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Original Research Paper

A Hyperstructures Approach to the Direct Limit and Tensor Product for Left(right) G-sets on Hypergroups

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Abstract. In this paper, we introduce the concept of left(right)-G sets by external hyperoperations and some examples presented. We construct quotient left(right)-G sets by regular (strongly) regular relations. Also, we consider fundamental relation as a smallest strongly regular relations and by complete parts concepts introduce an equivalence relation that is coincide with fundamental relation. The main purpose of this paper is to introduce the concepts of tensor product and direct limit on G-sets of n-ary semihypergroups that are non-additive modification of classical construction in module theory. This concept is crucially important in homological algebra and several properties are found and examples are presented.

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1 Introduction

The concept of n-group was introduced by Dörnte[9], which is a natural generalization of the notion of group. Since then, many papers concerning various n-ary algebras have appeared in literature. Another field which proved to be relevant was of algebraic hyperstructures. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element while in an algebraic hyperstructure, the composition of two elements is a set. The notion of hypergroup was introduced in 1934 by a French mathematician F. Marty [15], at the 8^{th} Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions and non-commutative groups. Since then, hundreds of papers and several books have been written on this topic and several kinds of hypergroups have been intensively studied, such as: regular hypergroups, reversible regular hypergroups, canonical hypergroups, cogroups, cyclic hypergroups, reduced hypergroups and associativity hypergroups(for example see [2, 3, 4]).

The recent book on hyperstructures [6] points out their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Moreover, Davvaz and Vougiouklis [7] have established a connection between the two domains in the form of an extension of the concept of n-ary groups to the concept of n-ary hypergroups, which has also proved to be of great interest and they were studied by Ghadiri and Waphare [11] and others [5, 8, 10, 17, 16, 20, 1].

In this paper, we define the left(right) G-sets in the context of n-ary semihypergroups. Furthermore, we define direct system and direct limit of G-sets and prove that some properties about them. We note that this concepts defined and considered only by binary operation and binary hyperoperation [13, 18]. We generalized this concept by external hyperoperation on n-ary semihypergroups. Finally, we introduce the concept of tensor product that is a non-additive modification of classical in module theory and play an important role in homological algebra [19]. Also, we prove that the tensor product exists and is unique up to isomorphism.

2 Basic definitions

In this paragraph, we present some definitions concerning n-ary semihypergroups. Let G be a non-empty set and f be a mapping $f : G\times G \longrightarrow \mathcal{P}^*(G)$, where $\mathcal{P}^*(G)$ is the set of all non-empty subsets of G. Then, f is called a *binary hyperoperation* on G. We denote by G^n the cartesian product $G \times G.... \times G$, where G appears n times. The couple (G, f) is called *hypergroupoid*. When, $n = 2$, for any two non-empty subsets G_1 and G_2 of G, we define

$$G_1 \circ G_2 = \bigcup_{g_1\in G_1, g_2\in G_2} g_1 \circ g_2.$$

In this case, a hypergroupoid (G, f) is called *semihypergroup* if for all g_1, g_2 and g_3 of G, we have $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

In general, $f : G^n \longrightarrow \mathcal{P}^*(G)$ is called an *n-ary hyperoperation* on G and (G, f) is called *n-ary hypergroupoid*.
Let G_1,G_2,...,G_n be non-empty subsets of G. Then, we define

$$f(G_1, G_2, ..., G_n) = \bigcup_{g_i\in G_i, i\in\{1,2,...,n\}} f(g_1, g_2, ..., g_n).$$

The sequence $g_i, g_{i+1}, ..., g_j$, will be denoted by g_i^j. For $j < i$, g_i^j is the empty set.

Definition 2.1. [7] The n-ary hypergroupoid (G, f) is called *n-ary semihypergroup* if for any $i, j \in \{1, 2, ..., n\}$ and $g_1^{2n-1} \in G$,

$$f(g_1^{i-1}, f(g_i^{n+i-1}), g_{n+i}^{2n-1}) = f(g_1^{j-1}, f(g_j^{n+j-1}), g_{n+j}^{2n-1}).$$

An n-ary semihypergroup (G, f) has an identity element if there is an element $e \in G$ such that

$$x \in f\left(e^{(i-1)}, x, e^{(i-1)}\right),$$

for all $x \in G$ and all $1 \leq i \leq n$.

An n-ary semihypergroup (G, f) is *commutative* if for all $g_1^n \in G$ and any permutation σ of $\{1, 2, ..., n\}$, we have

$$f(g_1^n) = f(g_{\sigma(1)}, g_{\sigma(2)}, ..., g_{\sigma(n)}).$$

Let (G, f) be an n-ary hypergroup such that there exists a unique $0 \in G$ such that $g = f\left(\overset{(i-1)}{0}, g, \overset{(n-i)}{0}\right)$. Also, there exists a unitary operation $-$ on G such that
$g \in f(g_1^n)$ implies that $g_i \in f(-g_{i-1}, ..., -g_1, g, -g_n, ..., -g_{i+1})$, for all $1 \leq i \leq n$. Then, (G, f) is called *n-ary polygroup* and a commutative n-ary polygroup is called *canonical n-ary hypergroup*.

Let (G, f) be an n-ary semihypergroup and H be a non-empty subset of G. Then, H is an n-ary sub semihypergroup of G if it is close under the n-ary hyperoperation f, i.e., for every $(h_1, h_2, ..., h_n) \in H^n$ implies that $f(h_1, h_2, ..., h_n) \subseteq H$.

The n-ary semihypergroup (G, f) is called *n-ary hypergroup*, when the equation $g \in f\left(g_1^{i-1}, x_i, g_{i+1}^n\right)$ has the solution $x_i \in G$ for any $g_1^{i-1}, g, g_{i+1}^n \in G$ and $1 \leq i \leq n$.

Let(G_1, f_1) and (G_2, f_2) be two n-ary semihypergroups. Then, a mapping $\varphi : G_1 \longrightarrow G_2$ is called a *homomorphism* if for all $x_1^n \in G_1$, we have

$$\varphi(f_1(x_1, x_2, ..., x_n)) = f_2(\varphi(x_1), \varphi(x_2), ..., \varphi(x_n)).$$

When G_1 and G_2 are n-ary semihypergroups with identity, $\varphi(e_1) = e_2$.

Example 2.2. Let $(G, +)$ be a semihypergroup and f be an n-ary hyperoperation on G as follows:

$$\forall g_1^n \in G, \quad f(g_1^n) = \sum_{i=1}^{n} g_i.$$

Then, (G, f) is an n-ary semihypergroup.

Example 2.3. Let G be a group and $< x, y >$ be a subgroup of G generated by x and y. Then, we define

$$f(g_1, g_2, ..., g_n) =< g_1, g_2, ..., g_n >,$$

where $g_1^n \in G$. We obtain that (G, f) is an n-ary hypergroup.

Example 2.4. Let G be a semigroup and N be a normal subsemigroup of G. Then, for all $g_1^n \in G$, we define $f(g_1, g_2, ..., g_n) = g_1 g_2 ... g_n N$. Hence, (G, f) is an n-ary semihypergroup.

Example 2.5. Let D be an integral domain and F be its field of fractions and snd U be the group of the invertible elements of D. Then, we define

$$f(\overline{g_1}, \overline{g_2}, ..., \overline{g_n}) = \{\overline{g} : \exists u_1^n \in U, g = u_1 g_1 + u_2 g_2 + ... + u_n g_n\},$$

where $\overline{g_i} \in F/U$ with $1 \leq i \leq n$. Hence, $(F/U, f)$ is an n-ary semihypergroup.

