A Note On LS-Category And Topological Complexity Of Real Grassmannian Manifolds

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Abstract. Let $G_{k,n}$ be the Grassmann manifold of $k$-planes in $\mathbb{R}^{n+k}$. The Lusternik-Schnirelmann category and topological complexity are important invariants of topological spaces. In this note we calculate the Lusternik-Schnirelmann category and topological complexity of certain products of Grassmannian manifolds by using cup and zero-cup length. Also we will find the lower and upper bounds of the topological complexity of some Grassmannian manifolds by the same method.

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1 Introduction

In 1934, L. Lusternik and L. Schnirelmann described a new invariant of a manifold called category. Their purpose in creating this concept was to obtain a lower bound on the number of critical points for each
smooth function on the manifold. This category examines the important concepts of geometry and dynamical systems. The topological complexity is a numerical homotopy invariant, introduced by M. Farber in 2001, and in [5], [6], [7] he examined the topological complexity of the robotics. Topological complexity has close relationship to classical invariant, Lusternik-Schnirelmann category. In [1] we have studied the product of projective spaces by here we are going to study real Grassmannian manifolds.

In Section 2 we calculate by known results the category of products of $G_2(\mathbb{R}^{2p+1})$ and $G_2(\mathbb{R}^{2p+2})$. In Section 3 first we calculate the topological complexity of $G_2(\mathbb{R}^3)$ and $G_2(\mathbb{R}^4)$ by different method in [11] following the products of them. In Section 4 we give upper and lower bounds for topological complexity of certain Grassmanian manifolds. Specially we show that $10 \leq TC(G_2(\mathbb{R}^5)) \leq 11$ and $12 \leq TC(G_2(\mathbb{R}^6)) \leq 13$.

**Definition 1.1.** The Lusternik-Schnirelmann category of a space $X$ is the least integer $n$ such that there exists an open covering $U_1, \cdots, U_{n+1}$ of $X$ with each $U_i$ contractible to a point in the space $X$. We denote this by $cat(X) = n$ and we call such a covering $U_i$ categorical. If no such integer exists, we write $cat(X) = \infty$.

In [5], Michael Farber, defined a numerical invariant $TC(X)$. We may outlined as follows: Let $PX$ denote the space of all continuous paths $\gamma : [0,1] \to X$ in $X$ and $\pi : PX \to X \times X$ denotes the map associating to any path $\gamma \in PX$ the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space $PX$ with the compact-open topology.

**Definition 1.2.** The topological complexity of a path-connected space $X$, denoted by $TC(X)$, is the least integer $n$ such that the Cartesian product $X \times X$ can be covered with $n$ open subsets $U_i$, $X \times X = U_1 \cup U_2 \cup \cdots \cup U_n$ such that for any $i = 1, 2, \cdots, n$ there exists a continuous local section $s_i : U_i \to PX$ of $\pi$, that is, $\pi \circ s_i = id$ over $U_i$. If no such $m$ exists we will set $TC(X) = \infty$.

**Theorem 1.3.** Let $G_{k,n}$ denote the Grassmann manifold of $k$-planes in $\mathbb{R}^{n+k}$. Then $H^*(G_{k,n}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k]/I_{k,n}$ where $I_{k,n}$ is the ideal generated by the dual Stiefel-Whitney classes $w_{n+1}, \ldots, w_{n+k}$. 


Remark 1.4. The set \( \{ w^a w^b : a + b \leq n \} \) is vector space basis for the cohomology ring \( H^*(G_{2,n}; \mathbb{Z}_2) \).

2 LS-category of the products of \( G_2(\mathbb{R}^{2p+1}) \), \( G_2(\mathbb{R}^{2p+2}) \)

This section is devoted to calculate LS-category of certain products of real Grassmannian manifolds by using cup-length.

Definition 2.1. Let \( R \) be a commutative ring and \( X \) be a space. The cup-length of \( X \) with coefficients in \( R \) is the least integer \( k \) (or \( \infty \)) such that all \( (k+1) \)-fold cup products vanish in the reduced cohomology \( \tilde{H}^*(X; R) \); we denote this integer by \( \text{cup}_R(X) \).

Proposition 2.2. The \( R \)-cuplength of a space is less than or equal to the category of the space for all coefficients \( R \). In notation, we write \( \text{cup}_R(X) \leq \text{cat}(X) \).

