Journal of Mathematical Extension
Vol. 16, No. 3, (2022) (8)1-14
URL: https://doi.org/10.30495/JME.2022.1780
ISSN: 1735-8299
Original Research Paper

The Product-Type Operators from the Besov Spaces into nth Weighted Type Spaces

E. Abbasi
Mahabad Branch, Islamic Azad University

Abstract. The main goal of this paper is the investigation of boundedness and compactness of a class of product-type operators $T^m_{u,v,\varphi}$ from Besov spaces into nth weighted type spaces.

AMS Subject Classification: 47B38; 30H25; 30H99
Keywords and Phrases: Besov spaces, boundedness, compactness, product-type operators, nth weighted type spaes.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denotes the space of all analytic functions on $\mathbb{D}$. Let $u, v \in H(\mathbb{D})$, φ be an analytic self-map of $\mathbb{D}$ ($\varphi(\mathbb{D}) \subseteq \mathbb{D}$) and $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$. In [11] Stević and co-authors defined a new product-type operator $T^m_{u,v,\varphi}$ as follows

$$T^m_{u,v,\varphi}g(z) = u(z)g^{(m)}(\varphi(z)) + v(z)g^{(m+1)}(\varphi(z)), \quad g \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

When $m = 0$, we obtain the Stević-Sharma type operator and for $v \equiv 0$, we get the generalized weighted composition operators $D^m_{u,\varphi}$. For more

Received: September 2020; Accepted: March 2021

details about product-type operators, we refer the interested reader to [7, 8, 13].

Let $dA(z)$ be the normalized area measure on $\mathbb{D}$ and $1 < p < \infty$. The Besov space B_p consists of all $g \in H(\mathbb{D})$ such that

$$b_p(g) = \Big(\int_{\mathbb{D}} |g'(z)|^p (1-|z|^2)^{p-2} dA(z) \Big)^{\frac{1}{p}} < \infty.$$

B_p is a Banach space with $\|g\|_{B_p} = |g(0)| + b_p(g)$ and it is a Möbius invariant space in the sense that $b_p(go\psi) = b_p(g)$ for all $g \in B_p$ and $\psi \in Aut(\mathbb{D})$, the Möbius group of $\mathbb{D}$ (see [15, 17]).

Let $n \in \mathbb{N}_0$ and $\mu(z)$ be a weight, positive and continuous function on $\mathbb{D}$. The nth weighted type space $\mathcal{W}_\mu^n(\mathbb{D}) = \mathcal{W}_\mu^n$, was introduced by Stević in [9], consists of all $g \in H(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^n}(g) = \sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| < \infty.$$

This space is a Banach space with the following norm

$$\|g\|_{\mathcal{W}_\mu^n} = \sum_{i=0}^{n-1} |g^{(i)}(0)| + b_{\mathcal{W}_\mu^n}(g).$$

Let $\alpha > 0$. Then $\mathcal{W}^{(0)}_{(1-|z|^2)^\alpha} = H^{-\alpha}$(growth space), $\mathcal{W}^{(1)}_{(1-|z|^2)^\alpha} = \mathcal{B}^\alpha$(Bloch type space) and $\mathcal{W}^{(2)}_{(1-|z|^2)^\alpha} = \mathcal{Z}^\alpha$(Zygmund type space). Also $\mathcal{W}_\mu^{(0)} = H_\mu$(weighted-type space), $\mathcal{W}_\mu^{(1)} = \mathcal{B}\mu$(weighted Bloch space), $\mathcal{W}_\mu^{(2)} = \mathcal{Z}_\mu$(weighted Zygmund space) and $\mathcal{W}^{(1)}_{(1-|z|^2)\log\frac{2}{1-|z|^2}}$ coincides with the logarithmic Bloch space $\mathcal{B}_{\log}$. More information about nth weighted type spaces can be found in [1, 2, 3, 9, 10, 12, 18].