Example 2.6. Let V be a vector space over an ordered field F and $x_1, x_2, ..., x_n \in V$. Then, we define

$$f(x_1, x_2, ..., x_n) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Hence, (V, f) is an n-ary semihypergroup.

Example 2.7. Let $G = \{a_1, a_2, a_3\}$. Then, G is a 3-ary semihypergroup by following hyperoperation:

$$f(a_1, a_1, a_1) = f(a_2, a_2, a_2) = \{a_1, a_2\}, \quad f(a_3, a_3, a_3) = \{a_3\},$$

$$f(a_3, a_1, a_1) = f(a_1, a_3, a_1) = f(a_1, a_1, a_3) = \{a_3\},$$

$$f(a_3, a_1, a_2) = f(a_1, a_3, a_2) = \{a_3\},$$

$$f(a_1, a_2, a_3) = \{a_3\}, f(a_3, a_2, a_2) = f(a_2, a_3, a_2) = f(a_2, a_2, a_3) = \{a_3\}.$$

3 Left(right) G-sets

In this section, we generalize the concept of tensor product of left(right) G-sets as a generalization of semigroups[12].

Let G be an n-ary semihypergroup and X be a non-empty set. Then, we say that X is a *left G-set* if there is an external hyperoperation $h : G^{n-1} \times X \longrightarrow \mathcal{P}^*(X)$ with the property

$$h\left(f(g_1^n), g_{n+1}^{2n-2}, x\right) = h\left(g_1, f(g_2^n, g_{n+1}), g_{n+2}^{2n-2}, x\right) = ... = h\left(g_1^{n-1}, h\left(g_n^{2n-2}, x\right)\right),$$

where $g_1^{2n-2} \in G$ and $x \in X$. If e is an scalar identity of G, we say that X has an unitary when $h\left(e^{(n-1)}, x\right) = x$, for every $x \in X$.

Dually, a non-empty set X is a right G-set if there is an external hyperoperation $h : X \times G^{n-1} \longrightarrow \mathcal{P}^*(X)$,

$$h\left(x, g_1^{n-2}, f\left(g_{n-1}^{2n-2}\right)\right) = ... = h\left(h\left(x, g_1^{n-1}\right), g_n^{2n-2}\right).$$

In the same way, we say that X has an unitary when $h\left(x, e^{(n-1)}\right) = x$, for every $x \in X$.

Let G and H be n-ary semihypergroups. Then, we say that X is a (G,H)*-set* if it is a left G-set by external hyperoperation $h_1 : G^{n-1} \times X \longrightarrow \mathcal{P}^*(X)$ and a right H-set by external hyperoperation $h_2 : X \times H^{n-1} \longrightarrow \mathcal{P}^*(X)$ and

$$h_2(h_1(g_1^{n-1}, x), t_1^{n-1}) = h_1(g_1^{n-1}, h_2(x, t_1^{n-1})),$$

where $g_1^{n-1} \in G, t_1^{n-1} \in H$ and $x \in X$.

Let G be a canonical n-ary hypergroup and X be a left G-set. Then, we say that X is *reversible* if $x_1 \in h(g_1, g_2, \cdots, g_{n-1}, x_2)$ implies that $x_2 \in h(-g_{n-1}, -g_{n-2}, \cdots, -g_1, x_1)$, where $x_1, x_2 \in X$ and $g_1, g_2, \cdots, g_{n-1} \in G$.

Let X be a left G-set, G be an n-ary semigroup and $h : G^{n-1} \times X \longrightarrow X$. Then, we say that X is a *multiplicative left G-set*.

Example 3.1. Let G be a canonical n-ary hypergroup and N be a sub canonical n-ary hypergroup of G. Then, we define the relation N^* on G as follows:

$$xN^*y \Longleftrightarrow\ f\left(x, -y, 0^{(n-2)}\right) \cap N \neq \emptyset.$$

It is not difficult to see that N^* is an equivalence relation. Hence, $N^*(x) = f\left(N, x, 0^{(n-2)}\right)$ and the set of all equivalence classes $G/N^* = \{N^*(x) : x \in G\}$ is a left G-set as follows:

$$\begin{aligned} h : G^{n-1} \times G/N^* &\longrightarrow \mathcal{P}^*(G/N^*) \\ (g_1^{n-1}, N^*(x)) &\longrightarrow \{N^*(t) : t \in f(g_1^{n-1}, x)\}. \end{aligned}$$

Definition 3.2. Let G be a canonical n-ary hypergroup and X be a reversible left G-set. Then, we define the relation $\equiv$ on X as follows:

$$x_1 \equiv x_2 \Longleftrightarrow\ \exists g_1^{n-1} \in G :\ x_1 \in h(g_1^{n-1}, x_2).$$

Proposition 3.3. *Let G be a canonical n-ary hypergroup and X be a reversible left G-set and has an unitary. Then, the relation $\equiv$ is an equivalence.*

Proof. Suppose that $x \in X$. Since $x = h(e^{(n-1}, x)$, it implies that the relation $\equiv$ is reflexive. Let $x_1, x_2 \in X$ and $x_1 \equiv x_2$. Then, $x_1 \in h(g_1^{n-1}, x_2)$. Since X is reversible, we have $x_2 \in h(-g_{n-1}, -g_{n-2}, \cdots, -g_1, x_1)$. Hence, $\equiv$ is symmetric. Let $x_1, x_2, x_3 \in X$ such that $x_1 \in h(g_1^{n-1}, x_2)$ and $x_2 \in h(k_1^{n-1}, x_3)$, where $g_1^{n-1}, k_1^{n-1} \in G$. This implies that

$$x_1 \in h(g_1^{n-1}, x_2) \subseteq h(g_1^{n-1}, h(k_1^{n-1}, x_3)) = h(f(g_1^{n-1}, k_1), k_2^{n-1}, x_3).$$

Then, there exists $g \in f(g_1^{n-1}, k_1)$ such that $x_1 \in h(g, k_2^{n-1}, x_3)$. This implies that the relation $\equiv$ is transitive. Therefore, the relation $\equiv$ is equivalence. □

We denote the equivalence class of $x \in X$ with respect to the equivalence relation $\equiv$ by $orb(x)$ and it is called orbital of x. Hence,

$$orb(x) = \left\{ t \in X : \ \exists g_1^{n-1} \in G, \quad t \in h(g_1^{n-1}, x) \right\}.$$

Definition 3.4. Let G be an n-ary semihypergroup and X be a left G-set and $x \in X$. Then, stabilizer x defined as follows:

$$Stab(x) = \left\{ g \in G : x = h(\underbrace{g, g, \cdots, g}_{n-1}, x) \right\}.$$

When X is a left G-set with unitary and $x \in X$. We have $x = h(e^{(n-1)}, x)$. Hence, $Stab(x) \neq \emptyset$.