Proof. See Proposition 1.5 in [4]. □

Theorem 2.3. For a path-connected locally contractible paracompact space, \( \text{cat}(X) \leq \text{dim}(X) \).

Proof. See Theorem 1.7 in [4]. □

Example 2.4. Since \( H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[a]/(a^{n+1}) \) with \( \text{deg}(a) = 1 \). Since \( a^n \neq 0 \), then \( \text{cup}(\mathbb{RP}^n) = n \leq \text{cat}(\mathbb{RP}^n) \leq \text{dim}(\mathbb{RP}^n) = n \). Thus, \( \text{cat}(\mathbb{RP}^n) = n \).

Theorem 2.5. Suppose \( X \) and \( Y \) are path-connected spaces such that \( X \times Y \) is completely normal. Then \( \text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y) \).

Proof. See Theorem 1.37 in [4]. □

Theorem 2.6. If \( X \) is a closed, connected \( n \)-manifold with \( \pi_1(X) \approx \mathbb{Z}_2 \), then \( \text{cat}(X) = \text{dim}(X) \) iff \( w^{\text{dim}(X)} \neq 0 \), where \( w \) is the nonzero element of \( H^1(X; \mathbb{Z}_2) \).
Proof. See a proof [2, 10]. □

From Theorem 2.6 we have the following corollary.

**Corollary 2.7.** $w^{\dim(X)} = 0$ if and only if $\text{cat}(X) < \dim(X)$.

**Theorem 2.8.** For any positive integers $p \geq 1$, we have:

$$\text{cat}(G_2(\mathbb{R}^{2p+1})) = 2^{p+1} - 2.$$ 


**Theorem 2.9.** For any positive integer $p_i \geq 1, m \geq 1$, we have:

$$\text{cat}(G_2(\mathbb{R}^{2p_1+1}) \times \cdots \times G_2(\mathbb{R}^{2p_m+1})) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m$$

Proof. Since $H^*(G_{k,n}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k]/I_{k,n}$, by Künneth formulas

$$H^*(G_2(\mathbb{R}^{2p_1+1}) \times \cdots \times G_2(\mathbb{R}^{2p_m+1})) = H^*(G_2(\mathbb{R}^{2p_1+1})) \otimes \cdots \otimes H^*(G_2(\mathbb{R}^{2p_m+1})) = \mathbb{Z}_2[w_1, w_2]/(\bar{w}_{2p_1}, \bar{w}_{2p_1+1}) \otimes \cdots \otimes \mathbb{Z}_2[w_1, w_2]/(\bar{w}_{2p_m}, \bar{w}_{2p_m+1}),$$

Since $\text{cat}(G_2(\mathbb{R}^{2p_1+1})) = \dim(G_2(\mathbb{R}^{2p_1+1}))$, then by Theorem 2.6; $w_1^{\dim} \neq 0$ and $w_1^{\dim+1} = 0$. Set,

$$\alpha_1 = w_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1$$
$$\alpha_2 = 1 \otimes w_1 \otimes 1 \otimes \cdots \otimes 1$$
$$\vdots$$
$$\alpha_m = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w_1.$$

Thus

$$\alpha_1^{2^{p_1+1}-2} = w_1^{2^{p_1+1}-2} \otimes \cdots \otimes 1$$
$$\alpha_2^{2^{p_2+1}-2} = 1 \otimes w_1^{2^{p_2+1}-2} \otimes \cdots \otimes 1$$
$$\vdots$$
$$\alpha_m^{2^{p_m+1}-2} = 1 \otimes 1 \otimes \cdots \otimes w_1^{2^{p_m+1}-2}.$$
Therefore
\[ \alpha_1^{2^{p_1+1}-2} \alpha_2^{2^{p_2+1}-2} \cdots \alpha_m^{2^{p_m+1}-2} = w_1^{2^{p_1+1}-2} \otimes w_1^{2^{p_2+1}-2} \otimes \cdots \otimes w_1^{2^{p_m+1}-2} \neq 0. \]

From which,
\[ \cup_{Z_2}(G_2(R^{2^{p_1+1}}) \times G_2(R^{2^{p_2+1}}) \times \cdots \times G_2(R^{2^{p_m+1}})) \geq (2^{p_1+1} - 2) + (2^{p_2+1} - 2) + \cdots + (2^{p_m+1} - 2) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m. \]

On the other hand, By Theorem 2.5,
\[ \text{cat}(G_2(R^{2^{p_1+1}}) \times G_2(R^{2^{p_2+1}}) \times \cdots \times G_2(R^{2^{p_m+1}})) \leq 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m. \]

