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A Hybrid DHFEA/AHP Method for Ranking Units with Hesitant Fuzzy Data

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Abstract. One of the attractive subjects in decision analysis is the investigating of the uncertain data which is inevitable in many real-world applications. Fuzzy sets theory has been introduced to investigate the uncertain data which formulates the uncertainty by using the membership functions. However, in many real world applications, it is difficult to determine the exact amount of the membership value and so the skepticism can be raised during the decision-making process. The new perspective manages the uncertainty caused by the skepticism and in this case, the most important issues are to collect the hesitant fuzzy information and to select the optimal alternative. This study develops the deviation-oriented hesitant fuzzy envelopment analysis (DHFEA) based on deviation values. The hybrid fuzzy DHFEA/AHP approach derives the AHP pair-wise comparisons by hesitant fuzzy DEA and utilizes AHP to fully rank units. It shows that the proposed approach generates a logical ranking of units that has perfect compatibility with

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hesitant fuzzy DEA ranking and there is no any form of subjective analysis engaged within the methodology. The potential application of the proposed method is illustrated with a numerical example.

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Keywords and Phrases: Hesitant fuzzy envelopment analysis, Efficiency, Analytic Hierarchy Process, Ranking.

1 Introduction

Data envelopment analysis (DEA) is a non-parametric methodology for assessing the relative efficiency of decision making units (DMUs) with multiple inputs and multiple outputs, which was presented by Charnes et al. [7] and Banker et al. [5]. DEA has been used in many application areas such as the technical efficiency analysis [9, 4] and the measurement of banks' effectiveness [19] and the measurement of Stochastic efficiency with correlated data [15]. Emrouznejad and Yang [10] reported DEA studies from 1978 to end of 2016. Tone [27] proposed a slack-based method, nemed SBM, to evaluate the units. Since the units may get the identical efficiency scores, therefore, the classical DEA models may not be able to discriminate among them. In this regard, several ranking methods have been proposed in the DEA literature. See Adler et al. [1] for more studies about the ranking methods in DEA.
The fuzzy sets theory was initially introduced by Lotfi zadeh [35] which is widely used in many real world applications[18, 36]. The introducing of fuzzy sets provided a new viewpoint to deal with the data uncertainty in an evaluation process. Since then, a large amount of studies has been done in fuzzy sets theory and practice. For example, the type-2 fuzzy set [35], the intuitionistic fuzzy set [3], the hesitant fuzzy set [29], the hesitant fuzzy linguistic term set [22], the interval-valued hesitant fuzzy set [11], the triangular hesitant fuzzy set [32], the interval-valued intuitionistic hesitant fuzzy set [6], the interval-valued dual hesitant fuzzy set [20], and the generalizations of the hesitant fuzzy set, such as the dual hesitant fuzzy set [33], the hesitant probabilistic fuzzy set [31]. Hence, the hesitant fuzzy sets (HFS) and their expanded forms are attractive subjects. This study develops a fuzzy DEA model and uses AHP method to rank the units. Also, Saaty [23] proposed the Analytic Hierarchy Process (AHP) method by expanding the existing methods and combining

them with multi-criteria decision-making.
On the other hand, DEA has developed in many directions and in numerous Applications. Whereas all the DEA basic models divide DMUs into two groups: efficient DMUs and inefficient DMUs, and lack of discrimination of efficient units is a serious problem. There have been attempts to fully rank DMUs in the context of DEA during the last researches. Some researchers attempt to fully rank DMUs in DEA utilizes AHP. Shang and Sueyoshi [25] used the subjective AHP results in DEA for selection of a flexible manufacturing system. However, this approach has the limitations of both methods, the subjectivity of AHP and the Pareto solutions of DEA.
Sinuany-Stern et al. [26] formulated a combination model to evaluate and rank the DMUs; and the Sinuany-Stern's model obtain pair-wise comparison of DMUs without using other DMUs, and also its ranking is incompatible with traditional model in DEA when there are multiple inputs and outputs. while in our propose model the evaluation of each decision making unit pair is obtained by comparing to the function of all the decision making units. Thus, there is no subjective evaluation [26]. This method was a new idea in ranking but has lots of problems. The most principal problem is that its ranking is incompatible with the traditional model in DEA when there are multiple inputs and outputs, and this incompatibility causes some efficient units to be ranked lower than inefficient units. To cover the incompatibility in [26], developed AHP/DEA approaches were studied by some researchers in [2, 21].
Cheng-Kai et al. [8] presented AHP/DEA for fully ranking DMUs with multiple fuzzy criteria.
Table 1 summaries the integrated AHP/DEA approaches proposed in literature.

Due to the vagueness involved in the real-world decision-making problems, different fuzzy modeling approaches are introduced. The fuzzy DEA (FDEA) models have been developed by some scholars for investigating the data uncertainty [16]. Recently, Hatami-Marbini et al. [13] and Liu and Lee [17] proposed the cross-efficiency evaluation method in FDEA. Recently, Hosseinazeh Lotfi et al. [14] Introduced the data envelopment analysis and fuzzy sets. HFS and DEA can be considered as the effective decision-making tools. Although the fuzzy sets and the

Table 1: Literature on the integrated AHP/DEA

Overview	Strength	Weakness
Shang and Sueyoshi [25]	This research attempts to fully rank DMUs in DEA utilizing AHP.	It includes the subjectivity of AHP and Pareto solutions.
Sinuany-Stern et al. [26]	The AHP pair-wise comparisons are generated by running pair-wise DEA. Thus, there is no subjective evaluation.	Its ranking is incompatible with traditional model in DEA when there are multiple inputs and outputs.
Alirezaee and Sani [2]	This improved approach overcomes the draw-backs of the AHP/DEA method in [26]	This AHP/DEA method can not reflect the vagueness of human thought while ranking units with multiple fuzzy criteria.
Rakhshan et al. [21]	The proposed approach generates the ranking of units which is compatible with traditional DEA ranking.	It has the limitation on dealing with human thoughts with uncertainty in the real world applications.
Cheng-Kai et al. [8]	This work combining fuzzy DEA and AHP is proposed to rank units with multiple fuzzy criteria.	This paper used the fuzzy information. But by providing more uncertain information, decision making becomes more flexible.

related models are flexible due to the assessment of units in the case of the data uncertainty, but they do not propose approaches to rank all units. DEA models consider the inputs and outputs to evaluate the DMUs and classify them into efficient and inefficient categories. On the other hand, it may not be possible to report the data as the certain data, for example, there is not enough time to access this type of data. Therefore, among the decision-making methods, the hesitant fuzzy envelopment analysis (HFEA) method eliminates the above mentioned drawbacks and improves the decision-making process by creating a connection between the HFS and DEA models. In this way, Recently, Zhou et al. [37] proposed HFEA model by combining the priority of criterion. Their proposed model was named the hesitant fuzzy priority envelopment analysis (HFPEA) model. Although this method can tackle the ranking problems with consider subjective criteria, it transformed subjective variables into prioritize, which may cause loss or distortion of information, therefore, we felt the need for further research to deal with the problems of decision making by subjective variables which can overcome this short coming. Although HFS models have been extensively developed, but the combination of HFS and DEA has not been widely reported. As a result, we were motivated to overcome the problems of the pervious methods of AHP/DEA and HFPEA, by presenting the proposed method under hesitant conditions with logical calculations.