Lemma 1.1 ([16], Proposition 8)**.** *For any $g \in \mathcal{B}$ and $n \in \mathbb{N}$,*

$$\|g\|_{\mathcal{B}} \approx \sum_{i=0}^{n-1} |g^{(i)}(0)| + \sup_{z \in \mathbb{D}} (1-|z|^2)^n |g^{(n)}(z)|.$$

Lemma 1.2 ([6], Lemma 2.1)**.** *The sequence $\{z^j\}_1^\infty$ is bounded in $\mathcal{B}$ and*

$$\lim_{j \to \infty} \|z^j\|_{\mathcal{B}} = \frac{2}{e}.$$

Lemma 1.3 ([4]). *Let $1 < p < \infty$. Then for any $g \in B_p$*

$$\|g\|_{\mathcal{B}} \preceq \|g\|_{B_p}.$$

For $n, k \in \mathbb{N}_0$ with $k \leq n$, the partial Bell polynomials are defined by

$$B_{n,k}(y_1, y_2, ..., y_{n-k+1}) = \sum \frac{n!}{j_1! j_2! ... j_{n-k+1}!} (\frac{y_1}{1!})^{j_1} (\frac{y_2}{2!})^{j_2} ... (\frac{y_{n-k+1}}{(n-k+1)!})^{j_{n-k+1}},$$

where $j_1, j_2, ..., j_{n-k+1} \in \mathbb{N}_0$ such that

$$j_1 + 2j_2 + ... + (n-k+1)j_{n-k+1} = n \quad \text{and} \quad j_1 + j_2 + ... + j_{n-k+1} = k.$$

More information about Bell polynomials can be found in [[5], pp 134]. From Lemma 4 of [10], we have the next lemma.

Lemma 1.4. *Let $g, \varphi, u, v \in H(\mathbb{D})$. Then for any $m, n \in \mathbb{N}_0$,*

$$\begin{aligned}(T^m_{u,v,\varphi} g)^{(n)}(z) = &\sum_{i=0}^{n} g^{(m+i)}(\varphi(z)) \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi'(z), ..., \varphi^{(l-i+1)}(z)) \\ &+ \sum_{i=0}^{n} g^{(m+1+i)}(\varphi(z)) \sum_{l=i}^{n} \binom{n}{l} v^{(n-l)}(z) B_{l,i}(\varphi'(z), ..., \varphi^{(l-i+1)}(z)).\end{aligned}$$

For simplicity in calculation, we set

$$I^{n,u}_{i,\varphi}(z) := \begin{cases} \sum_{l=i}^{n} \binom{n}{l} u^{(n-l)}(z) B_{l,i}(\varphi'(z), ..., \varphi^{(l-i+1)}(z)) & i, n \in \mathbb{N}_0 \text{ and } i \leq n \\ 0 & \text{otherwise} \end{cases}$$

By applying the above notion, we can rewrite the previous lemma as follows

$$(T^m_{u,v,\varphi} g)^{(n)}(z) = \sum_{i=0}^{n+1} g^{(m+i)}(\varphi(z)) (I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z). \tag{1}$$

Recently, Liu and Yu in [8] studied the boundedness and compactness of operator $T^m_{u,v,\varphi}$ from the Logarithmic Bloch spaces to Zygmund type spaces. Also Zhu and the author of this paper, have found some characterizations for boundedness and compactness of $T^0_{u,v,\varphi} : B_p \to \mathcal{B}$ in [19]. Motivated by previous works, in this paper some characterizations for boundedness and compactness of operator $T^m_{u,v,\varphi} : B_p \to \mathcal{W}^n_\mu$ are given. As application some new characterizations for the boundedness, compactness of generalized weighted composition operators from Besov spaces into nth weighted type spaces are found.

Throughout this paper, if there exists a constant c such that $a \leq cb$ we use the notation $a \preceq b$. The symbol $a \approx b$ means that $a \preceq b \preceq a$.