Proposition 3.5. *Let G be a commutative n-ary semihypergroup, X be a left G-set with unitary and $x \in X$. Then, $Stab(x)$ is a commutative n-ary sub semihypergroup of G.*

Proof. Since $e \in Stab(x)$, we have $Stab(x)$ is a non-empty set. Let $g_1, g_2, \cdots, g_n \in Stab(x)$. Then,

$$h(\underbrace{g_1, g_1, \cdots, g_1}_{n-1}, x) = h(\underbrace{g_2, g_2, \cdots, g_2}_{n-1}, x) = \cdots = h(\underbrace{g_n, g_n, \cdots, g_n}_{n-1}, x) = x.$$

Since G is a commutative n-ary semihypergroup, we have

$$h(\underbrace{f(g_1, g_2, \cdots, g_n), f(g_1, g_2, \cdots, g_n), \cdots, f(g_1, g_2, \cdots, g_n)}_{n-1}, x) = x.$$

This implies that $h(\underbrace{g, g, \cdots, g}_{n-1}, x) = x$, for every $g \in f(g_1, g_2, \cdots, g_n)$. Therefore, $f(g_1, g_2, \cdots, g_n) \subseteq Stab(x)$. This completes the proof. □

Example 3.6. Let G be a canonical hypergroup and $X = G$. Then, X is a reversible left G-set by external hyperoperation $h : G \times X \longrightarrow \mathcal{P}^*(X)$ such that $h(g, x) = g^{-1}xg$, where $g \in G$ and $x \in X$. Indeed,

$$h(g_1, h(g_2, x)) = g_1^{-1}h(g_2, x)g_1 = g_1^{-1}g_2^{-1}xg_2g_1 = h(g_1g_2, x).$$

Let $x_1 \in h(g, x_2)$, where $x_1, x_2 \in X$ and $g \in G$. Then, $x_1 \in g^{-1}x_2g$. Hence, there exists $k \in g^{-1}x_2$ such that $x_1 \in kg$. This implies that $k \in x_1g^{-1}$ and $x_2 \in gk$. Then, $x_2 \in gx_1g^{-1} = h(g^{-1}, x_1)$. Therefore, X is a reversible left G-set. Also, for every $x \in X$, we have

$$orb(x) = \{t \in X : \exists\, g \in G,\ t \in g^{-1}xg\}, \quad Stab(x) = \{g \in G : gx = xg\}.$$

Proposition 3.7. *Let G be a commutative n-ary semihypergroup and X be a left G-set. Then, X is a (G, G)-set.*

Proof. Since X is a left G-set, there exists an external hyperoperation $h : G^{n-1} \times X \longrightarrow \mathcal{P}^*(X)$ such that

$$h\left(f(g_1^n), g_{n+1}^{2n-2}, x\right) = h\left(g_1, f(g_2^n, g_{n+1}), g_{n+2}^{2n-2}, x\right) = ... = h\left(g_1^{n-1}, h\left(g_n^{2n-2}, x\right)\right).$$

Let right external hyperoperation $\widehat{h} : X \times G^{n-1} \longrightarrow \mathcal{P}^*(X)$ defined by $\widehat{h}(x, g_1^{n-1}) = h(g_1^{n-1}, x)$, where $g_1^{n-1} \in G$ and $x \in X$. Then, one can see that X is a right G-set and (G, G)-set. □

Example 3.8. Let K_4 be a Kelain group and $S = \{a_1, a_2, a_3, a_4, a_5\}$. Then, S is a left K_4-set by following hyperoperation:

	a_1	a_2	a_3	a_4	a_5
e	$\{a_1\}$	$\{a_2, a_3\}$	$\{a_2, a_3\}$	$\{a_4\}$	$\{a_5\}$
a	$\{a_2, a_3\}$	$\{a_1\}$	$\{a_1\}$	$\{a_5\}$	$\{a_4\}$
b	$\{a_4\}$	$\{a_5\}$	$\{a_5\}$	$\{a_1\}$	$\{a_2, a_3\}$
ab	$\{a_5\}$	$\{a_4\}$	$\{a_4\}$	$\{a_2, a_3\}$	$\{a_1\}$

Where $K_4 = \{e, a, b, ab\}$ such that $a^2 = b^2 = e$ and $ab = ba$. Since K_4 is an abelian group, we can consider S as a right K_4-set.

Definition 3.9. A map $\varphi : X \longrightarrow Y$ from a left G-set X into a left G-set Y is called *G-map* if

$$\varphi(h_1(g_1^{n-1}, x)) = h_2(g_1^{n-1}, \varphi(x)).$$

When X and Y are (G, H)-sets and $\varphi : X \longrightarrow Y$ is G-map and H-map, then φ is called *(G, H)-map.* A G-map φ is called *isomorphism*, when it is both one to one and onto.

Let $Mor(X, Y)$ be the set of all G-maps from X into Y such that X and Y be left G-sets and $h_1 : G^{n-1} \times X \longrightarrow \mathcal{P}^*(X)$, $h_2 : G^{n-1} \times Y \longrightarrow \mathcal{P}^*(Y)$. Then, we define

$$\begin{aligned} h : G^{n-1} \times Mor(X, Y) &\longrightarrow Mor(X, Y) \\ (g_1^{n-1}, \varphi) &\longmapsto \overline{\varphi}, \end{aligned}$$

where $\overline{\varphi} : X \longrightarrow Y$ and $\overline{\varphi}(x) = h_2\left(g_1^{n-1}, \varphi(x)\right)$. Hence,

$$\begin{aligned} \overline{\varphi}(h_1\left(g_1^{n-1}, x\right)) &= h_2\left(g_1^{n-1}, \varphi(h_1\left(g_1^{n-1}, x\right))\right) \\ &= h_2\left(g_1^{n-1}, h_2\left(g_1^{n-1}, \varphi(x)\right)\right) \\ &= h_2\left(g_1^{n-1}, \overline{\varphi}(x)\right). \end{aligned}$$

This implies that $\overline{\varphi} \in Mor(X, Y)$. Moreover, for every $x \in X$,

$$\begin{aligned} h(f(g_1^n), g_{n+1}^{2n-2}, \varphi)(x) &= h_2(f(g_1^n), g_{n+1}^{2n-2}, \varphi(x)) \\ &= h_2(g_1, f(g_2^n, g_{n+1}), g_{n+2}^{2n-2}, \varphi(x)) \\ &= h(g_1, f(g_2^n, g_{n+1}), g_{n+2}^{2n-2}, \varphi)(x). \end{aligned}$$

This implies that $h(f(g_1^n), g_{n+1}^{2n-2}, \varphi) = h\left(g_1, f(g_2^n, g_{n+1}), g_{n+2}^{2n-2}, \varphi\right)$. In the same way, we can see

$$h\left(f(g_1^n), g_{n+1}^{2n-2}, \varphi\right) = h(g_1, f(g_2^n, g_{n+1}), g_{n+2}^{2n-2}, \varphi) = \cdots = h\left(g_1^{n-1}, h(g_n^{2n-2}, \varphi)\right).$$

Hence, $Mor(X, Y)$ is a left G-set.

Let X be a left G-set, ρ be an equivalence relation on X and $A, B \subseteq X$. Then, we define

$$A\overline{\rho}B \Longleftrightarrow \ \forall a \in A, \ \exists \ b \in B : a\rho b \ \text{ and } \ \forall b \in B, \ \exists \ a \in A : a\rho b.$$

Also,

$$A\overline{\overline{\rho}}B \Longleftrightarrow \ \forall \ a \in A, \forall \ b \in B, \ a\rho b.$$

An equivalence relation ρ is called *regular* on a left G-set X, when

$$x_1\rho \ x_2 \Longrightarrow h(g_1^{n-1}, x_1)\overline{\rho} \ h(g_1^{n-1}, x_2), \forall g_1^{n-1} \in G.$$

An equivalence relation ρ is called *strongly regular* on a left G-set X, when

$$x_1\rho \ x_2 \Longrightarrow h(g_1^{n-1}, x_1)\overline{\overline{\rho}} \ h(g_1^{n-1}, x_2), \forall g_1^{n-1} \in G.$$

The quotient set $[X : \rho]$ is a left G-set by $\overline{h} : G^{n-1} \times [X : \rho] \longrightarrow \mathcal{P}^*([X : \rho])$, where

$$\overline{h}(g_1^{n-1}, \rho(x)) = \{\rho(t) : t \in h(g_1^{n-1}, x)\}.$$

We note that there is a G-map $\varphi : X \longrightarrow [X : \rho]$ by $\varphi(x) = \rho(x)$.