This paper aims to establish a relationship between these two decision-making tools and uses them to solve the optimization problems. For this purpose, we develop a HFEA model and combine it with AHP method. The proposed method considers the mental information of decision maker (DM) which is the main advantage of it. The proposed method measures the efficiency of DMUs in terms of the deviation from the mean and finally, ranks all units by using the obtained weights.

The rest of this paper is organized as follows: Sect. 2 reviews the basic concepts such as HFE and HFS and the related concepts. In Sect. 3, we summarize the HFEA model. Sect. 4 proposes the deviation-oriented HFEA model based on SBM and AHP to evaluate and to rank the decision making units. An algorithm of the proposed approach and its validation are provided in Sect. 5. An application from a real-life decision making is provided in Sect. 6. Sect. 7 carries out comparison analyses to show the superiority of the proposed method. Finally, conclusions are furnished in Sect. 8.

2 Preliminaries and basic definitions

Torra and Narukawa [29] introduced the concept of the hesitant fuzzy sets (HFS) to illustrate the membership value and to overcome the difficulty of the qualitative evaluation. These sets define the membership degree of each element as a set of several possible values between 0 and 1. It is defined as follows:

Definition 2.1. *[29]. Let X be a fixed set, a HFS on X is in terms of a function that when applied to X returns a subset of [0, 1].*

Xia and Xu [30] provided the mathematical symbol $E = \{< x,\ h_E(x) >| x \in X\}$ to express the HFS and make it easily understood, where $h_E(x)$ is a set of values in [0,1], denoting the possible membership degrees of the element $x \in X$ to the set E. Also, Xia and Xu [30] named $h = h_E(x)$ as a hesitant fuzzy element (HFE) and H as the set of all HFEs.

Note that, several membership degrees can be assigned to an element by applying the hesitant fuzzy sets. This means that, the number of members can vary in different HFEs.

Example 2.2. Suppose $X = \{1, 2\}$ be a reference set.
$A = \{\langle 1, \{0.2, 0.4, 0.6\}\rangle, \langle 2, \{0.4, 08\}\rangle\}$ is a hesitant fuzzy set with two hesitant fuzzy elements. In fuzzy sets, each member with a membership degree belongs to [0,1] characterized, since in hesitant fuzzy sets there is hesitation in determining the degree of membership of each element, the degree of membership of each element is expressed by a set of values belonging to [0,1].

Xia and Xu [30] introduced the score function to compare HFEs. It is defined as follows:

Definition 2.3. *Suppose that* $h = \bigcup_{\gamma\in h}\{\gamma\}$ *is a HFE and* γ *is the possible membership degree of* h *in [0,1] and* $N(h)$ *is the number of the elements in* h*. The score function of* h *is defined as*

$$S(h) = \frac{1}{N(h)}\sum_{\gamma\in h}\gamma. \tag{1}$$

Therefore, if h_1 *and* h_2 *are two HFEs and* $s(h_1) > s(h_2)$ *then* $h_1 \succ h_2$ *and* $s(h_1) = s(h_2)$ *results in* $h_1 \sim h_2$*.*

A few years later, Zhou and Xu [38] proposed the deviation function to compare HFEs. The score function and the deviation function are defined as follows:

Definition 2.4. *Suppose that* $h = \bigcup_{\gamma\in h}\{\gamma\}$ *is a HFE and* γ *is the possible membership degree of* h *in [0,1],* $N(h)$ *is the number of the elements in* h *and* $S(h) = \frac{1}{N(h)}\sum_{\gamma\in h}\gamma$ *is the score function of* h*. The deviation function of* h *is defined as*

$$d(h) = \frac{1}{N(h)}\sum_{\gamma\in h}|\gamma - s(h)| = \frac{1}{N(h)}\sum_{\gamma\in h}\sqrt{(\gamma - s(h))^2}. \tag{2}$$

Suppose that h_1 and h_2 are two HFEs. The main operations to aggregate h_1 and h_2 were defined as follows by Xia and Xu [30]:

(1) $h_1^{\lambda} = \bigcup_{\gamma_1\in h_1}\{{\gamma_1}^{\lambda}\}, \quad \lambda > 0;$
(2) $\lambda h_1 = \bigcup_{\gamma_1\in h_1}\left\{1-(1-\gamma_1)^{\lambda}\right\}, \ \lambda > 0;$
(3) $h_1 \bigotimes h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\ \gamma_2\};$
(4) $h_1 \bigoplus h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1 + \ \gamma_2 - \gamma_1\ \gamma_2\}.$

These operations can be used for decision-making under the hesitant fuzzy environment.

3 An overview of HFEA

Using the above calculations and developing them for ranking the DMUs are usually complex and time consuming. On the other hand, there is no explanation for inefficient units. Hence, this section reviews HFS envelopment analysis called the hesitant fuzzy envelopment analysis (HFEA) which was proposed by Zhou et al. [37]. The main equation of HFS envelopment analysis has been based on the definition of efficiency in DEA and the efficiency in the hesitant fuzzy envelopment analysis is defined in equation (3):

$$\frac{\sum_{i=1} p_i \times Output}{\sum_{i=1} q_i \times Intput} \Longleftrightarrow \frac{\sum_{i=1} p_i \times Score}{\sum_{i=1} q_i \times Deviation} \tag{3}$$

Where p_i and q_i are the weight values.

Definition 3.1. *If k alternatives $(x_1, x_2, \ldots, x_k)$ with n attributes $(y_1, y_2, \ldots, y_n)$, are evaluated by k HFSs showed as H_j $(j = 1, \ldots, k)$, then any H_e includes n HFE and the enveloped efficiency of H_e is defined as follows:*

$$m_e = \frac{p_1 s_{1e} + p_2 s_{2e} + \cdots + p_n s_{ne}}{q_1 d_{1e} + q_2 d_{2e} + \cdots + q_n d_{ne}} = \frac{\sum_{i=1}^{n} p_i s_{ie}}{\sum_{i=1}^{n} q_i d_{ie}} \tag{4}$$

Where $H_e = \{h_{1e}, h_{2e}, \ldots, h_{ne}\}$ is a HFS. $h_{ie} = \cup_{\gamma \in h_{ie}} \{\gamma\}$ is a HFE, p_i s_{ie} and q_i d_{ie} are the weighted score and the deviation values, respectively, and also, $s_{ie}, d_{ie} \in [0,1]$ for all $e = \{1, \ldots, k\}$ and $i = 1, \ldots, n$.
Since $p_i \geq 0$ and $q_i \geq 0$, then equation (5) can be obtained:

$$\sum_{i=1}^{n} p_i s_{ij} / \sum_{i=1}^{n} q_i d_{ij} \leq 1, \quad j \in \{1, 2, \ldots, k\}. \tag{5}$$