2 Boundedness

In this section, some equivalent conditions for boundedness of the operator $T^m_{u,v,\varphi} : B_p \to \mathcal{W}^n_\mu$ are obtained.

Lemma 2.1 ([19], Lemma 2.5)**.** *Let $1 < p < \infty$. For any $a \in \mathbb{D}$ and $j \in \{1, ..., k\}$, set*

$$f_{j,a}(z) = \Big(\frac{1-|a|^2}{1-\bar{a}z}\Big)^j, \qquad z \in \mathbb{D}. \tag{2}$$

Then $f_{j,a} \in B_p$ and $\sup_{a\in\mathbb{D}} \|f_{j,a}\|_{B_p} < \infty$.

By using the functions defined in (2), we get the next lemma. Since the proof of it resembles to the proof of Lemma 2.1 [1], hence it is omitted.

Lemma 2.2. *Let δ_{ik} be Kronecker delta. For any $0 \neq a \in \mathbb{D}$, $m \in \mathbb{N}$ and $i \in \{0, 1, ..., n+1\}$ there exists a function $g_{i,a} \in B_p$ such that*

$$g^{(m+k)}_{i,a}(a) = \frac{\delta_{ik}\bar{a}^{m+k}}{(1-\mid a\mid^2)^{m+k}}.$$

In this case $g_{i,a}(z) = \sum_{j=1}^{n+2} c^i_j f_{j,a}(z)$, where $f_{j,a}$ are defined in (2) and c^i_j are independent of the choice of a.

Theorem 2.3. *Let $m, n \in \mathbb{N}$, $1 < p < \infty$, μ be a weight, $u, v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The following assertions are pairwise equivalent.*

(a) The operator $T^m_{u,v,\varphi} : B_p \to \mathcal{W}^n_\mu$ *is bounded.*

(b) If $p_j(z) = z^j$ *then* $\sup_{j\geq 1} \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^n_\mu} < \infty$.

(c) For each $i \in \{0, ..., n+1\}$,

$$\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} < \infty, \quad \sup_{z\in\mathbb{D}} \mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)| < \infty,$$

where $f_{i,a}$ *are defined in* (2).

(d) For each $i \in \{0, 1, ..., n+1\}$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)|}{(1 - |\varphi(z)|^2)^{i+m}} < \infty.$$

Proof. $(b) \Rightarrow (c)$ For any $i \in \{0, 1, ..., n+1\}$ and $a \in \mathbb{D}$

$$f_{i+1,a}(z) = (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} \binom{i+j}{j} \bar{a}^j z^j.$$

So,

$$\begin{aligned} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} &\leq (1 - |a|^2)^{i+1} \sum_{j=0}^{\infty} \binom{i+j}{j} |\bar{a}|^j \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^n_\mu} \\ &\leq 2^{i+1} \sup_{j\geq 1} \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^n_\mu}. \end{aligned}$$

Therefore, $\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} < \infty$.

Applying the operator $T^m_{u,v,\varphi}$ for $p_m(z) = z^m$, so by employing (1), we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|I^{n,u}_{0,\varphi}(z)| = \frac{1}{m!} \sup_{z\in\mathbb{D}} \mu(z)|(T^m_{u,v,\varphi} p_m)^{(n)}(z)| < \infty.$$

Now suppose that the following inequalities hold for $0 \leq i \leq j - 1$,

$$\sup_{z\in\mathbb{D}} \mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)| < \infty,$$

where $j \leq n+1$. Applying the operator $T^m_{u,v,\varphi}$ for $p_{m+j}(z) = z^{m+j}$ and using (1), we have

$$\sup_{z\in\mathbb{D}} \mu(z)\Big|\frac{(m+j)!}{j!}\varphi^j(z)I^n_0(z) + \sum_{k=1}^{j}\frac{(m+j)!}{(j-k)!}(\varphi(z))^{j-k}\ (I^{n,u}_{k,\varphi} + I^{n,v}_{k-1,\varphi})(z)\Big|$$
$$\leq \|T^m_{u,v,\varphi}p_{m+j}\|_{\mathcal{W}^n_\mu} < \infty.$$