Now by Proposition 2.2,
\[ \text{cat}(G_2(R^{2^{p_1+1}}) \times G_2(R^{2^{p_2+1}}) \times \cdots \times G_2(R^{2^{p_m+1}})) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - 2m. \]

Corollary 2.10. For any positive integer \( p \), we have;
\[ \text{cat}(G_2(R^{2^{p+1}}) \times G_2(R^{2^{p+1}}) \times \cdots \times G_2(R^{2^{p+1}})) = m(2^{p+1} - 2). \]

Theorem 2.11. For any positive integer \( p \), \( \text{cat}(G_2(R^{2^{p+2}})) = 2^{p+1} - 1. \)


Theorem 2.12. For any positive integer \( p_i \geq 1, m \geq 1 \), we have:
\[ \text{cat}(G_2(R^{2^{p_1+2}}) \times G_2(R^{2^{p_2+2}}) \times \cdots \times G_2(R^{2^{p_m+2}})) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - m. \]

Proof. Since \( H^*(G_{k,n}; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k]/I_{k,n} \), by Künneth formulas

\[ H^*(G_2(R^{2^{p_1+2}}) \times \cdots \times G_2(R^{2^{p_m+2}})) = \mathbb{Z}_2[w_1, w_2]/\langle \bar{w}_{2^{p_1+1}}, \bar{w}_{2^{p_1+2}} \rangle \otimes \cdots \otimes \mathbb{Z}_2[w_1, w_2]/\langle \bar{w}_{2^{p_m+1}}, \bar{w}_{2^{p_m+2}} \rangle. \]
Where
\[ \bar{w}_{2^{p_i}+1} = w_1^{2^{p_i}+1} + \cdots + w_1 w_2^{2^{p_i}-1}; \]
\[ \bar{w}_{2^{p_i}+2} = w_1^{2^{p_i}+2} + w_1^{2^{p_i}} w_2 + \cdots + w_2^{2^{p_i}-1+1}. \]

Since \( \text{cat}(G_2(\mathbb{R}^{2^{p_i}+2})) < \text{dim}(G_2(\mathbb{R}^{2^{p_i}+2})) \), then by Corollary 2.7, \( w_1^{2^{p_i}+1-1} = 0 \) but \( w_1^{2^{p_i}+1-2} \neq 0 \). Set:

\[ \alpha_1 = \ w_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \]
\[ \alpha_2 = \ 1 \otimes w_1 \otimes 1 \otimes \cdots \otimes 1 \]
\[ \vdots \]
\[ \alpha_m = \ 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w_1. \]

Thus

\[ \alpha_1^{2^{p_i}+1-2} = \ w_1^{2^{p_i}+1-2} \otimes \cdots \otimes 1 \]
\[ \alpha_2^{2^{p_i}+1-2} = \ 1 \otimes w_1^{2^{p_i}+1-2} \otimes \cdots \otimes 1 \]
\[ \vdots \]
\[ \alpha_m^{2^{p_i}+1-2} = \ 1 \otimes 1 \otimes \cdots \otimes w_1^{2^{p_i}+1-2}. \]

Also let,

\[ \beta_1 = \ w_2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \]
\[ \beta_2 = \ 1 \otimes w_2 \otimes 1 \otimes \cdots \otimes 1 \]
\[ \vdots \]
\[ \beta_m = \ 1 \otimes 1 \otimes \cdots \otimes 1 \otimes w_2. \]

Therefore for \( i = 1, \cdots, m \)

\[ \alpha_1^{2^{p_i}+1-2} \cdots \alpha_m^{2^{p_i}+1-2} \beta_1 \cdots \beta_m = w_1^{2^{p_i}+1-2} w_2 \otimes w_1^{2^{p_i}+1-2} w_2 \otimes \cdots \otimes w_1^{2^{p_i}+1-2} w_2 \neq 0. \]

From which,

\[ \cup_{\mathbb{F}_p} (G_2(\mathbb{R}^{2^{p_1}+2}) \times \cdots \times G_2(\mathbb{R}^{2^{p_m}+2})) \geq (2^{p_1+1} - 1) + \cdots + (2^{p_m+1} - 1) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - m. \]
Now by Theorem 2.5 and Proposition 2.2 we have
\[
\text{cat}(G_2(\mathbb{R}^{2p_1+2}) \times \cdots \times G_2(\mathbb{R}^{2p_m+2})) = 2^{p_1+1} + 2^{p_2+1} + \cdots + 2^{p_m+1} - m.
\]
□