The HFEA model can be formulated as follows by using the equations (4) and (5):

$$
\begin{aligned}
&\max m_e = \tfrac{p_1 s_{1e} + p_2 s_{2e} + \cdots + p_n s_{ne}}{q_1 d_{1e} + q_2 d_{2e} + \cdots + q_n d_{ne}} = \tfrac{\sum_{i=1}^{n} p_i s_{ie}}{\sum_{i=1}^{n} q_i d_{ie}} \\
&s.t. \\
&\sum_{i=1}^{n} p_i s_{ij} / \sum_{i=1}^{n} q_i d_{ij} \leq 1 \qquad j = 1, 2, \ldots, k, \\
&s_{ij} = \frac{1}{N(h_{ij})} \sum_{\gamma \in h_{ij}} \gamma \qquad i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, k, \\
&d_{ij} = \frac{1}{N(h_{ij})} \sum_{\gamma \in h_{ij}} \sqrt{(\gamma - s_{ij})^2} \qquad i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, k, \\
&p_i \geq 0, \; q_i \geq 0, \qquad i = 1, 2, \ldots, n, e \in \{1, 2, \ldots, k\}.
\end{aligned}
\tag{6}
$$

Where $h_{ij} = \cup_{\gamma \in h_{ij}} \{\gamma\}$ is a HFE. p_i and q_i are the weight values, $p_i\ s_{ij}$ and $q_i\ d_{ij}$ are the weighted score and the deviation values, respectively, and also, $s_{ij}, d_{ij} \in [0, 1]$ for all $j = \{1, \ldots, k\}$ and $i = 1, \ldots, n$.

Note that, the equation (6) is a nonlinear programming where even determining the optimal solutions is difficult in general. This model can be converted into its equivalent linear form, model (7). This model is called the deviation-oriented hesitant fuzzy envelopment analysis (DHFEA) model and it is formulated by considering the following settings:

$$
f = (\sum_{i=1}^{n} q_i d_{ie})^{-1}, \; \xi_i = f p_i, \quad and \quad \tau_i = f q_i
$$

$$
\begin{aligned}
&\max m_e = f \sum_{i=1}^{n} p_i s_{ie} = \sum_{i=1}^{n} f p_i s_{ie} = \sum_{i=1}^{n} \xi_i s_{ie} \\
&s.t. \\
&\sum_{i=1}^{n} \xi_i s_{ij} - \sum_{i=1}^{n} \tau_i d_{ij} \leq 0 \qquad j = 1, 2, \ldots, k, \\
&\sum_{i=1}^{n} \tau_i d_{ij} = 1, \qquad i = 1, 2, \ldots, n, j = 1, 2, \ldots, k, \\
&\xi_i \geq 0, \; \tau_i \geq 0, \qquad i = 1, 2, \ldots, n, e \in \{1, 2, \ldots, k\}.
\end{aligned}
\tag{7}
$$

Where $s_{ij} = \frac{1}{N(h_{ij})} \sum_{\gamma \in h_{ij}} \gamma$ and $d_{ij} = \frac{1}{N(h_{ij})} \sum_{\gamma \in h_{ij}} \sqrt{(\gamma - s_{ij})^2}$, in other words, s_{ij} and d_{ij} are the score and the deviation values, respectively, and also, $s_{ij}, d_{ij} \in [0,1]$ for all $j = \{1, \ldots, k\}$ and $i = 1, \ldots, n$.

The dual of model (7) is as follows:

$$
\begin{array}{ll}
\min \pi_e & \\
s.t. & \\
\sum_{j=1}^{k} \sigma_j d_{ij} \leq \pi_e d_{ie} & i = 1, 2, \ldots, n, e \in \{1, 2, \ldots, k\}, \\
\sum_{j=1}^{k} \sigma_j s_{ij} \geq s_{ie} & i = 1, 2, \ldots, n, e \in \{1, 2, \ldots, k\}, \\
\sigma_j \geq 0, & j = 1, 2, \ldots, k, e \in \{1, 2, \ldots, k\}.
\end{array}
$$

Where $s_{ij} = \frac{1}{N(h_{ij})} \sum_{\gamma \in h_{ij}} \gamma$ and $d_{ij} = \frac{1}{N(h_{ij})} \sum_{\gamma \in h_{ij}} \sqrt{(\gamma - s_{ij})^2}$.

The enveloped efficiency measure, π_e, can be determined by equation (6) and can be used in the decision- making process. Zhou et al. [37] pointed out that the following cases are held:

1. $0 < \pi_e \leq 1$
2. If $\pi_{e1} > \pi_{e2}$ then $H_{e1} \succ H_{e2}$ and also the enveloped efficiency measure of $e1$ is higher than the enveloped efficiency measure of $e2$.
3. If $\pi_e = 1$, then the corresponding alternative is efficient.
4. If $\pi_e < 1$ then the corresponding alternative is relatively inefficient.

4 The Methodology

In this section, a two- stage model is proposed to evaluate and to rank the decision making units. We use SBM model to formulate the proposed model which is based on the deviation-oriented hesitant fuzzy envelopment analysis.

On the other hand, production possibility set (PPS) is one of the important definitions in DEA and is defined as

$$T = \{(X,Y)|\ \text{A non}-\text{negative X can produce a non}-\text{negative Y}\}$$

Each production technology has its own subject principles. The CCR production possibility set that presented in [7] is based on following five principles: 1-Non-empty, 2-Possibility, 3-Unbounded Ray, 4-Convexity, 5-Minimality.

Firstly suppose that $T^{\mathrm{p,q}}$ is the proposed production possibility set as follows:

$$T^{\mathrm{p,q}} = \{(x,y)|x \geq \sum_{j=1,\ j\neq p,q}^{n} \lambda_j x_j\ ,\ y \leq \sum_{j=1,\ j\neq p,q}^{n} \lambda_j y_j\ ,\ \lambda_j \geq 0, j = 1,\dots,n,\ j \neq p,q\} \tag{8}$$

Now, we consider that $T^{\mathrm{p,q}}$ holds true in the five principles. For example, since $(\sum_{j=1,\ j\neq p,q}^{n} \lambda_j x_j\ ,\ y \leq \sum_{j=1,\ j\neq p,q}^{n} \lambda_j y_j) \in T,$ then $\forall j = 1,\dots,n,\ j \neq p,q,\ \ \lambda_j = 1, \exists\lambda_k \neq j = 0$, then $(x_j,\ y_j) \in T^{\mathrm{p,q}},\ \ j = 1,\dots,n,\ j \neq p,q$. Therefore, all observations belong to this set. On the other hand, subscribing to any number of convex sets is a convex set since $T^{\mathrm{p,q}}$ is the Subscription of a finite number of half-spaces, then it is convex.

We also take $T^{\mathrm{p,q}}$ is the smallest set that holds on the 1 to 4 principles,

Suppose $T^{'}$ holds for principles 1,2,3,4. We want to prove, $T^{\mathrm{p,q}} \subseteq T^{'}$. In other words, we want to prove if $(X,\ Y) \in T^{\mathrm{p,q}}$ then $(X,\ Y) \in T^{'}$. From $(X,\ Y) \in T^{\mathrm{p,q}}$,there is a vector like $\lambda^{'\geq 0}$ such that $X \geq \sum_{j=1,\ j\neq p,q}^{n} \lambda_j^{'} X_j\ ,\ y \leq \sum_{j=1,\ j\neq p,q}^{n} \lambda_j^{'} Y_j$. Consider the vector $\overline{\lambda}$ as follows: $\overline{\lambda}_j = \frac{\lambda_j^{'}}{d}\ ,\ j = 1,\dots,n,\ j \neq p,q$, where in $d = \sum_{j=1,\ j\neq p,q}^{n} \lambda_j^{'}$, $\lambda^{'} \neq 0$, then $\overline{\lambda} \geq 0,\ \sum_{j=1,\ j\neq p,q}^{n} \overline{\lambda}_j = 1$. Since $T^{'}$ are satisfy of the nonempty and convexity principles, then $(\sum_{j=1,\ j\neq p,q}^{n} \overline{\lambda}_j X_j\ ,\ \sum_{j=1,\ j\neq p,q}^{n} \overline{\lambda}_j Y_j) \in$ $\mathrm{T}^{'}$.