Since $\varphi(\mathbb{D}) \subset \mathbb{D}$, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|(I^{n,u}_{j,\varphi} + I^{n,v}_{j-1,\varphi})(z)| < \infty.$$

$(c) \Rightarrow (d)$ For any $\varphi(a) \neq 0$ and $i \in \{0, ..., n+1\}$, employing (1) and Lemma 2.2, we obtain

$$\begin{aligned}\frac{\mu(a)|\varphi(a)|^{m+i}|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(a)|}{(1-|\varphi(a)|^2)^{m+i}} &\leq \sup_{a\in\mathbb{D}}\|T^m_{u,v,\varphi}g_{i,\varphi(a)}\|_{\mathcal{W}^n_\mu}\\ &\leq \sum_{j=1}^{n+2}|c^i_j|\sup_{a\in\mathbb{D}}\|T^m_{u,v,\varphi}f_{j,a}\|_{\mathcal{W}^n_\mu} < \infty.\end{aligned}$$

From the last inequality,

$$\sup_{|\varphi(a)|>\frac{1}{2}} \frac{\mu(a)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(a)|}{(1-|\varphi(a)|^2)^{m+i}} < \infty,$$

and from (c)

$$\sup_{|\varphi(a)|\leq\frac{1}{2}} \frac{\mu(a)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(a)|}{(1-|\varphi(a)|^2)^{m+i}} \preceq \sup_{|\varphi(a)|\leq\frac{1}{2}} \mu(a)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(a)| < \infty.$$

So, for any $i \in \{0, ..., n+1\}$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{m+i}} < \infty.$$

$(d) \Rightarrow (b)$ Setting $g(z) = p_j(z) = z^j (j \geq m+n+1)$ in (1), so from Lemmas 1.1 and 1.2, we obtain

$$\begin{aligned}
&\mu(z)|(T^m_{u,v,\varphi}p_j)^{(n)}(z)| \leq \\
&\mu(z)\sum_{i=0}^{n+1}\frac{j!}{(j-m-i)!}(1-|\varphi(z)|^2)^{i+m}|\varphi(z)|^{j-m-i}\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{i+m}} \preceq \\
&\|z^j\|_{\mathcal{B}}\sum_{i=0}^{n+1}\sup_{z\in\mathbb{D}}\frac{\mu(z)|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{i+m}} \preceq \\
&\frac{2}{e}\sum_{i=0}^{n+1}\sup_{z\in\mathbb{D}}\frac{\mu(z)|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{i+m}}. \qquad (3)
\end{aligned}$$

On the other hand, for any $k < n$, we get

$$\begin{aligned}
&|(T^m_{u,v,\varphi}p_j)^{(k)}(0)| \leq \\
&\sum_{i=0}^{k+1}\frac{j!}{(j-m-i)!}(1-|\varphi(0)|^2)^{i+m}|\varphi(0)|^{j-m-i}\frac{|(I^{k,u}_{i,\varphi}+I^{k,v}_{i-1,\varphi})(0)|}{(1-|\varphi(0)|^2)^{i+m}} \\
&\preceq \frac{2}{e}\sum_{i=0}^{k+1}\frac{|(I^{k,u}_{i,\varphi}+I^{k,v}_{i-1,\varphi})(0)|}{(1-|\varphi(0)|^2)^{i+m}}. \qquad (4)
\end{aligned}$$

Hence, by using (3) and (4), we get (b).