Example 3.10. Let G be an n-ary semigroup. Then, G is a (G, G)-set, where the action of G on G is defined by means of multiplication.

Example 3.11. Let H be an n-ary subsemigroup of G. Then, G is a (H, H)-set in the obvious way.

Example 3.12. Let G be an n-ary semihypergroup and X be an n-ary subsemihypergroup of G and

$$\begin{aligned} h : G^{n-1} \times X \ &\longrightarrow X \\ (g_1^{n-1}, x) \ &\longmapsto e, \end{aligned}$$

where e is a scalar identity and $g_1^{n-1} \in G$ and $x \in X$. Then, X is a left G-set.

Example 3.13. Let $G = \bigcup_{n \geq 0} A_n$ such that $A_0 = \{0\}$, $A_n = [n, n + 1)$ and X be positive integers numbers. We define

$$\begin{aligned} f : G^n &\longrightarrow \mathcal{P}^*(G) \\ (g_1^n) &\longmapsto A_t, \end{aligned}$$

where $t = max\{m_1, m_2, ..., m_n\}$ and $g_i \in A_{m_i}$. Then, (G, f) is n-ary semihypergroup. Also,

$$\begin{aligned} h : G^{n-1} \times X &\longrightarrow X \\ (g_1^{n-1}, x) &\longmapsto max\{m_1, m_2, ..., m_{n-1}, x\}. \end{aligned}$$

Then, X is a left G-set.

It is clear that the cartesian product $X \times Y$ of a left G_1-set X and a right G_2-set Y becomes (G_1, G_2)-set by the following definitions:

$$\overline{h}_1(g_1^{n-1}, (x, y)) = (h_1(g_1^{n-1}, x), y),$$

$$\overline{h}_2((x, y), t_1^{n-1}) = (x, h_2(y, t_1^{n-1})),$$

where $x \in X$, $y \in Y$, $g_i \in G_1$ and $t_j \in G_2$, for $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$.

Definition 3.14. Let X be a left G-set with unitary. Then, we define

$$a\beta_m b \iff \exists\ g_{ij} \in G,\ 1 \leq i \leq m, 1 \leq j \leq n-1,\ x \in X$$

$$:\ \{a, b\} \subseteq h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right),$$

where $a, b \in X$ and $m \geq 1$. Let $\beta = \bigcup_{m \geq 1} \beta_m$. Clearly, the relation β is reflexive and symmetric. We denote by β^* the transitive closure of β.

Let X be a left G-set. Then, we define

$$P = \left\{h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right) : x \in X, m \in N\right\}.$$

Theorem 3.15. *Let X be a left G-set with unitary. Then, β^* is the smallest strong regular relation on X.*

Proof. Suppose that $a\beta^*b$ and $k_1^{n-1} \in G$. It follows that there exist $x_0 = a, x_1, x_2, ..., x_n = b$ such that for all $i \in \{0, 1, 2, \cdots, n-1\}$, we have $x_i\beta x_{i+1}$. Let $u_1 \in h(k_1^{n-1}, a)$ and $u_2 \in h(k_1^{n-1}, b)$. We check that $u_1\beta^*u_2$. From $x_i\beta x_{i+1}$, it follows that there exists a hyperproduct P_i such that $\{x_i, x_{i+1}\} \subseteq P_i$. Hence, $h(k_1^{n-1}, x_i) \subseteq h(k_1^{n-1}, P_i)$ and $h(k_1^{n-1}, x_{i+1}) \subseteq h(k_1^{n-1}, P_i)$, which means that $h(k_1^{n-1}, x_i)\overline{\overline{\beta}}h(k_1^{n-1}, x_{i+1})$. Hence, for all $i \in \{0, 1, 2, \cdots, m-1\}$ and for all $s_i \in h(k_1^{n-1}, x_i)$, we have $s_i\beta s_{i+1}$. We consider $s_0 = u_1$ and $s_m = u_2$. It follows that, $u_1\beta^*u_2$. Then, the equivalence relation β^* is strongly regular.

Let ρ be a strongly regular relation. Since ρ is reflexive $\beta_1 \subseteq \rho$. Suppose that $\beta_{m-1} \subseteq \rho$ and $a\beta_m b$. Hence

$$\{a, b\} \subseteq h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right).$$

There exist $u, v \in h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, h\left(g_{(m-3)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right)$ such that $a \in h\left(g_1^{n-1}, u\right)$ and $b \in h\left(g_1^{n-1}, v\right)$. We have $u\beta_{m-1}v$ and according to the hypothesis, we obtain $u\rho v$. Since ρ is strongly regular, it follows that $a\rho b$. Hence, $\beta_m \subseteq \rho$. Therefore, $\beta^* \subseteq \rho$. □

Definition 3.16. Let X be a left G-set. Then, β^* is called *fundamental relation* on X and the set of all equivalence classes $[X : \beta^*]$ is called the *fundamental G-set*.

Definition 3.17. Let G be an n-ary semihypergroup with scalar identity and X be a left G-set. Then, we define

$$a/b = \{x \in X : a \in h(e^{(n-2)}, b, x)\},$$

where $a \in X$ and $b \in G$.

Definition 3.18. Let G be an n-ary semihypergroup with scalar identity and X be a left G-set. Then, (X, h) is called a *join space* if the following condition holds for all $a, b \in X$ and all $c, d \in G$:

$$a/c \cap b/d \neq \emptyset \quad \Rightarrow \quad h(e^{(n-2)}, a) \cap h(e^{(n-1)}, b) \neq \emptyset.$$

Definition 3.19. Let G be an n-ary semihypergroup with scalar identity and a left G-set X be *join space*. Then, we define the following relation on X:

$$xJ_Gy \Longleftrightarrow h(e^{(n-2)}, G, x) \cap h(e^{(n-2)}, G, y) \neq \emptyset.$$

Theorem 3.20. *Let G be an n-ary semihypergroup with scalar identity and a left G-set X be join space. Then, the relation J_G is an equivalence relation on X and the equivalence class of an element $a \in X$ is as follows:*

$$J_G(a) = h(e^{(n-2)}, G, a)/G.$$

Proof. It's obvious that J_G is reflexive and symmetric. Now we prove that J_G is transitive: Suppose that aJ_Gb and bJ_Gc. Hence, $h(e^{(n-2)}, G, a) \cap h(e^{(n-2)}, G, b) \neq \emptyset$ and $h(e^{(n-2)}, G, b) \cap h(e^{(n-2)}, G, c) \neq \emptyset$. There exists $d_1 \in h(e^{(n-2)}, G, a) \cap h(e^{(n-2)}, G, b)$ and $d_2 \in h(e^{(n-2)}, G, b) \cap h(e^{(n-2)}, G, c)$. We have

$$\exists g \in G : d_1 \in h(e^{(n-2)}, g, b) \quad \longrightarrow b \in d_1/g \subseteq h(e^{(n-2)}, G, a)/G,$$

$$\exists g' \in G : d_2 \in h(e^{(n-2)}, g', b) \quad \longrightarrow b \in d_2/g' \subseteq h(e^{(n-2)}, G, c)/G.$$

Hence,

$$h(e^{(n-2)}, G, a)/G \cap h(e^{(n-2)}, G, c)/G \neq \emptyset,$$

$$h(e^{(n-2)}, G, a) \cap h(e^{(n-2)}, G, c) \neq \emptyset,$$

which completes the proof. $\square$

Definition 3.21. Let X be a left G-set and A be a non-empty subset of X. Then, we say that A is a *complete part* of X if for any non-zero natural number m and $g_{ij} \in G$, where $1 \leq i \leq m,\ 1 \leq j \leq n-1$ and $x \in X$, the following implication holds:

$$A \cap h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right) \neq \emptyset \Longrightarrow$$

$$h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right) \subseteq A.$$

If B is a subset of X, denote by $C(B)$ the complete closure of B, which is the smallest complete part of X that contains B.