According to the third principle and $d \geq 0$ then, $d(\sum_{j=1,\ j\neq p,q}^{n} \overline{\lambda}_j X_j\ ,$ $\sum_{j=1,\ j\neq p,q}^{n} \overline{\lambda}_j Y_j) \in T^{'}$ and we have $(\sum_{j=1,\ j\neq p,q}^{n} \lambda_j^{'} X_j\ ,\ \sum_{j=1,\ j\neq p,q}^{n} \lambda_j^{'} Y_j)$

$\in T^{'}$. From $X \geq \sum_{j=1,\ j\neq p,q}^{n} \lambda_j^{'} X_j$, $Y \leq \sum_{j=1,\ j\neq p,q}^{n} \overline{\lambda}_j Y_j$ and according to possibility principle, we have $(X,\ Y) \in T^{\mathrm{p,q}}$.

In this method, the units are evaluated by applying the pair-wise comparisons of other DMUs. Sect. 4.1 presents the construction of different production possibility set (PPS), and Pair-wise comparisons by DHFEA model, and Sect. 4.2 presents the ranking by AHP.

4.1 The first stage: The pair-wise comparisons by using DHFEA model

Definition 4.1. *We consider the input index as the deviation function and the output index as the score function. Therefore, we have:*

$y_{ij} = \frac{1}{N(h)} \sum_{\gamma\in h} \gamma$ *and* $x_{\mathrm{ij}}=\frac{1}{N(h)} \sum_{\gamma\in h} \sqrt{(\gamma - y_{ij})^2}$, *in which* γ *is the possible member of* h *in [0,1],* $N(h)$ *is the number of the elements in* h *and* x_{ij} , $y_{\mathrm{ij}}\epsilon[0,1]$ *for all* $i = 1, \ldots, m$, $j = 1, \ldots, n$.

We consider SBM model and the production possibility set defined in equation (8); therefore, we have:

$$\begin{array}{ll}
E\left(p, T^{p,q}\right) = \min\ t - \frac{1}{m}\ \sum_{i=1}^{m} \frac{s_i^-}{x_{ip}} & \\
s.t. & \\
t + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r^+}{y_{rp}} = 1 & \\
\sum_{j=1,j\neq p,q}^{n} \lambda_j x_{ij} + s_i^- = t\ x_{ip} & i = 1, \ldots, m, \\
\sum_{j=1,j\neq p,q}^{n} \lambda_j y_{rj} - s_r^+ = t\ y_{rp} & ,\ r = 1, \ldots, s \\
t > 0,\ x_{ij} > 0,\ y_{rj} > 0,\ \lambda_j \geq 0,\ j \neq p,q, s_i^- \geq 0,\ s_r^+ \geq 0, & \\
i = 1, \ldots, m\ ,\ j = 1, \ldots, n,\ ,\ p\epsilon \left\{1, \ldots, n\right\},\ q\epsilon\left\{1, \ldots, n\right\}, r = 1, \ldots, s\ , &
\end{array} \tag{9}$$

Where $x_{\mathrm{ij}}=\frac{1}{N(h)} \sum_{\gamma\in h} \sqrt{(\gamma - y_{ij})^2}$ and $y_{ij} = \frac{1}{N(h)} \sum_{\gamma\in h} \gamma$.
In model (9), $E\left(p, T^{p,q}\right)$ is the relative evaluation of DMU_p.

Similarly, $E\left(q,T^{p,q}\right)$ is defined as follows:

$$
\begin{array}{ll}
E\left(q,T^{p,q}\right) = \min\ t - \frac{1}{m}\ \sum_{i=1}^{m} \frac{s_i^-}{x_{iq}} & \\
s.t. & \\
t + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{rq}} = 1 & \\
\sum_{j=1,j\neq p,q}^{n} \lambda_j x_{ij} + s_i^- = t\ x_{iq} \qquad i = 1,\ldots,m & \\
\sum_{j=1,j\neq p,q}^{n} \lambda_j y_{rj} - s_r^+ = t\ y_{rq} \qquad ,\ r = 1,\ldots,s & \quad (10)\\
t > 0,\ x_{ij} > 0,\ y_{rj} > 0,\ \lambda_j \geq 0,\ j \neq p,q, s_i^- \geq 0,\ s_r^+ \geq 0, & \\
i = 1,\ldots,m\ ,\ j = 1,\ldots,n,,\ p\epsilon\left\{1,\ldots,n\right\},\ q\epsilon\left\{1,\ldots,n\right\}, r = 1,\ldots,s\ , &
\end{array}
$$

Where $x_{\mathrm{ij}}=\frac{1}{N(h)}\sum_{\gamma\in h}\sqrt{\left(\gamma - y_{ij}\right)^2}$ and $y_{ij} = \frac{1}{N(h)}\sum_{\gamma\in h}\gamma$.

In the other word, at each evaluation, we eliminate the units DMU_{p} and $DMU_q$from the production possibility set and solve models (9) and (10) to make the pairwise comparisons and to evaluate the units.

4.2 Stage 2: Ranking by AHP

In this stage, the pair-wise comparisons matrix is introduced for each pair of DMUs, e.g. p and q, by using the obtained results of the SBM-oriented DHFEA:

$$A = \left[a_{pq}\right]_{n\times n}$$

$$a_{pq} = \frac{E(p,T^{p,q})}{E(q,T^{p,q})},\quad \mathrm{p,q} = 1,2,\ldots,\mathrm{n} \qquad (11)$$

$a_{p,q}$ is defined as a fraction in which the numerator is the obtained results of the evaluation of the alternative $p, E\left(p,T^{p,q}\right)$, and the denominator is the obtained results of the evaluation of the alternative $q,\ E\left(q,T^{p,q}\right)$. It is clear that:

$$a_{pq} = \frac{1}{a_{qp}}\qquad ,\ p,q = 1,2,\ldots,n \qquad (12)$$

The elements of matrix A are determined by using the obtained results of DHFEA model. Therefore, the relative weight vector w can be determined by the pairwise comparisons of A. The priority of the alternatives and their ranks can be determined by using the relative weight vector w.

5 An Algorithm and Validation of the Hybrid DHFEA/AHP Method

Based on the discussion in the previous section, an algorithm of the ranking method by the hybrid DHFEA/AHP can be organized as below (Algorithm 5.1).