$(d) \Rightarrow (a)$ From (1) and Lemmas 1.2, 1.3, we have

$$\begin{aligned}
&\mu(z)|(T^m_{u,v,\varphi}f)^{(n)}(z)| \leq \\
&\mu(z)\sum_{i=0}^{n+1}(1-|\varphi(z)|^2)^{m+i}|f^{(m+i)}(\varphi(z))|\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{m+i}} \preceq \\
&\|f\|_{B_p}\sum_{i=0}^{n+1}\sup_{z\in\mathbb{D}}\frac{\mu(z)|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{m+i}}. \qquad (5)
\end{aligned}$$

Also for any $k < n$, with similar calculation in the (4), we obtain

$$|(T^m_{u,v,\varphi}f)^{(k)}(0)| \preceq \|f\|_{B_p}\sum_{i=0}^{k+1}\frac{|(I^{k,u}_{i,\varphi}+I^{k,v}_{i-1,\varphi})(0)|}{(1-|\varphi(0)|^2)^{i+m}}. \qquad (6)$$

From (5) and (6), we get (a).

$(a) \Rightarrow (c)$ For each $i \in \{0, 1, ..., n+1\}$, from Lemma 2.1, we obtain

$$\sup_{a\in\mathbb{D}} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} \leq \|T^m_{u,v,\varphi}\|_{B_p\to\mathcal{W}^n_\mu} \sup_{a\in\mathbb{D}} \|f_{i+1,a}\|_{B_p} < \infty.$$

For any $j \in \mathbb{N}$, $z^j \in B_p$. So, the proof of the second part resembles to the proof of second part of $(b) \Rightarrow (c)$ and hence it is dropped. The proof is completed. □

3 Compactness

In this section, some new characterizations for compactness of the operator $T^m_{u,v,\varphi} : B_p \to \mathcal{W}^n_\mu$ are given. The proof of the following lemma resembles to the proof of Lemma 2.10 [14], therefore it is dropped.

Lemma 3.1. *Let $1 < p < \infty$, μ be a weight and $S : B_p \to \mathcal{W}^n_\mu$ be bounded. Then S is compact if and only if whenever $\{f_k\}$ is bounded in B_p and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$,*

$$\lim_{k\to\infty} \|Sf_k\|_{\mathcal{W}^n_\mu} = 0.$$

Theorem 3.2. *Let $m, n \in \mathbb{N}$, $1 < p < \infty$, μ be a weight, $u, v \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Let the operator $T^m_{u,v,\varphi} : B_p \to \mathcal{W}^n_\mu$ be bounded then the following assertions are pairwise equivalent.*

(a) The operator $T^m_{u,v,\varphi} : B_p \to \mathcal{W}^n_\mu$ is compact.

(b) If $p_j(z) = z^j$ then $\lim_{j\to\infty} \|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^n_\mu} = 0$.

(c) For each $i \in \{0, ..., n+1\}$,

$$\lim_{|a|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} = 0,$$

where $f_{i,a}$ are defined in (2).

(d) For each $i \in \{0, 1, ..., n+1\}$,

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)|}{(1 - |\varphi(z)|^2)^{i+m}} = 0.$$

Proof. $(b) \Rightarrow (c)$ For given ϵ, there exists $M \in \mathbb{N}$ such that for $j \geq M$,

$$\|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^n_\mu} < \epsilon.$$

Hence, for each $i \in \{0, ..., n+1\}$

$$f_{i+1,a}(z) = (1-|a|^2)^{i+1}\Big(\sum_{k=0}^{M-1}\binom{i+k}{k} + \sum_{k=M}^{\infty}\binom{i+k}{k}\Big)\overline{a}^k z^k.$$

So,

$$\|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} \leq$$
$$2\max\{\|T^m_{u,v,\varphi} p_j\|_{\mathcal{W}^n_\mu}\}_{j=0}^{M-1}(1-|a|^2)^i(1-|a|^M)\binom{i+M-1}{M-1} + 2^{i+1}\epsilon.$$

Hence,

$$\limsup_{|a|\to 1}\|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} \leq \epsilon.$$

Since ϵ is arbitrary, so $\limsup_{|a|\to 1}\|T^m_{u,v,\varphi} f_{i+1,a}\|_{\mathcal{W}^n_\mu} = 0$.