Let $K_1(A) = A$ and for all $n \geq 1$, we define

$$K_{n+1}(A) = \Big\{ y \in X : \exists g_{m1}^{n-1}, g_{(m-1)1}^{n-1}, ..., g_{11}^{n-1} \in G, x \in X,$$

$$y \in h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right),$$

$$K_n(A) \cap h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) ...\right)\right)\right) \neq \emptyset \Big\}.$$

Also, we define $K(A) = \bigcup_{n \geq 1} K_n(A)$.

Theorem 3.22. *Let X be a left G-set and A be a non-empty subset of X. Then, $C(A) = K(A)$.*

Proof. Suppose that

$$K(A) \cap h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) \cdots\right)\right)\right) \neq \emptyset.$$

Then, there exists $n \geq 1$ such that

$$K_n(A) \cap h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) \cdots\right)\right)\right) \neq \emptyset.$$

This means that

$$h\left(g_{m1}^{n-1}, h\left(g_{(m-1)1}^{n-1}, h\left(g_{(m-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) \cdots\right)\right)\right) \subseteq K_{n+1}(A) \subseteq K(A).$$

Hence, $K(A)$ is a complete part. Now, if $A \subseteq B$ and B is a complete part of X, then we show that $K(A) \subseteq B$. We have $K_1(A) = A \subseteq B$ and suppose that $K_n(A) \subseteq B$. We check that $K_{n+1}(A) \subseteq B$. Let $z \in K_{n+1}(A)$. Then, there exists a hyperproduct

$$h\left(g_{p1}^{n-1}, h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) \cdots\right)\right)\right),$$

such that

$$z \in h\left(g_{p1}^{n-1}, h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) \cdots\right)\right)\right).$$

Hence,

$$B \cap h\left(g_{p1}^{n-1}, h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right) \cdots\right)\right)\right) \neq \emptyset.$$

Since B is a complete part, we obtain

$$h\left(g_{p1}^{n-1}, h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right)\cdots\right)\right)\right) \subseteq B.$$

We obtain $z \in B$. Therefore, $K(A)$ is a smallest complete part contains A. This implies that $C(A) = K(A)$. $\square$

Proposition 3.23. *Let X be a left G-set and x be an arbitrary element of X. Then,*

(1) For all $n \geq 2$, we have $K_n(K_2(x)) = K_{n+1}(x)$,

(2) $x \in K_n(y) \Longleftrightarrow y \in K_n(x)$.

Proof. (1) We proof this proposition by induction. We have

$$K_2(K_2(x)) = \Big\{y \in X : \exists g_{p1}^{n-1}, g_{(p-1)1}^{n-1}, ..., g_{11}^{n-1} \in G,$$

$$.y \in h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right)\right)\right)$$

$$K_2(x) \cap h\left(g_{(p-2)}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right)\right) \neq \emptyset\Big\} = K_3(x).$$

Suppose that $K_{n-1}(K_2(x)) = K_n(x)$. Then,

$$K_n(K_2(x)) = \Big\{z \in X : \exists g_{p1}^{n-1}, g_{(p-1)1}^{n-1}, ..., g_{11}^{n-1} \in G,$$

$$z \in h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right)\right)\right),$$

$$K_{(n-1)}(K_2(x)) \cap h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)}^{n-1}, \cdots, h\left(g_{11}^{n-1}, x\right)\right)\right) \neq \emptyset\Big\} = K_{(n+1)}(x).$$

(2)We check the equivalence by induction. For $n = 2$, we have

$$x \in K_2(y) = \Big\{z \in X : g_{p1}^{n-1}, g_{(p-1)1}^{n-1}, ..., g_{11}^{n-1} \in G, t \in X$$

$$z \in h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, t\right)\cdots\right)\right)$$

$$K_1(y) \cap h\left(g_{(p-1)1}^{n-1}, h\left(g_{(p-2)1}^{n-1}, \cdots, h\left(g_{11}^{n-1}, t\right)\cdots\right)\right) \neq \emptyset\Big\}.$$

Hence, $\{y,x\} \subseteq h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right)$. This implies that $y \in K_2(x)$. Let $x \in K_{(n-1)}(y)$ if and only if $y \in K_{(n-1)}(x)$. If $x \in K_n(y)$, then there exists $h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right)$ such that $x \in h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right)$ and

$$h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right) \cap K_{(n-1)}(y) \neq \emptyset.$$

Hence, there exists $v \in X$ such that

$$v \in h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right) \cap K_{(n-1)}(y).$$

Also,

$$\begin{aligned} K_2(x) = \Big\{ & z \in X : g^{n-1}_{p1}, g^{n-1}_{(p-1)1}, ..., g^{n-1}_{11} \in G, t \in X, \\ & z \in h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right) \\ & K_1(x) \cap h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, t\right)\cdots\right)\right) \neq \emptyset \Big\}. \end{aligned}$$

By definition $K_2(x)$ we have $v \in K_2(x)$. Also, by induction hypothesis $v \in K_{(n-1)}(y)$ implies that $y \in K_{(n-1)}(v)$.

It follows that $v \in K_2(x)$ and $y \in K_{(n-1)}(v)$. Hence, $y \in K_{(n-1)}(K_2(x)) = K_n(x)$. Similarly, we obtain the converse implication. □

Corollary 3.24. *Let X be a left G-set. Then, the following relation is an equivalence:*

$$xK\ y \Longleftrightarrow \exists n \geq 1 \ : \ x \in K_n(y).$$

Theorem 3.25. *Let X be a left G-set. Then, the equivalence relation K and β^* are coincide.*

Proof. Suppose that $x, y \in X$ and $x\beta y$. This implies that

$$\{x,y\} \subseteq h\left(g^{n-1}_{p1}, h\left(g^{n-1}_{(p-1)1}, \cdots, h\left(g^{n-1}_{11}, x\right)\cdots\right)\right),$$

for some $g_{ij} \in G$, where $1 \leq i \leq p$ and $1 \leq j \leq n-1$. This implies that $x \in K_2(y) \subseteq K(y)$. Hence, $\beta \subseteq K$ whence $\beta^* \subseteq K$. Now, if we have

xKy and $x \neq y$, then there exists $n \geq 1$ such that xK_ny, which means that there exists

$$h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, x\right)\cdots\right)\right)$$

such that

$$x \in h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, x\right)\cdots\right)\right).$$

Let

$$x_1 \in h\left(g^{n-1}_{(p-1)1}, h\left(g^{n-1}_{(p-2)1}, \cdots, h\left(g^{n-1}_{11}, x\right)\cdots\right)\right) \cap K_n(y).$$

Then, $x\beta x_1$. Thus, $x_1 \in K_n(y)$. In the same way, after a finite number of steps, we obtain that there exist x_{n-1} and x_n such that $x_{n-1}\beta x_n$ and $x_n \in K_{n-(n-1)}(y) = \{y\}$. Therefore, $x\beta^* y$. $\square$