Algorithm 5.1: The hybrid DHFEA/AHP ranking method

Step 1. *Construct the different PPS, $T^{p,q}$, and the pair-wise comparison matrix by DHFEA based on SBM.*
Step 1.1 Decision makers provide the DMUs under the hesitant fuzzy environmental, and assign the
hesitant fuzzy value as the deviation function(x_{ij}) and the score function(y_{ij}),where x_{ij} , $y_{\mathrm{ij}} \epsilon [0,1]$.
Step 1.2 Solve problem (9) and obtain the efficiency of DMU_p, that is the relative evaluation of the
unit DMU_p, i.e. $E(p, T^{p,q})$.
Step 1.3 Solve problem (11) and obtain the efficiency of DMU_q that is the relative evaluation of the
unit DMU_q, i.e. $E(q, T^{p,q})$.
Step 1.4 Construct the pair-wise comparison matrix $A = [a_{pq}]_{n\times n}$ by Equations (11) and (12) using
the results obtained in Steps 1.2 and 1.3.
Step 2. *Rank units by AHP*
Step 2.1 Obtain the weight vector $\boldsymbol{W} = (w_1, \ldots, w_n)^T$ of the pair-wise comparison matrix
$A = [a_{pq}]_{n\times n}$ generated in Step 1.
Step 2.2 Assign the rank 1 to the DMU with the maximal value of w_j and stop. The DMU which has
higher corresponded value of w_j has higher ranking.

The flow chart with the steps of the proposed algorithm is presented in Figure 1 .
To show that there is perfect compatibility between the rank derived from the proposed method and efficient/inefficient classification of DEA, we have the following result.

Theorem 5.2. *If DMU_p is efficient and DMU_q is inefficient according to the result of the efficiency score in DEA, and w_p and w_q are corresponding weights obtained by the hybrid DHFEA/AHP method, then $w_p > w_q$.*

Proof. According to [24], To show that the weights $w_p > w_q$ with DMU_p are efficient and DMU_q is inefficient, we have to prove that $a_{pk} \geq a_{qk}$, $k = 1, 2, \ldots, n$, in the pair-wise comparison matrix and for at least one k, $k = 1, 2, \ldots, n$ it is a restrict inequality [21]. Denote $E\left(p, T^{p,s}\right)$, $s = 1, 2, \ldots, n$, the efficiency of the DMU_p obtained by solving the equation (9) and $E\left(q, T^{q,s}\right)$, $s = 1, 2, \ldots, n$, the efficiency of the DMU_q obtained by solving equation (10). □ For each efficient

DMU_p and each inefficient DMU_q, we have:

$$E\left(q, T^{q,s}\right) \leq E\left(p, T^{p,s}\right), \quad s = 1, 2, \ldots, n \tag{13}$$

We consider a unit k, where $k \neq q$, then $E\left(k, T^{q,s}\right) \geq E\left(k, T^{p,s}\right).$
It implies that:

$$\frac{1}{E\left(k, T^{p,s}\right)} \geq \frac{1}{E\left(k, T^{q,s}\right)} \tag{14}$$

Combining (13) and (14), we have:

$$a_{pk} = \frac{E(p,T^{p,s})}{E(k,T^{p,s})} \geq \frac{E(q,T^{q,s})}{E(k,T^{q,s})} = a_{qk} \tag{15}$$

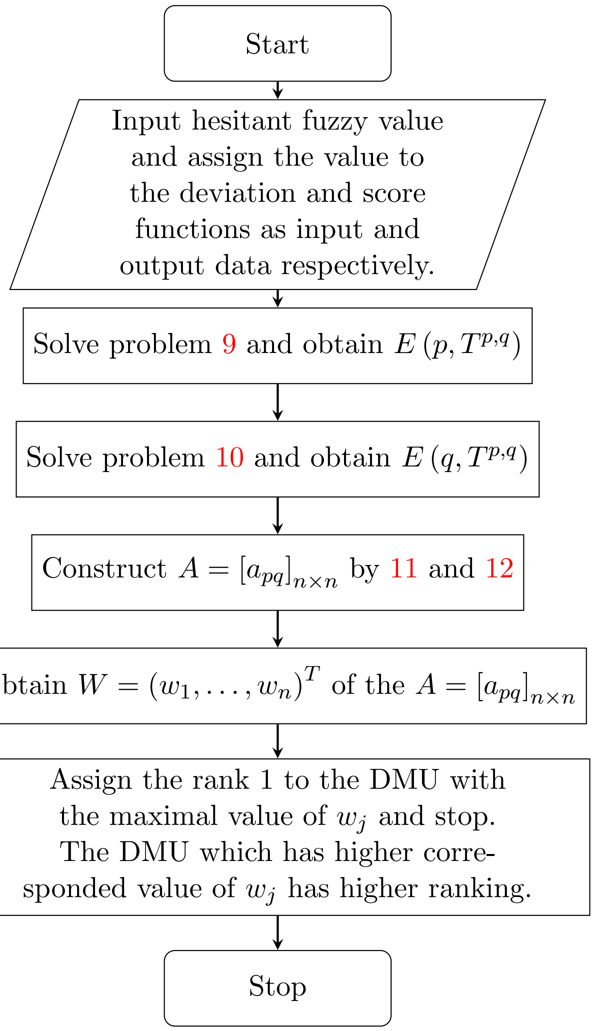

Figure 1: The flow chart with the steps of the proposed algorithm

Moreover, since DMU_p is efficient and DMU_q is inefficient, then we have for at least one s, $s = 1, 2, \ldots, n$:

$$E(p, T^{p,s}) > E(q, T^{q,s}) > E(q, T^{p,s})$$

Then $$E(p, T^{p,s}) > E(q, T^{p,s})$$

Consequently, we have:

$$a_{pq} = \frac{E(p, T^{p,s})}{E(q, T^{p,s})} > 1 = \frac{E(q, T^{q,s})}{E(q, T^{q,s})} = a_{qq} \tag{16}$$

According to the eigen-vector method and equations (15) and (16) imply that $w_p > w_q$.

According to Theorem 5.2, the integrated DHFEA/AHP method ranks efficient DMUs, which are not ranked by DEA, and also ranks inefficient DMUs, assuring at the same time that efficient DMUs have the better position than the inefficient DMUs. That shows that in the proposed method there is no problem of non-compliance with the ranking by DEA.

6 An application from a real-life decision making

In this section, a real case of decision-making under the hesitant fuzzy environment with four criteria is examined to demonstrate the application of the proposed method.

Case description. The Chinese government held a tender to buy emergency supplies in an unpredictable disaster such as an earthquake. Many companies participated in the project. After comparing the proposals, the experts selected four companies (A_1, A_2, A_3, A_4) to provide emergency supplies. Selected experts considered the selected criterion and

Table 2: The hesitant fuzzy information matrix

Criteria \ Company	A_1	A_2	A_3	A_4
r_1	$\{0.20, 0.50, 0.80\}$	$\{0.40, 0.80\}$	$\{0.60, 0.80\}$	$\{0.10, 0.50, 0.70\}$
r_2	$\{0.10,\ 0.60\}$	$\{0.32, 0.45, 0.70\}$	$\{0.25, 0.40, 0.55\}$	$\{0.30, 0.80\}$
r_3	$\{0.50, 0.90\}$	$\{0.40, 0.80\}$	$\{0.25, 0.40, 0.55\}$	$\{0.50, 0.90\}$
r_4	$\{0.20, 0.80, 0.90\}$	$\{0.10, 0.40, 0.60\}$	$\{0.40, 0.50, 0.70\}$	$\{0.20, 0.80\}$

data. We can use 4 attributes to select the most suitable company in this decision-making process. r_1 are the prices related to the government's budget, r_2 indicates the quality of products, r_3 shows the specific supplying plan which involves the amount of emergency supplies, the required time for delivery and the transportation, r_4 is the credit of each company. Table 1 shows a hesitant fuzzy evaluation matrix to represent all the evaluation information provided by the selected experts.