$(c) \Rightarrow (d)$ Let $\{a_k\}$ be any sequence in $\mathbb{D}$, such that $\lim_{k\to 1}|\varphi(a_k)| = 1$. For each $i \in \{0, ..., n+1\}$, applying (1) and Lemma 2.2, we have

$$\frac{\mu(a_k)|\varphi(a_k)|^{m+i}|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(a_k)|}{(1-|\varphi(a_k)|^2)^{m+i}} \leq \sup_{\varphi(a_k)\in\mathbb{D}}\|T^m_{u,v,\varphi} g_{i,\varphi(a_k)}\|_{\mathcal{W}^n_\mu} \leq$$
$$\sum_{j=1}^{n+2}|c^i_j|\sup_{\varphi(a_k)\in\mathbb{D}}\|T^m_{u,v,\varphi} f_{j,\varphi(a_k)}\|_{\mathcal{W}^n_\mu}.$$

Taking the limit when $k \to \infty$, we get

$$\lim_{|\varphi(a)|\to 1}\frac{\mu(a)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(a)|}{(1-|\varphi(a)|^2)^{m+i}} \leq \sum_{j=1}^{n+2}|c^i_j|\limsup_{|\varphi(a)|\to 1}\|T^m_{u,v,\varphi} f_{j,\varphi(a)}\|_{\mathcal{W}^n_\mu} = 0.$$

$(d) \Rightarrow (b)$ For given ϵ, there exists a $0 < \delta < 1$ such that

$$\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|(I^{n,u}_{i,\varphi} + I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{m+i}} < \epsilon, \quad i \in \{0, 1, ..., n+1\}. \tag{7}$$

Let $p_j(z) = z^j (j \geq m+n+1)$. So from (1) and Lemmas 1.1, 1.2, we obtain

$$\mu(z)|(T^m_{u,v,\varphi}p_j)^{(n)}(z)| \leq \tag{8}$$
$$\mu(z)\sum_{i=m}^{m+n+1}\frac{j!}{(j-i)!}|\varphi(z)|^{j-i}|(I^{n,u}_{i-m,\varphi}+I^{n,v}_{i-m-1,\varphi})(z)| \leq$$
$$\sum_{i=0}^{n+1}\underbrace{\sup_{|\varphi(z)|\leq\delta}\mu(z)\frac{j!}{(j-m-i)!}(1-|\varphi(z)|^2)^{i+m}|\varphi(z)|^{j-m-i}\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{i+m}}}_{F_i}+$$
$$\sum_{i=0}^{n+1}\underbrace{\sup_{|\varphi(z)|>\delta}\mu(z)\frac{j!}{(j-m-i)!}(1-|\varphi(z)|^2)^{i+m}|\varphi(z)|^{j-m-i}\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{i+m}}}_{E_i}.$$

Since $\{p_j\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, so $\{p_j^{(t)}\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, hence from Theorem 2.3, we get

$$\limsup_{j\to\infty} F_i = 0. \tag{9}$$

From Lemmas 1.1, 1.2, 1.3 and (7)

$$E_i \preceq \|z^j\|_{\mathcal{B}}\sum_{i=0}^{n+1}\sup_{|\varphi(z)|>\delta}\frac{\mu(z)|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{i+m}} \preceq \frac{2(n+2)}{e}\epsilon.$$

Since ϵ is arbitrary, so

$$\limsup_{j\to\infty} E_i = 0. \tag{10}$$

Also by simple calculation for each $k < n$, we have

$$\limsup_{j\to\infty} |(T^m_{u,v,\varphi}p_j)^{(k)}(0)| = 0. \tag{11}$$

Therefore from (8), (9), (10) and (11), we obtain (b).