Proposition 3.26. *Let X and Y be left G-sets and $\alpha : X \longrightarrow Y$ be a G-map. Then, a map $\widehat{\alpha} : [X : \beta^*] \longrightarrow [Y : \beta^*]$, defined by $\widehat{\alpha}(\beta^*(x)) = \beta^*(\alpha(x))$ is a G-map.*

Proof. Suppose that $\beta^*(a) = \beta^*(b)$, where $a, b \in X$. Then, there exist $x_1, x_2, \cdots, x_n \in X$ such that $x_1 = a$ and $x_n = b$ and $x_i\beta x_{i+1}$, for $1 \leq i \leq n-1$. This implies that

$$\{x_i, x_{i+1}\} \subseteq h\left(g^{n-1}_{m_i 1}, h\left(g^{n-1}_{(m_i-1)1}, h\left(g^{n-1}_{(m_i-2)1}, \cdots, h\left(g^{n-1}_{11}, x\right)\cdots\right)\right)\right),$$

where $x \in X$, $m_i \in N$ and $g^{n-1}_{m_i 1} \in G$. Since α is a G-map, we have

$$\{\alpha(x_i), \alpha(x_{i+1})\} \subseteq h\left(g^{n-1}_{m_i 1}, h\left(g^{n-1}_{(m_i-1)1}, \cdots, h\left(g^{n-1}_{11}, \alpha(x)\right)\cdots\right)\right).$$

This implies that $\beta^*(\alpha(a)) = \beta^*(\alpha(b))$ and $\widehat{\alpha}$ is well-defined. Also, for every $\beta^*(x) \in [X : \beta^*]$ and $g_1, g_2, \cdots, g_{n-1} \in G$, we have

$$\begin{aligned}
\widehat{\alpha}(\overline{h}\left(g^{n-1}_1, \beta^*(x)\right)) &= \widehat{\alpha}(\left\{\beta^*(t) : t \in h\left(g^{n-1}_1, x\right)\right\}) \\
&= \left\{\beta^*(\alpha(t)) : t \in h\left(g^{n-1}_1, x\right)\right\} \\
&= \left\{\beta^*(a) : a = \alpha(t),\ t \in h\left(g^{n-1}_1, x\right)\right\} \\
&= \left\{\beta^*(a) : a \in h\left(g^{n-1}_1, \alpha(x)\right)\right\} \\
&= \overline{h}\left(g^{n-1}_1, \widehat{\alpha}(\beta^*(x))\right).
\end{aligned}$$

Therefore, the map $\widehat{\alpha}$ is a G-map. $\square$

4 Direct limit of (G, H)-sets

Let $(I, \leq)$ be a partially ordered set and $\{X_i : i \in I\}$ be a collection of (G, H)-sets, where G and H be n-ary semihypergroups. Also, for every $i, j \in I$ such that $i \leq j$, there are (G, H)-maps $\alpha_{ij} : X_i \longrightarrow X_j$ such that

(1) $\alpha_{ii} = I_{X_i}$,

(2) $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$.

Then, we say that $(X_i, \alpha_{ij})_{i,j \in I}$ is a *direct system* of (G, H)-sets.

A (G, H)-set X is called a *direct limit* of $(X_i, \alpha_{ij})_{i,j \in I}$ if there exists (G, H)-maps $\beta_i : X_i \longrightarrow X$ such that for all $i \leq j$, $\beta_j \circ \alpha_{ij} = \beta_i$. Also, if there exists a (G, H)-set Y with the property that there exists (G, H)-maps $\gamma_i : X_i \longrightarrow Y$ such that $\gamma_j \circ \alpha_{ij} = \gamma_i$, where $i \leq j$, then there is a unique (G, H)-map $\delta : X \longrightarrow Y$ such that $\delta \circ \beta_i = \gamma_i$, for every $i \in I$. We write $lim_{i \in I} X_i = X$.

Theorem 4.1. *Let $(X_i, \alpha_{ij})_{i,j \in I}$ be a direct system. Then, the direct limit exists and is unique up to isomorphism.*

Proof. Suppose that $(X_i, \alpha_{ij})_{i,j \in I}$ is a direct system. Then, there is no loss of generality in supposing that the sets $X_i (i \in I)$ are pairwise disjoint. Let Z be the union of all the sets X_i. Then, Z is a (G, H)-set in an obvious way. We define

$$\rho = \{(x_i, y_j) : x_i \in X_i, y_j = \alpha_{ij}(x_i),\ i, j \in I,\ i \leq j\},$$

and ρ^* be an equivalence relation generated by ρ. Let $x_i \rho y_j$ and $g_1^{n-1} \in G$. Then, $\alpha_{ij}(x_i) = y_j$. This implies that

$$h(g_1^{n-1}, y_j) = h(g_1^{n-1}, \alpha_{ij}(x_i)) = \alpha_{ij}(h(g_1^{n-1}, x_i)).$$

Hence, for every $b \in h(g_1^{n-1}, y_j)$ there exists $a \in h(g_1^{n-1}, x_i)$ such that $\alpha_{ij}(a) = b$. Thus, $a\rho b$. Also, $x\rho^* y$ implies that there exist $x_1, x_2, ..., x_n \in Z$ such that $x_i \rho x_{i+1}$. Hence, $h(g_1^{n-1}, x_i)\overline{\rho} h(g_1^{n-1}, x_{i+1})$ implies that $h(g_1^{n-1}, x_i)\overline{\rho^*} h(g_1^{n-1}, x_{i+1})$. We have a (G, H)-map $\beta_i : X_i \longrightarrow [Z : \rho^*]$ defined by $\beta_i(x_i) = \rho^*(x_i)$. Then, $\beta_j(\alpha_{ij}(x_i)) = \rho^*(\alpha_{ij}(x_i)) = \rho^*(x_i) = \beta_i(x_i)$, for every $x_i \in X_i$ and so $\beta_j \circ \alpha_{ij} = \alpha_i$. If Y is a (G, H)-set and

$\gamma_i : X_i \longrightarrow Y$ be G-maps such that $\gamma_j \circ \alpha_{ij} = \gamma_i$, for each $i, j \in I,\ i \leq j$, then we have a (G, H)-map $\varphi : Z = \cup_{i\in I} X_i$ into Y such that $\varphi(x_i) = \gamma_i(x_i)$, where $x_i \in X_i$ and $i \in I$. Also,

$$\varphi(\alpha_{ij}(x_i)) = \gamma_j(\alpha_{ij}(x_i)) = \gamma_i(x_i) = \varphi(x_i).$$

Hence, there exists a (G, H)-map $\omega : [Z : \rho^*] \longrightarrow Y$ defined by $\omega(\rho^*(x_i)) = \gamma_i(x_i)$. Moreover, $\omega \circ \beta_i(x_i) = \omega(\rho^*(x_i)) = \gamma_i(x_i)$, for all $x_i \in X_i$. Finally, the (G, H)-map ω is unique: If ω_1 is another (G, H)-map with the same properties, then for all $x_i \in X_i$ we have

$$\omega_1(\rho^*(x_i)) = \omega_1(\beta_i(x_i)) = \gamma_i(x_i) = \omega(\rho^*(x_i)).$$

Hence $\omega = \omega_1$. Let X and Y be direct limit of the direct system $(X_i, \alpha_{ij})_{i,j\in I}$. Then, we have a unique (G, H)-map $\omega_1 : X \longrightarrow Y$ and $\omega_2 : Y \longrightarrow X$ such that $\beta_i \circ \omega_1 = \gamma_i$ and $\gamma_i \circ \omega_2 = \beta_i$. Hence $\beta_i \circ \omega_1 \circ \omega_2 = \beta_i$ and $\gamma_i \circ \omega_2 \circ \omega_1 = \gamma_i$. This implies that $\omega_1 \circ \omega_2 = Id$ and $\omega_2 \circ \omega_1 = Id$. Therefore, the direct limit is unique up to isomorphism. □