**Corollary 2.13.** For any positive integer \( p \), we have:
\[
\text{cat}(\underbrace{G_2(\mathbb{R}^{2p+2}) \times \cdots \times G_2(\mathbb{R}^{2p+2})}_{m \text{-times}}) = m(2^{p+1} - 1).
\]

### 3 Topological complexity of products of \( G_2(\mathbb{R}^3) \), \( G_2(\mathbb{R}^4) \)

In this section we will calculate the topological complexity of \( G_2(\mathbb{R}^3) \), \( G_2(\mathbb{R}^4) \) following the product of them. We briefly recall a result from [3] giving a lower bound on \( TC(X) \). It is quite useful since it allows us an effective computation of \( TC(X) \) in many examples. A lower bound for topological complexity is obtained by using the zero-divisor-cup-length of \( X \).

**Definition 3.1.** Let \( k \) be a field. The kernel of homomorphism
\[
\cup : H^*(X; k) \otimes H^*(X; k) \longrightarrow H^*(X; k)
\]
is called the ideal of the zero-divisors of \( H^*(X; k) \). The zero-divisors-cup-length of \( H^*(X; k) \) is the length of the longest nontrivial product in the ideal of the zero-divisors of \( H^*(X; k) \). This number will be denoted by \( zcl(X) \).

**Theorem 3.2.** The number \( TC(X) \) is greater than the zero-divisors-cup-length of \( H^*(X; K) \).

**Proof.** See Theorem 7 in [6]. □

**Theorem 3.3.** If \( X \) is path-connected and paracompact then
\[
\text{cat}(X) \leq TC(X) \leq 2\cdot \text{cat}(X) - 1.
\]
Proof. See Theorem 5 in [6].  

Theorem 3.4. For any path-connected metric spaces $X$ and $Y$,

$$TC(X \times Y) \leq TC(X) + TC(Y) - 1.$$ 

Proof. See Theorem 11 in [6].  

Lemma 3.5. $TC(G_2(\mathbb{R}^3)) = 4$.

Proof. Since $G_2(\mathbb{R}^3)$ is in fact $\mathbb{R}P^2$, so by [5], $TC(\mathbb{R}P^2) = 4 = TC(G_2(\mathbb{R}^3))$.  

We may give another proof with the method of zero divisor cup length.  

Since $H^*((G_2(\mathbb{R}^3)); Z_2) = Z_2[w_1, w_2]/\langle \bar{w}_2, \bar{w}_3 \rangle$ and $\bar{w}_2 = w_1^2 + w_2, \bar{w}_3 = w_1^3$, we have $H^*((G_2(\mathbb{R}^3)); Z_2) = Z_2[w_1, w_2]/\langle w_1^2 + w_2, w_1^3 \rangle$.  

Now define $\alpha, \beta \in H^*(G_2(\mathbb{R}^3) \otimes H^*(G_2(\mathbb{R}^3))$, by: $\alpha = (w_1 \otimes 1) + (1 \otimes w_1)$, $\beta = (w_2 \otimes 1) + (1 \otimes w_2)$.  

Since $\alpha^2 = (w_1^2 \otimes 1) + (1 \otimes w_1^2)$, $\alpha^3 = (w_1^3 \otimes 1) + (1 \otimes w_1^3)$, $\beta^2 = (w_2^2 \otimes 1) + (1 \otimes w_2^2) = 0$, but $\alpha^3 \beta = 0$ on the other hand $\alpha^2 \beta = (w_1^2 \otimes w_2) + (w_2 \otimes w_1^2) \neq 0$ consequently $zcl(G_2(\mathbb{R}^3)) \geq 3$, by Theorem 3.3, $3 < TC(G_2(\mathbb{R}^3)) \leq 4$, as a result $TC(G_2(\mathbb{R}^3)) = 4$. 