According to Table 2, we can use the deviation and score functions of these four companies to assess the deviation and score values of them which reported in Tables 3 and 4, respectively.
The scores of the hesitant fuzzy number of the alternatives A_j (j=1,2,3,4) on the four attributes r_i (i=1,2,3,4) can be obtained by using Eq.1 as follows:

$$s_{11} = \frac{1}{N(h_{11})} \sum_{\gamma \in h_{ij}} \gamma \;=\; \frac{0.20+0.50+0.80}{3} \;= 0.500$$

$$s_{12} = \frac{1}{N(h_{12})} \sum_{\gamma \in h_{ij}} \gamma \;=\; \frac{0.40+0.80}{2} \;= 0.600$$

$$s_{13} = \frac{1}{N(h_{13})} \sum_{\gamma \in h_{ij}} \gamma \;=\; \frac{0.60+0.80}{2} \;= 0.700$$

$$s_{14} = \frac{1}{N(h_{14})} \sum_{\gamma \in h_{ij}} \gamma \;=\; \frac{0.10+0.50+0.70}{3} \;= 0.433$$

$$s_{21} = \frac{1}{N(h_{21})} \sum_{\gamma \in h_{ij}} \gamma \;=\; \frac{0.10+0.60}{2} \;= 0.350$$

$$s_{22} = \frac{1}{N(h_{22})} \sum_{\gamma \in h_{ij}} \gamma \;=\; \frac{0.32+0.45+0.70}{3} \;= 0.490$$

$$s_{23} = \frac{1}{N(h_{23})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.25+0.40+0.55}{3} \;=0.400$$

$$s_{24} = \frac{1}{N(h_{24})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.30+0.80}{2} \;=0.550$$

$$s_{31} = \frac{1}{N(h_{31})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.50+0.90}{2} \;=0.700$$

$$s_{32} = \frac{1}{N(h_{32})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.40+0.80}{2} \;=0.600$$

$$s_{33} = \frac{1}{N(h_{33})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.20+0.40+0.55}{3} \;=0.400$$

$$s_{34} = \frac{1}{N(h_{34})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.50+0.90}{2} \;=0.700$$

$$s_{41} = \frac{1}{N(h_{41})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.20+0.80+0.90}{3} \;=0.633$$

$$s_{42} = \frac{1}{N(h_{42})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.10+0.40+0.80}{3} \;=0.367$$

$$s_{43} = \frac{1}{N(h_{43})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.40+0.50+0.70}{3} \;=0.533$$

$$s_{44} = \frac{1}{N(h_{44})}\sum_{\gamma\in h_{ij}}\gamma \;=\frac{0.20+0.80}{2} \;=0.500$$

Also the deviation values of the hesitant fuzzy number of the alternatives A_j (j=1,2,3,4) on the four attributes r_i (i=1,2,3,4) can be obtained by using Eq.2 as follows:

$$\mathrm{d}_{11} = \frac{1}{N(h_{11})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{11}\ )^2} = \frac{1}{3}\left[\sqrt{(0.2-0.5)^2}+\sqrt{(0.5-0.5)^2}+\sqrt{(0.8-0.5)^2}\right] = \frac{0.6}{3} = 0.2$$

$$\mathrm{d}_{12} = \frac{1}{N(h_{12})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{12}\ )^2} = \frac{1}{2}\left[\sqrt{(0.4-0.6)^2}+\sqrt{(0.8-0.6)^2}\right] = \frac{0.4}{2} = 0.2$$

$$\mathrm{d}_{13} = \frac{1}{N(h_{13})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{13}\ )^2} = \frac{1}{2}\left[\sqrt{(0.6-0.7)^2}+\sqrt{(0.8-0.7)^2}\right] = \frac{0.2}{2} = 0.1$$

$$\mathrm{d}_{14} = \frac{1}{N(h_{14})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{14}\ )^2} = \frac{1}{3}\left[\sqrt{(0.1-0.433)^2}+\sqrt{(0.5-0.433)^2}+\sqrt{(0.7-0.433)^2}\right] = \frac{0.667}{3} = 0.222$$

$$\mathrm{d}_{21} = \frac{1}{N(h_{21})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{21}\ )^2} = \frac{1}{2}\left[\sqrt{(0.1-0.35)^2}+\sqrt{(0.6-0.35)^2}\right] = \frac{0.5}{2} = 0.25$$

$$\mathrm{d}_{22} = \frac{1}{N(h_{22})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{22}\ )^2} = \frac{1}{3}\left[\sqrt{(0.32-0.49)^2}+\sqrt{(0.45-0.49)^2}+\sqrt{(0.7-0.49)^2}\right] = \frac{0.42}{3} = 0.14$$

Table 3: The deviation value matrix as input value of DEA model

Criteria \ Company	A_1	A_2	A_3	A_4
r_1	0.200	0.200	0.100	0.222
r_2	0.250	0.140	0.100	0.250
r_3	0.200	0.200	0.100	0.200
r_4	0.289	0.178	0.111	0.300

$$\mathrm{d}_{23} = \frac{1}{N(h_{23})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{23}\ )^2} = \frac{1}{3}\left[\sqrt{(0.25-0.4)^2}+\sqrt{(0.4-0.4)^2}+\sqrt{(0.55-0.4)^2}\right] = \frac{0.3}{3} = 0.1$$

$$\mathrm{d}_{24} = \frac{1}{N(h_{24})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{24}\ )^2} = \frac{1}{2}\left[\sqrt{(0.3-0.55)^2}+\sqrt{(0.8-0.55)^2}\right] = \frac{0.5}{2} = 0.25$$

$$\mathrm{d}_{31} = \frac{1}{N(h_{31})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{31}\ )^2} = \frac{1}{2}\left[\sqrt{(0.5-0.7)^2}+\sqrt{(0.9-0.7)^2}\right] = \frac{0.4}{2} = 0.2$$

$$\mathrm{d}_{32} = \frac{1}{N(h_{32})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{32}\ )^2} = \frac{1}{2}\left[\sqrt{(0.4-0.6)^2}+\sqrt{(0.8-0.6)^2}\right] = \frac{0.4}{2} = 0.2$$

$$\mathrm{d}_{33} = \frac{1}{N(h_{33})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{33}\ )^2} = \frac{1}{3}\left[\sqrt{(0.25-0.4)^2}+\sqrt{(0.4-0.4)^2}+\sqrt{(0.55-0.4)^2}\right] = \frac{0.3}{3} = 0.1$$

$$\mathrm{d}_{34} = \frac{1}{N(h_{34})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{34}\ )^2} = \frac{1}{2}\left[\sqrt{(0.5-0.7)^2}+\sqrt{(0.9-0.7)^2}\right] = \frac{0.4}{2} = 0.2$$