Theorem 4.2. *Let $(X_i, \alpha_{ij})_{i,j\in I}$ be a direct system of left G-subsets of X, G be a commutative n-ary semihypergroup and β^* be a fundamental relation on X. Then, $([X_i : \beta^*], \widehat{\alpha}_{ij})_{i,j\in I}$ is a direct system of $[X : \beta^*]$. Also, $lim_{i\in I}[X_i : \beta^*] = [lim_{i\in I} X_i : \beta^*]$.*

Proof. Suppose that $(X_i, \alpha_{ij})_{i,j\in I}$ is a direct system and $\alpha_{ij} : X_i \longrightarrow X_j$ be (G, G)-maps for $i \leq j$. By Proposition 3.26, $\widehat{\alpha}_{ij} : [X_i : \beta^*] \longrightarrow [X_j : \beta^*]$ are (G, G)-maps, where $i, j \in I$ and $i \leq j$. Also,

$$\widehat{\alpha}_{ii}(\beta^*(x)) = \beta^*(\alpha_{ii}(x)) = \beta^*(x),$$

where $\beta^*(x) \in [X_i : \beta^*]$ and

$$\begin{aligned}\widehat{\alpha}_{ij} \circ \widehat{\alpha}_{jk}(\beta^*(x)\quad) &= \widehat{\alpha}_{ij}(\widehat{\alpha}_{jk}(\beta^*(x))) \\ &= \beta^*(\alpha_{ij} \circ \alpha_{jk}(x)) \\ &= \beta^*(\alpha_{ik}(x)) \\ &= \widehat{\alpha}_{ik}(\beta^*(x))\end{aligned}$$

,

where $\beta^*(x) \in [X_i : \beta^*]$. Hence, $([X_i, \beta^*], \widehat{\alpha}_{ij})_{i,j\in I}$ is a direct system.

Let X' be the direct limit of the direct system $(X_i, \alpha_{ij})_{i,j\in I}$. Then, $\alpha_{ij} : X_i \longrightarrow X_j$ and $\sigma_i : X_i \longrightarrow X'$ such that $\sigma_j \circ \alpha_{ij} = \sigma_i$. We define $\widehat{\sigma_i} : [X_i : \beta^*] \longrightarrow [X' : \beta^*]$ by $\widehat{\sigma_i}(\beta^*(x_i)) = \beta^*(\sigma_i(x_i))$, where $x_i \in X_i$. By Proposition 3.26, $\widehat{\sigma}_i$ are well-defined and (G, G)-maps. Also, the maps $\widehat{\alpha}_{ij} : [X_i : \beta^*] \longrightarrow [X_j : \beta^*]$ are well-defined and (G, G)-maps. Thus,

$$\widehat{\sigma}_j \circ \widehat{\alpha}_{ij}(\beta^*(x_i)) = \beta^*(\sigma_j \circ \alpha_{ij}(x_i)) = \beta^*(\sigma_i(x_i)) = \widehat{\sigma}_i(\beta^*(x_i)).$$

Let T be an another (G, G)-set and $\gamma_i : X_i \longrightarrow T$ be (G, G)-maps such that $\gamma_j \circ \alpha_{ij} = \gamma_i$. Then, $\widehat{\gamma_j} \circ \widehat{\alpha_{ij}} = \widehat{\gamma_i}$. Hence, there exists a unique (G, G)-map $\delta : X' \longrightarrow T$ such that $\delta \circ \sigma_i = \gamma_i$. By Proposition 3.26, $\widehat{\delta} : [X' : \beta^*] \longrightarrow [T : \beta^*]$ is a (G, G)-map and easy to see that $\widehat{\delta} \circ \widehat{\sigma_i} = \widehat{\gamma_i}$.
Now, $\widehat{\delta}$ is unique:
If $\widehat{\delta}_1$ is another (G, G)-map with the same properties of $\widehat{\delta}$, then

$$\begin{aligned} \widehat{\delta}_1(\widehat{\sigma}_i(\beta^*(x_i))) &= \beta^*(\delta_1 \circ \sigma_i(x_i)) \\ &= \beta^*(\delta \circ \sigma_i(x_i)) \\ &= \widehat{\delta} \circ \widehat{\sigma}_i(\beta^*(x_i)) \end{aligned}$$

This completes the proof. □

5 Tensor product

In this section, we generalize the concept of tensor product of n-ary semihypergroups as a generalization of semigroups[12].

Let X, Y and Z be (G_1, G_2)-set, (G_2, G_3)-set and (G_1, G_3)-set,respectively. Then, we know that $X \times Y$ is a (G_1, G_3)-set. We say that a map $\varphi : X \times Y \longrightarrow Z$ is called *bimap* if for all $x \in X$, $g_1^{n-1} \in G_2$ and $y \in Y$, we have

$$\varphi\left(h_1(x, g_1^{n-1}), y\right) = \varphi\left(x, h_2(g_1^{n-1}, y)\right).$$

A pair (P, ψ) consists of a (G_1, G_3)-set P and a bimap $\psi : X \times Y \longrightarrow P$ will be called a *tensor product* of X and Y over G_2 if for every (G_1, G_3)-set Z and every bimap $\beta : X \times Y \longrightarrow Z$, there exists a unique bimap $\overline{\beta} : P \longrightarrow Z$ such that $\overline{\beta} \circ \psi = \beta$.

Suppose that ρ^* is the equivalence relation on $X \times Y$ generated by the relation

$$\rho = \left\{ \left((t_1, y), (x, t_2) \right), x \in X, y \in Y, g_1^{n-1} \in G_2, t_1 = h_1\left(x, g_1^{n-1}\right), t_2 = h_2(g_1^{n-1}, y) \right\}.$$

where $g_1^{n-1} \in G_2$ are scalar elements. This means that for every $x \in X$ and $y \in Y$ we have $| h_1\left(x, g_1^{n-1}\right) |= 1$ and $| h_2(g_1^{n-1}, y) |= 1$. We denote a typical element $\rho^*(x,y)$ of $[X \times Y : \rho^*]$ by $x \otimes y$ and we define $X \otimes Y$ to be $[X \times Y : \rho^*]$. By definition of ρ^* we immediately have that

$$h_1(x, g_1^{n-1}) \otimes y = x \otimes h_2(g_1^{n-1}, y).$$

Example 5.1. Let G be an commutative n-ary semihypergroup and X be a non-empty set. Then, X is a left G-set as follows:

$$\begin{aligned} h : G^{n-1} \times X &\longrightarrow \mathcal{P}^*(X) \\ (g_1^{n-1}, x) &\longrightarrow \{x\} \end{aligned}$$

Hence for every $x, y \in X$, we have $\rho^*(x,y) = \{(x,y)\}$. Also, if we define $h(g_1^{n-1}, x) = X$, then we obtain $\rho^*(x,y) = X \times X$.

Example 5.2. Let $G = \{a_1, a_2, a_3\}$ be a canonical hypergroup by following hyperoperation and $X = G$. Then, X is a left G-set as follows:

	a_1	a_2	a_3
a_1	a_1	a_2	a_3
a_2	a_2	a_2	$\{a_1, a_2, a_3\}$
a_3	a_3	$\{a_1, a_2, a_3\}$	a_3

$$\begin{aligned} h : G \times X &\longrightarrow \mathcal{P}^*(X) \\ (g, x) &\longrightarrow g^{-1} x g \end{aligned}$$

Hence $[X \times Y : \rho^*] = X \times Y$.