$(d) \Rightarrow (a)$ For given ϵ, there exists a $0 < \delta < 1$ such that

$$\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{m+i}} < \epsilon, \quad i \in \{0,1,...,n+1\}. \tag{12}$$

Let $\{f_k\}$ be any bounded sequence in B_p such that converges to 0 uniformly on compact subsets of $\mathbb{D}$.

$$\begin{aligned}\mu(z)\Big|\Big(T^m_{u,v,\varphi}f_k\Big)^{(n)}(z)\Big| &\le \mu(z)\sum_{i=0}^{n+1}|f_k^{(i+m)}(z)||(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)| \\ &\le \sum_{i=0}^{n+1}\underbrace{\sup_{|\varphi(z)|\le\delta}\mu(z)|f_k^{(i+m)}(z)||(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}_{H_i} \\ &+\sum_{i=0}^{n+1}\underbrace{\sup_{|\varphi(z)|>\delta}\mu(z)|f_k^{(i+m)}(z)||(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}_{L_i}\end{aligned} \tag{13}$$

Since $f_k \to 0$ converge to 0 uniformly on compact subsets of $\mathbb{D}$, so $f_k^{(t)} \to 0$ converge to zero uniformly on compact subsets of $\mathbb{D}$. Hence, from the boundedness of $T^m_{u,v,\varphi}: B_p \to \mathcal{W}^n_\mu$ and Theorem 2.3, we obtain

$$\limsup_{k\to\infty} H_i = 0, \quad i \in \{0, ..., n+1\}. \tag{14}$$

Also by using Lemmas 1.1, 1.3 and (12), for any $i \in \{0, ..., n+1\}$, we have

$$\begin{aligned}L_i &= \sup_{|\varphi(z)|>\delta}\mu(z)(1-|\varphi(z)|^2)^{m+i}|f_k^{(i+m)}(z)|\frac{|(I^{n,u}_{i,\varphi}+I^{n,v}_{i-1,\varphi})(z)|}{(1-|\varphi(z)|^2)^{m+i}} \\ &\preceq \epsilon\|f_k\|_{B_p}\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty} L_i = 0, \quad i \in \{0, ..., n+1\}. \tag{15}$$

For any $j < n$, we get

$$|(T^m_{u,v,\varphi}f_k)^{(j)}(0)| \le \sum_{i=0}^{j+1}|f_k^{(i+m)}(\varphi(0))||(I^{k,u}_{i,\varphi}+I^{k,v}_{i-1,\varphi})(0)|.$$

Since $\lim_{k\to\infty} f_k^{(i+m)}(\varphi(0)) = 0$, so

$$\limsup_{k\to\infty} |(T^m_{u,v,\varphi} f_k)^{(j)}(0)| = 0. \tag{16}$$

Therefore, from (13), (14), (15), (16) and Lemma 3.1, we have (a).

$(a) \Rightarrow (c)$ Let $\{a_k\}$ be arbitrary sequence such that $\lim_{k\to 1} |a_k| = 1$. It is obvious that for each $i \in \{0, 1, ..., n+1\}$, $f_{i+1,a_k} \to 0$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. So by using Lemmas 2.1 and 3.1, we get

$$\lim_{|a_k|\to 1} \|T^m_{u,v,\varphi} f_{i+1,a_k}\|_{\mathcal{W}^n_\mu} = 0.$$

The proof is completed. $\square$

Remark 3.3. Setting $\mu(z) = (1-|z|^2)^\alpha$ and $n = 1(n = 2)$ in Theorems 2.3 and 3.2, some equivalent conditions for boundedness and compactness of operator $T^m_{u,v,\varphi} : B_p \to \mathcal{B}^\alpha (T^m_{u,v,\varphi} : B_p \to \mathcal{Z}^\alpha)$ are obtained.

Remark 3.4. Putting $n = 1$ and $\mu(z) = (1-|z|^2)\log\frac{2}{1-|z|^2}$ in Theorems 2.3 and 3.2, similar results are given for operator $T^m_{u,v,\varphi} : B_p \to \mathcal{B}_{\log}$.