$$\mathrm{d}_{41} = \frac{1}{N(h_{41})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{41}\ )^2} = \frac{1}{3}\left[\sqrt{(0.2-0.633)^2}+\sqrt{(0.8-0.633)^2}+\sqrt{(0.9-0.633)^2}\right] = \frac{0.867}{3} = 0.289$$

$$\mathrm{d}_{42} = \frac{1}{N(h_{42})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{42}\ )^2} = \frac{1}{3}\left[\sqrt{(0.1-0.367)^2}+\sqrt{(0.4-0.367)^2}+\sqrt{(0.6-0.367)^2}\right] = \frac{0.534}{3} = 0.178$$

$$\mathrm{d}_{43} = \frac{1}{N(h_{43})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{43}\ )^2} = \frac{1}{3}\left[\sqrt{(0.4-0.533)^2}+\sqrt{(0.5-0.533)^2}+\sqrt{(0.7-0.533)^2}\right] = \frac{0.333}{3} = 0.111$$

$$\mathrm{d}_{44} = \frac{1}{N(h_{44})}\sum_{\gamma\in h_{ij}}\sqrt{(\gamma - s_{44}\ )^2} = \frac{1}{2}\left[\sqrt{(0.2-0.5)^2}+\sqrt{(0.8-0.5)^2}\right] = \frac{0.6}{2} = 0.3$$

The deviation matrix of table 3 and score matrix of table 4 which are determined by the hesitant fuzzy matrix can be considered as the input index (x_{ij}) and the output index (y_{ij}) in models 9 and 10, respectively:

Table 4: The score value matrix as output value of DEA model

Criteria \ Company	A_1	A_2	A_3	A_4
r_1	0.500	0.600	0.700	0.433
r_2	0.350	0.490	0.400	0.550
r_3	0.700	0.600	0.400	0.700
r_4	0.633	0.367	0.533	0.500

$$\begin{aligned}
&E\left(A_1, T^{A_1,A_2}\right) = \min\ t - \tfrac{1}{4}\ \left(\tfrac{S_1^-}{0.200} + \tfrac{S_2^-}{0.250} + \tfrac{S_3^-}{0.200} + \tfrac{S_4^-}{0.289}\right)\\
&s.t. \qquad t + \tfrac{1}{4}\left(\tfrac{S_1^+}{0.500} + \tfrac{S_2^+}{0.350} + \tfrac{S_3^+}{0.700} + \tfrac{S_4^+}{0.633}\right) = 1\\
&0.1\ \lambda_{A_3} + 0.222\lambda_{A_4} + S_1^- = 0.200\ t\\
&0.1\ \lambda_{A_3} + 0.222\lambda_{A_4} + S_2^- = 0.250\ t\\
&0.1\ \lambda_{A_3} + 0.2\lambda_{A_4} + S_3^- = 0.200\ t\\
&0.111\ \lambda_{A_3} + 0.3\lambda_{A_4} + S_4^- = 0.289\ t\\
&0.7\ \lambda_{A_3} + 0.433\lambda_{A_4} - S_1^+ = 0.500\ t\\
&0.4\ \lambda_{A_3} + 0.55\lambda_{A_4} - S_2^+ = 0.350\ t\\
&0.4\ \lambda_{A_3} + 0.7\lambda_{A_4} - S_3^+ = 0.700\ t\\
&0.533\ \lambda_{A_3} + 0.5\lambda_{A_4} - S_4^+ = 0.633\ t\\
&t > 0,\ \lambda_{A_3} \geq 0,\ \lambda_{A_4} \geq 0,\ \ s_1^-,\ s_2^-,\ s_3^-\ , s_4^- \geq 0, s_1^+,\ s_2^+, s_3^+, s_4^+ \geq 0.
\end{aligned} \tag{17}$$

Similarly, structure of $E\left(q, T^{p,q}\right)$, where $p, q \in \{A_1, A_2, A_3, A_4\}$ is determined as follows:

$$\begin{aligned}
&E\left(A_2, T^{A_1,A_2}\right) = \min\ t - \tfrac{1}{4}\ \left(\tfrac{S_1^-}{0.200} + \tfrac{S_2^-}{0.140} + \tfrac{S_3^-}{0.200} + \tfrac{S_4^-}{0.178}\right)\\
&s.t. \qquad t + \tfrac{1}{4}\left(\tfrac{S_1^+}{0.600} + \tfrac{S_2^+}{0.490} + \tfrac{S_3^+}{0.600} + \tfrac{S_4^+}{0.367}\right) = 1\\
&0.1\ \lambda_{A_3} + 0.222\lambda_{A_4} + S_1^- = 0.200\ t\\
&0.1\ \lambda_{A_3} + 0.222\lambda_{A_4} + S_2^- = 0.140\ t\\
&0.1\ \lambda_{A_3} + 0.2\lambda_{A_4} + S_3^- = 0.200\ t\\
&0.111\ \lambda_{A_3} + 0.3\lambda_{A_4} + S_4^- = 0.178\ t\\
&0.7\ \lambda_{A_3} + 0.433\lambda_{A_4} - S_1^+ = 0.600\ t\\
&0.4\ \lambda_{A_3} + 0.55\lambda_{A_4} - S_2^+ = 0.490\ t\\
&0.4\ \lambda_{A_3} + 0.7\lambda_{A_4} - S_3^+ = 0.600\ t\\
&0.533\ \lambda_{A_3} + 0.5\lambda_{A_4} - S_4^+ = 0.367\ t\\
&t > 0,\ \lambda_{A_3} \geq 0,\ \lambda_{A_4} \geq 0,\ \ s_1^-,\ s_2^-,\ s_3^-\ , s_4^- \geq 0, s_1^+,\ s_2^+, s_3^+, s_4^+ \geq 0.
\end{aligned} \tag{18}$$

Therefore, $E\left(A_1, T^{A_1,A_2}\right) = 0.39$ and $E\left(A_2, T^{A_1,A_2}\right) = 0.4$ are determined by solving models (17) and (18), respectively. By placing the results in Equations 11 and 12, we have:

$$a_{A_1A_2} = \frac{E\left(A_1, T^{A_1,A_2}\right)}{E\left(A_2, T^{A_1,A_2}\right)} = \frac{0.39}{0.43} = 0.907$$

Furthermore

$$a_{A_2A_1} = \frac{1}{a_{A_1A_2}} = 1.1026$$

In the other word, the efficiency of company C_2 is higher than the efficiency of company C_1. The pair-wise comparisons matrix can be constructed by using models (17) and (18) and the pair-wise comparisons of all DMUs as follows:

$$A = \begin{pmatrix} 1 & 0.9070 & 0.8958 & 1.1143 \\ 1.1026 & 1 & 0.8736 & 1.2286 \\ 1.1163 & 1.1447 & 1 & 1.2564 \\ 0.8974 & 0.8140 & 0.7959 & 1 \end{pmatrix}$$

Matrix A is the pair-wise comparisons matrix obtained by the proposed method. The inconsistency rate of matrix A can be determined by Saaty's [23] as follows:

$$I.R = \frac{I.I}{I.I.R} = \frac{0.0013}{0.9} = 0.0015$$

According to Saaty [23], since $0.0015 < 0.1$, then the inconsistency rate of matrix A is acceptable.
Then, we can obtain the weight vector w by using the minimum squares method. The enveloped efficiency and the corresponding weight vector of each DMU are reported in the following table:

According to Table 5, two companies A_2 and A_3 are efficient. Therefore, these companies can be selected to provide the emergency supplies while the project should only select one company as the most suitable alternative. On the other hand, two companies A_1 and A_4 are inefficient. The

Table 5: The results of the proposed method DHFEA/AHP

Companies	$^{*}_{CCR}$	Weight vector (w^*)	Rank
A_1	0.8750	0.2428	3
A_2	1.0000	0.2597	2
A_3	1.0000	0.2803	1
A_4	0.8750	0.2173	4

results of the proposed method DHFEA/AHP show that the optimal alternative is company A_3. According to the weight vector determined by the proposed DHFEA/AHP, we can prioritize the DMUs. The obtained results of the ranking methods AP, MAJ and LJK are compared in the following table:

Table 6: Ranking by different methods

DMUs (Companies)	$^{*}_{CCR}$	EFF and Rank-AP	EFF and Rank-MAJ	EFF and Rank-LJK	Ranking by Method in [37]	w^* -Ranking by new method
A_1	0.8750	0.8750(-)	0.8874(-)	1.0000(-)	0.7817(3)	0.2428(3)
A_2	1.0000	1.0714(2)	1.0400(2)	1.0100(2)	0.8809(2)	0.2597(2)
A_3	1.0000	2.4400(1)	1.7201(1)	1.5664(1)	1.0000(1)	0.2803(1)
A_4	0.8750	0.8750(-)	0.8750(-)	1.0000(-)	0.7683(4)	0.2173(4)

Generally, we can rank all companies and then select the best one by using the aggregation operations of the HFEs and aforementioned comparison rules $A_3 \succ A_1 \succ A_4 \succ A_2$ [33]. Whereas, as seen as the first column of Table 5 shows the CCR-efficiency of units; where C_1 and C_4 have efficiency score less than 1, then they are inefficient. The other columns reports the obtained results of the ranking methods AP, MAJ,LJK, Method in [37] and the proposed DHFEA/AHP method. As can be seen, the companies A_1 and A_4 do not get the allowed super efficiency score for ranking and so, they are not ranked by AP, MAJ and LJK. However, all companies are ranked by the weight vector determined by the proposed method and the result of the ranking corresponds to the method in [37], and efficient and inefficient units in DEA. The company A_3 is selected as the optimal company in the project. The efficient company A_2 obtains the second place. If the disturbance in the units' performance or other events occur, then companies A_1 and A_4 can obtain the third and the fourth places. In general, the companies are ranked $A_3 \succ A_2 \succ A_1 \succ A_4$ in this study.

7 Further comparative analysis

To show advantages of the proposed method, this section further compares the proposed method with AP, MAJ, LJK and method in [37]. The detailed comparison results are described in Table 6. In addition, to intuitively compare the ranking results of alternatives obtained by different methods, we depict these results in Figure 2.

(1) Compared with methods AP, MAJ and LJK , the proposed method is able to rank all units because the former only can rank efficient units. while the latter can handle rank problem with acquiring weight vector. Although method [37] also can tackle the ranking problems with consider subjective criteria, it transformed subjective variables into prioritize, which may cause loss or distortion of information. The proposed method deals with decision making problems by subjective variables which can effectively overcome this shortcoming.

(2) The proposed method determines efficiency scores of units by extended SBM model, and calculates weight vector, which can avoid the subjective randomness. However, method [37] gave criteria weights in advance by decision maker subjective judgments and did not consider the determination of criteria weights. Although method [37] employed priority relationships of criteria by decision makers to derive priority of criteria, there are two limitations: 1) it is supposed that criteria are independent on each other; 2) it

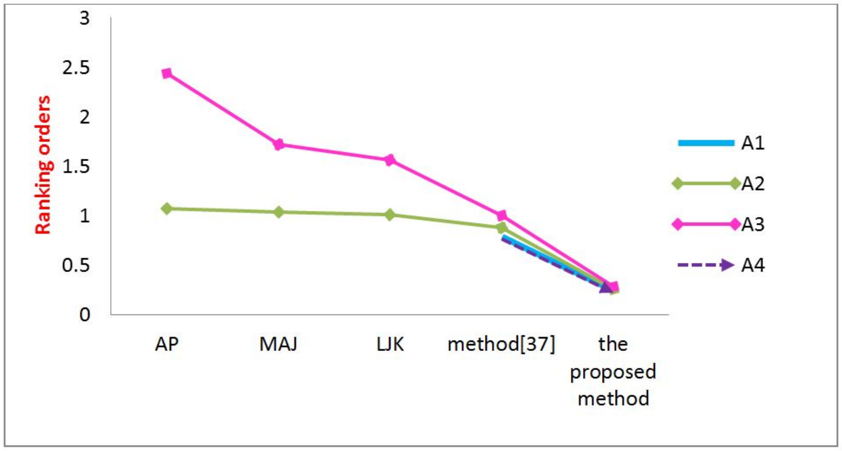

Figure 2: Ranking orders of alternatives obtained by different methods.

did not discuss how to repair the consistency of preference relations when preference relations are unacceptable consistent. On the other hand, the proposed method not only considers interactions among criteria, but also constructs consistent preference relations.

(3) As for the decision making approach, the proposed method utilizes DHFEA/AHP to rank alternatives. Compared with decision making approaches used in other methods AP, MAJ, LJK and [37], the conditions of DHFEA/AHP (i.e., alternatives are compared on proposed PPS and the comparison scores used for pair-wise comparison matrix) are more accurate. Therefore, the results obtained by DHFEA/AHP are more cautious and more reliable.

8 Conclusion

This paper used the hesitant fuzzy information and reviewed HFS and HFEA models. We used the deviation and score values to propose a two-stage deviation-oriented hesitant fuzzy decision-making method (DHFEA/AHP) based on the SBM method. Then, the obtained results were applied to construct the pair-wise comparisons matrix and finally, the DMUs were ranked. In general, most of the existing decision-making methods tend to focus on quantitative data to make more accurate decisions. However, it may not be possible to report the data as the certain data, for example, there is not enough time to access this type of data. This paper presented a method which was more flexible for the gathering data by experts and decision makers. On the other hand, the final evaluation process in the proposed method was not based on the mental judgments of the decision maker, hence, the ranking results were based on mathematical calculations and the decision-making process was more accurate. In most previous researches, the basic model has been used of the type of constant return to *scale as the CCR model,* or have been based on the variable return to scale as the BCC model type. Whereas, the presented model in the article is based on slack variables and designed based on the SBM model. It has a high dimension compared to changes in input and output variables. Also we chose this model to better express the efficiency of decision making units with poor efficiency, and if we intend to act to improve the efficiency of inefficient units, we can get slack variables directly from this model and measure the depth of inefficiency. This study considered the tender evaluation and compared the obtained results with the existing ranking methods AP, MAJ and LJK. In addition to the tender evaluation, the DHFEA/AHP approach can also be used as an effective decision-making tool for many investment strategies such as the banking industry, the stock market and the insurance industry. A possible extension of this research would be to deal with other external factors to compare criterion. Also, the traditional DEA model and AHP method can be developed for further research.

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