Lemma 3.6. For any positive integer $m$, we have:

$$zcl\left(\underbrace{G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3)}_{m \text{-times}}\right) \geq 3m.$$ 

Proof. Remember by Theorem 2.6, $w_1^2 \neq 0$. Let $\alpha_i, \beta_i \in H^*(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3)) \otimes H^*(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3))$, for $i = 1, 2, \cdots, m$, defined by:

$$\begin{align*}
\alpha_1 &= (w_1 \otimes 1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (w_1 \otimes 1 \otimes \cdots \otimes 1), \\
\alpha_2 &= (1 \otimes w_1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes w_1 \otimes \cdots \otimes 1), \\
& \vdots \\
\alpha_m &= (1 \otimes 1 \otimes \cdots \otimes w_1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes 1 \otimes \cdots \otimes w_1) \\
\text{and} \\
\beta_1 &= (w_2 \otimes 1 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (w_2 \otimes 1 \otimes \cdots \otimes 1), \\
\beta_2 &= (1 \otimes w_2 \otimes \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes w_2 \otimes \cdots \otimes 1), \\
& \vdots \\
\beta_m &= (1 \otimes 1 \otimes \cdots \otimes w_2) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes 1 \otimes \cdots \otimes w_2). 
\end{align*}$$
We may show by easy calculation that $\alpha_i$ and $\beta_i$ are in the kernel of $\cup : H^*(X) \otimes H^*(X) \to H^*(X)$. Clearly $\alpha_i^2 \neq 0$ and calculation shows that

$$\alpha_2^2 \alpha_2 \cdots \alpha_m^2 \beta_1 \cdots \beta_m = w_1^2 w_2 \otimes w_1^2 w_2 \otimes \cdots \otimes w_1^2 w_2 \neq 0.$$ 

Consequently,

$$zcl(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3)) \geq 2m + m = 3m.$$

\[\square\]

**Theorem 3.7.** For any positive integer $m \geq 1$, we have:

$$TC(G_2(\mathbb{R}^3) \times G_2(\mathbb{R}^3) \times \cdots \times G_2(\mathbb{R}^3)) = 3m + 1.$$ 

**Proof.** This proof follows by Theorems 3.4 and Lemmas 3.5, 3.6. \[\square\]

**Lemma 3.8.** $TC(G_2(\mathbb{R}^4)) = 5$.

**Proof.** First, we calculate the zero divisor cup length of $G_2(\mathbb{R}^4)$. Since $H^*((G_2(\mathbb{R}^4)); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/(\bar{w}_3, \bar{w}_4)$ and $\bar{w}_3 = w_3^1 + w_1^2 w_2 + w_2^1$, we have $H^*((G_2(\mathbb{R}^4)); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2]/(w_3, w_1^2 w_2 + w_2^1)$. Now let $\alpha, \beta \in H^*(G_2(\mathbb{R}^4)) \otimes H^*(G_2(\mathbb{R}^4))$, defined by:

$$\alpha = (w_1 \otimes 1) + (1 \otimes w_1), \quad \beta = (w_2 \otimes 1) + (1 \otimes w_2).$$

By an easy calculation we see that

$$\alpha^3 \beta = (w_1^2 w_2 \otimes w_1) + (w_1^2 \otimes w_1 w_2) + (w_1 w_2 \otimes w_1^2) + (w_1 \otimes w_1^2 w_2) \neq 0.$$ 

Consequently $zcl(G_2(\mathbb{R}^4)) \geq 4$, on the other hand by Theorem 3.3, 4 $< TC(G_2(\mathbb{R}^4)) \leq 5$, as a result $TC(G_2(\mathbb{R}^4)) = 5$. \[\square\]

K. J. Pearson and Tan Zhang in [11] used the equality $TC(X) = cat(X \times X)$, to compute the topological complexity of $G_2(\mathbb{R}^4)$, which is not true in general. In fact we have $TC(X) \leq cat(X \times X)$. See the following example.

**Example 3.9.** Let $X = G_2(\mathbb{R}^4)$ by Lemma 3.11 $TC(X) = 5$ and by Corollary 2.13 $cat(X \times X) = 6$. This shows that the equality $TC(X) = cat(X \times X)$ is not true in general.
Lemma 3.10. For any positive integer \( m \), we have:
\[
\text{zcl} \left( G_2(\mathbb{R}^4) \times G_2(\mathbb{R}^4) \times \cdots \times G_2(\mathbb{R}^4) \right) \geq 4m.
\]
\[
\text{m-times}
\]