Remark 3.5. Setting $v \equiv 0$ in Theorems 2.3 and 3.2, we have similar results for operator $D^m_{u,\varphi} : B_p \to \mathcal{W}^n_\mu$.

Acknowledgment

The author would like to express their sincere gratitude to the referee for a very careful reading of the paper and for all the valuable suggestions, which led to improvement in this paper.

References

[1] E. Abbasi and H. Vaezi, Estimates of essential norm of generalized weighted composition operators from Bloch type spaces to nth weighted type spaces, *Math. Slovaca*, 70(1) (2020), 71–80.

[2] E. Abbasi, S. Li and H. Vaezi, Weighted composition operators from the Bloch space to nth weighted-type, *Turk. J. Math.*, 44(1) (2020), 108-117.

[3] E. Abbasi, H. Vaezi and S. Li, Essential norm of weighted composition operators from H^∞ to nth weighted type spaces, *Mediterr. J. Math.*, 16 (2019), 133.

[4] F. Colonna and S. Li, Weighted composition operators from the Besov spaces to the Bloch spaces, *Bull. Malays. Math. Sci. Soc.*, 36 (2013), 1027–1039.

[5] L. COMTET, *Advanced Combinatorics*, D. Reidel, Dordrecht, (1974)

[6] O. Hyvärinen and M. Lindström, Estimates of essential norms of weighted composition operators between Bloch-type spaces, *J. Math. Anal. Appl.*, 393 (2012), 38–44.

[7] Y. Liu and Y. Yu, Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball, *J. Math. Anal. Appl.*, 423(1) (2015), 76–93.

[8] Y. Liu and Y. Yu, The product-type operators from Logarithmic Bloch spaces to Zygmund-type spaces, *Filomat*, 33(12) (2019), 3639–3653.

[9] S. Stević, Composition operators from the weighted Bergman space to the nth weighted spaces on the unit disc, *Discrete Dyn. Nat. Soc.*, 2009 (2009), 11p.

[10] S. Stević, Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk, *Appl. Math. Comput.*, 216 (2010), 3634–3641.

[11] S. Stević, A. Sharma and R. Krishan, Boundedness and compactness of a new product-type operator from general space to Bloch-type spaces, *J. Inequal. Appl.*, 2016(219) (2016), 32 p.

[12] S. Stević, Essential norm of some extensions of the generalized composition operators between k-th weighted-type spaces, *J. Ineq. Appl.*, 2017(220) (2017), 13 p.

[13] S. Stević and A. Sharma, On a product-type operator between Hardy and α-Bloch spaces of the upper half-plane, *J. Inequal. Appl.*, 2018 (273) (2018), 18 p.

[14] M. Tjani, Compact composition operators on some Möbius invariant Banach space, PhD dissertation, Michigan State university, (1996).

[15] K. Zhu, Analytic Besov spaces. *J. Math. Anal. Appl.*, 157(2) (1991), 318–336.

[16] K. Zhu, Bloch type spaces of analytic functions, *Rocky Mountain J. Math.*, 23(3) (1993), 1143–1177.

[17] K. Zhu, *Operator Theory in Function Spaces*, Amer. Math. Soc., second edition, (2007)

[18] X. Zhu, Weighted composition operators from the minimal möbius invariant space into nth weighted-type spaces, *Ann. Funct. Anal.,* 11 (2020), 379–390.

[19] X. Zhu, E. Abbasi and A. Ebrahimi, A class of operator-related composition operators from the Besov spaces into the Bloch space, *Bull. Iran. Math. Soc.*, 47 (2021), 171–184.

Ebrahim Abbasi
Assistant Professor of Mathematics
Department of Mathematics
Mahabad Branch, Islamic Azad University
Mahabad, Iran
E-mail: ebrahimabbasi81@gmail.com, e.abbasi@iau-mahabad.ac.ir