1} &= (w_2^{2^p} \otimes w_2^{2^p-1}) + (w_2^{2^p-1} \otimes w_2^{2^p}).
\end{aligned}$$

Not that α, β are in the kernel of $\cup : H^*(X) \otimes H^*(X) \longrightarrow H^*(X)$. And best possbility for zero cup-length comes from the element $\alpha^{2^{p+1}-2}\beta^{2^p+1} \neq 0$. Consequently,

$$zcl(G_2(\mathbb{R}^{2^p+2})) \geq (2^{p+1} - 2) + (2^p + 1) = 3(2^p) - 1$$

. □

Corollary 4.5. *For any positive integer, $p \geq 2$, we have:*

$$3(2^p) \leq TC(G_2(\mathbb{R}^{2^p+2})) \leq 2^{p+2} - 3.$$

Proof. It follows from Theorems 3.2, 3.3, 4.4. □

Remark 4.6. For $p = 2$, $12 \leq TC(G_2(\mathbb{R}^6)) \leq 13$, We see there is a gape between lower and upper bounds. For $p \geq 3$ we find a gape between lower and upper bound by $2^p - 3$.

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