Proof. Let \( \alpha_i, \beta_i \in H^*(G_2(\mathbb{R}^4) \times \cdots \times G_2(\mathbb{R}^4)) \otimes H^*(G_2(\mathbb{R}^4) \times \cdots \times G_2(\mathbb{R}^4)) \), for \( i = 1, 2, \cdots, m \), defined by:
\[
\begin{align*}
\alpha_1 &= (w_1 \otimes 1 \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (w_1 \otimes 1 \cdots \otimes 1) \\
\alpha_2 &= (1 \otimes w_1 \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes w_1 \cdots \otimes 1) \\
&\vdots \\
\alpha_m &= (1 \otimes 1 \cdots \otimes w_1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes 1 \cdots \otimes w_1)
\end{align*}
\]

and
\[
\begin{align*}
\beta_1 &= (w_2 \otimes 1 \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (w_2 \otimes 1 \cdots \otimes 1) \\
\beta_2 &= (1 \otimes w_2 \cdots \otimes 1) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes w_2 \cdots \otimes 1) \\
&\vdots \\
\beta_m &= (1 \otimes 1 \cdots \otimes w_2) \otimes (1 \otimes \cdots \otimes 1) + (1 \otimes \cdots \otimes 1) \otimes (1 \otimes 1 \cdots \otimes w_2)
\end{align*}
\]

We may show by an easy calculation that \( \alpha_i \)'s and \( \beta_i \)'s are in the kernel of \( \cup : H^*(X) \otimes H^*(X) \rightarrow H^*(X) \). Since \( w_1^2 \neq 0 \) and \( w_2 \neq 0 \), then calculation shows that
\[
\alpha_1^3 \alpha_2^3 \cdots \alpha_m^3 \beta_1 \cdots \beta_m = w_1^3 w_2 \otimes w_1^3 w_2 \otimes \cdots \otimes w_1^3 w_2 \neq 0.
\]

Consequently,
\[
\text{zcl}(G_2(\mathbb{R}^4) \times G_2(\mathbb{R}^4) \times \cdots \times G_2(\mathbb{R}^4)) \geq 3m + m = 4m.
\]
\[
\square
\]

Corollary 3.11. For any positive integer \( m \geq 1 \), we have:
\[
\text{TC} \left( G_2(\mathbb{R}^4) \times G_2(\mathbb{R}^4) \times \cdots \times G_2(\mathbb{R}^4) \right) \text{m-times} = 4m + 1.
\]

Proof. The proof follows by Theorems 3.4 and Lemmas 3.8 and 3.10.
\[
\square
\]
4 Lower and upper bounds on Topological complexity of certain real Grassmannian manifolds

In this section we calculate lower and upper bounds of Topological complexity of \(G_2(\mathbb{R}^{2p+1})\) and \(G_2(\mathbb{R}^{2p+2})\).

**Theorem 4.1.** For any positive integer \(p \geq 2\) we have:

\[ zcl(G_2(\mathbb{R}^{2p+1})) \geq 3(2^p - 1). \]

**Proof.** Let \(w_1, w_2 \in H^*(G_2(\mathbb{R}^{2p+1}); \mathbb{Z}_2)\) be generators. Then \(w_1^{2p+1-2} \neq 0\), but \(w_1^{2p+1-1} = 0\) and \(w_2^{2p-1} \neq 0\) but \(w_2^{2p} = 0\). Let \(\alpha, \beta \in H^*(G_2(\mathbb{R}^{2p+1}) \otimes H^*(G_2(\mathbb{R}^{2p+1}))\) defined by:

\[ \alpha = (w_1 \otimes 1) + (1 \otimes w_1), \quad \beta = (w_2 \otimes 1) + (1 \otimes w_2). \]

By an easy calculation,

\[ \alpha^{2p+1-1} = (w_1^{2p+1-2} \otimes w_1) + (w_1 \otimes w_1^{2p+1-2}) + \cdots \neq 0, \]

\[ \beta^{2p-2} = (w_2^{2p-2} \otimes 1) + (1 \otimes w_2^{2p-2}) + (w_2^{2p-4} \otimes w_2^2) + (w_2^2 \otimes w_2^{2p-4}) + \cdots \neq 0, \]

\[ \beta^{2p-1} = (w_2^{2p-1} \otimes 1) + (1 \otimes w_2^{2p-1}) + (w_2^{2p-2} \otimes w_2) + (w_2 \otimes w_2^{2p-2}) + \cdots \neq 0. \]

Clearly \(\alpha, \beta\) are in the kernel of \(\cup : H^*(X) \otimes H^*(X) \to H^*(X)\). And the calculation shows that \(\alpha^{2p+1-1} \beta^{2p-1} = 0\) but \(\alpha^{2p+1-1} \beta^{2p-2} \neq 0\). Consequently,

\[ zcl(G_2(\mathbb{R}^{2p+1})) \geq (2^p + 1 - 1) + (2^p - 2) = 3(2^p - 1). \]

\[ \Box \]

**Corollary 4.2.** For any positive integer \(p \geq 2\), we have:

\[ 3(2^p) - 2 \leq TC(G_2(\mathbb{R}^{2p+1})) \leq 2^{2p+2} - 5. \]

**Proof.** It follows from Theorem 3.2, Theorem 3.3. \( \Box \)
Remark 4.3. If $p = 1$ then $TC(G_2(\mathbb{R}^3)) = 4$. Note that $G_2(\mathbb{R}^3)$ is infact $\mathbb{R}P^2$ which is consistent with previous calculations. For $p = 2$, $10 \leq TC(G_2(\mathbb{R}^5)) \leq 11$. We see there is a gape between lower and upper bounds. For $p \geq 3$ we find a gape between lower and upper bounds by $2^p - 7$.

At the end we calculate topological complexity of $G_2(\mathbb{R}^{2p+2})$ for $p \geq 2$ using the same method of Theorem 3.8, but we see there is a gape between lower and upper bounds.

**Theorem 4.4.** For any positive integer, $p \geq 2$, we have:

$$zcl(G_2(\mathbb{R}^{2p+2})) \geq (2^{p+1} - 2) + (2^p + 1) = 2^{p+1} + 2^p - 1 = 3(2^p) - 1.$$

**Proof.** Let $w_1, w_2 \in H^*(G_2(\mathbb{R}^{2p+2}); \mathbb{Z}_2)$ be generators. Clearly $w_1^{2^{p+1} - 2} \neq 0$, $w_1^{2^{p+1} - 1} = 0$ and $w_2^{2p} \neq 0$, $w_2^{2^{p+1}} = 0$. Let $\alpha, \beta \in H^*(G_2(R^{2p+2}) \otimes H^*(G_2(\mathbb{R}^{2p+2}))$, defined by:

$$\alpha = (w_1 \otimes 1) + (1 \otimes w_1), \quad \beta = (w_2 \otimes 1) + (1 \otimes w_2).$$

By an easy calculation,

$$\alpha^{2^{p+1} - 2} = (w_1^{2^{p+1} - 2} \otimes 1) + (w_1^{2^{p+1} - 4} \otimes w_1^2) + \cdots + (w_1^2 \otimes w_1^{2^{p+1} - 4}) + (1 \otimes w_1^{2^{p+1} - 2})$$

$$\alpha^{2^{p+1} - 1} = (w_1^{2^{p+1} - 1} \otimes 1) + (w_1^{2^{p+1} - 2} \otimes w_1) + \cdots + (w_1 \otimes w_1^{2^{p+1} - 2}) + (1 \otimes w_1^{2^{p+1} - 1})$$

and also

$$\beta^{2^{p+1}} = (w_2^{2p} \otimes w_2) + (w_2 \otimes w_2^{2p})$$

$$\beta^{2^{p+2}} = (w_2^{2p} \otimes w_2^2) + (w_2^2 \otimes w_2^{2p})$$

$$\vdots$$

$$\beta^{2^{p+1} - 1} = (w_2^{2p} \otimes w_2^{2^{p-1}}) + (w_2^{2^{p-1}} \otimes w_2^{2p}).$$

Not that $\alpha, \beta$ are in the kernel of $\cup : H^*(X) \otimes H^*(X) \to H^*(X)$. And best possbility for zero cup-length comes from the element $\alpha^{2^{p+1} - 2} \beta^{2^{p+1}} \neq 0$. Consequently,

$$zcl(G_2(\mathbb{R}^{2p+2})) \geq (2^{p+1} - 2) + (2^p + 1) = 3(2^p) - 1.$$

$\Box$
Corollary 4.5. For any positive integer, $p \geq 2$, we have:

$$3(2^p) \leq TC(G_2(\mathbb{R}^{2p+2})) \leq 2^{p+2} - 3.$$ 

Proof. It follows from Theorems 3.2, 3.3, 4.4. □

Remark 4.6. For $p = 2$, $12 \leq TC(G_2(\mathbb{R}^6)) \leq 13$, We see there is a gap between lower and upper bounds. For $p \geq 3$ we find a gap between lower and upper bound by $2^p - 3$.

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References


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