Example 5.3. Let $(G, +)$ be a canonical hypergroup and N be a non-zero subcanonical hypergroup of G. Then, we define an equivalence relation N^* on G as follows:

$$x N^* y \Longleftrightarrow (x - y) \cap N \neq \emptyset,$$

where $x, y \in G$. The set of all equivalence classes $G/N^* = \{N^*(x) : x \in G\}$ is a left G-set as follows:

$$\begin{aligned} h : G \times G/N^* &\longrightarrow \mathcal{P}^*(G/N^*) \\ (g, N^*(x)) &\longrightarrow \{N^*(t) : t \in g + x\}. \end{aligned}$$

Indeed,

$$\begin{aligned} h(g_1, h(g_2, N^*(x))) &= \{h(g_1, N^*(t)) \ : \ N^*(t) \in h(g_2, N^*(x))\} \\ &= \{h(g_1, N^*(t)) \ : \ t \in g_2 + x\} \\ &= \{N^*(t') : t' \in g_1 + t \subseteq g_1 + (g_2 + x) = (g_1 + g_2) + x\} \\ &= \{N^*(t') : t' \in (g_1 + g_2) + x\} \\ &= h(g_1 + g_2, N^*(x)). \end{aligned}$$

Also,

$$h(0, N^*(x)) = \{N^*(t) : t \in 0 + x\} = N^*(x).$$

Let g be a scalar element of G. Then, for every $N^*(x) \in G/N^*$, we have $\mid h(g, N^*(x)) \mid= 1$. We claim, $g = 0$. Let $g \neq 0$. Then, $\mid h(g, N^*(0)) \mid= 1$. We have $\mid N^*(g) \mid= 1$. Since $g \in N^*(g)$ and for every $a \in g + N$, $a \in N^*(g)$, we have $a = g$. This implies that $N = \{0\}$ which is contradiction. Thus, the only scalar element of G is $\{0\}$. Therefore,

$$\begin{aligned} G/N^* \otimes G/N^* &= \{\rho^*(N^*(g_1), N^*(g_2)) : g_1, g_2 \in G\} \\ &= \{(N^*(g_1), N^*(g_2) : g_1, g_2 \in G\} \\ &= G/N^* \times G/N^*. \end{aligned}$$

Proposition 5.4. *Let X and Y be (G_1, G_2)- and (G_2, G_3)- sets, respectively, then $x_1 \otimes y_1 = x_2 \otimes y_2$ if and only if there exist $a_1, a_2, ..., a_n \in X$, $b_1, b_2, ..., b_{n-1} \in Y$ and scalar elements $t_{i1}^{n-1}, s_{j1}^{n-1} \in G_2$ where $1 \leq i \leq n-1$ and $1 \leq j \leq n$ such that*

$$\begin{aligned} x_1 &= h_1\left(a_1, s_{11}^{n-1}\right), \\ h_1\left(a_1, t_{11}^{n-1}\right) &= h_1\left(a_2, s_{21}^{n-1}\right), \\ &\vdots \\ h_1\left(a_i, t_{i1}^{n-1}\right) &= h_1\left(a_{i+1}, s_{(i+1)1}^{n-1}\right), \\ &\vdots \\ h_1\left(a_{n-1}, t_{(n-1)1}^{n-1}\right) &= h_1\left(x_2, s_{n1}^{n-1}\right). \end{aligned}$$

$$\begin{aligned} h_2\left(s_{11}^{n-1}, y_1\right) &= h_2\left(t_{11}^{n-1}, b_1\right), \\ h_2\left(s_{21}^{n-1}, b_1\right) &= h_2\left(t_{21}^{n-1}, b_2\right), \\ &\vdots \\ h_2(s_{n1}^{n-1}, b_{n-1}) &= y_2, \end{aligned}$$

Proof. Suppose that we have given sequence of equivalence equations. Then,

$$\begin{aligned} x_1 \otimes y_1 = h_1\left(a_1, s_{11}^{n-1}\right) \otimes y_1 = a_1 \otimes h_2(t_{11}^{n-1}, y_1) &= h_1(a_1, t_{11}^{n-1}) \otimes y_1 \\ &= h_1\left(a_2, s_{21}^{n-1}\right) \otimes y_1 \\ &\vdots \\ &= x_2 \otimes h_2(s_{n1}^{n-1}, b_{n-1}) \\ &= x_2 \otimes y_2. \end{aligned}$$

Conversely, suppose that $x_1 \otimes y_1 = x_2 \otimes y_2$. Hence there is a sequence $(p_1, h_1), (p_2, h_2), ..., (p_n, h_n)$ such that $(x_1, y_1) = (p_1, h_1)$,$(p_n, h_n) = (x_2, y_2)$ and $((p_i, h_i), (p_{i+1}, h_{i+1})) \in \rho$, $i = 1, \cdots, n-1$. By the definition ρ, we have the given sequence of equations. This completes the proof. □

Proposition 5.5. *Let X and Y be (G_1, G_2)-set and (G_2, G_3)-set, respectively. Then, $X \otimes Y$ is a (G_1, G_3)-set.*

Proof. Since X and Y are (G_1, G_2)- and (G_2, G_3)-sets, respectively, by definition

$$h_1 : X \times G_2^{n-1} \longrightarrow \mathcal{P}^*(X), \qquad h_1' : G_1^{n-1} \times X \longrightarrow \mathcal{P}^*(X),$$

$$h_2 : G_2^{n-1} \times Y \longrightarrow \mathcal{P}^*(Y), \qquad h_2' : Y \times G_3^{n-1} \longrightarrow \mathcal{P}^*(Y).$$

We define

$$\overline{h}_2\left(x \otimes y, s_1^{n-1}\right) = x \otimes h_2'\left(y, s_1^{n-1}\right),$$

$$\overline{h}_1\left(k_1^{n-1}, x \otimes y\right) = h_1'\left(k_1^{n-1}, x\right) \otimes y,$$

where $s_1^{n-1} \in G_3$, $k_1^{n-1} \in G_1$ and $x \in X, y \in Y$.

Suppose that $x_1 \otimes y_1 = x_2 \otimes y_2$. By the Proposition 5.4, we have

$$\begin{aligned} x_1 &= h_1\left(a_1, s_{11}^{n-1}\right), \\ h_1\left(a_1, t_{11}^{n-1}\right) &= h_1\left(a_2, s_{21}^{n-1}\right), \\ &\vdots \\ h_1\left(a_i, t_{i1}^{n-1}\right) &= h_1\left(a_{i+1}, s_{(i+1)1}^{n-1}\right), \\ &\vdots \\ h_1\left(a_{n-1}, t_{(n-1)1}^{n-1}\right) &= h_1\left(x_2, s_{n1}^{n-1}\right). \end{aligned}$$

$$\begin{aligned} h_2\left(s_{11}^{n-1}, y_1\right) &= h_2\left(t_{11}^{n-1}, b_1\right), \\ h_2\left(s_{21}^{n-1}, b_1\right) &= h_2\left(t_{21}^{n-1}, b_2\right), \\ &\vdots \\ h_2\left(s_{i1}^{n-1}, b_{i-1}\right) &= h_2\left(t_{i1}^{n-1}, b_i\right), \\ &\vdots \\ h_2\left(s_{(n-1)1}^{n-1}, b_{n-2}\right) &= h_2\left(t_{(n-1)1}^{n-1}, b_{n-1}\right), \\ h_2(s_{n1}^{n-1}, b_{n-1}) &= y_2, \end{aligned}$$

where $a_1, a_2, ..., a_n \in X$, $b_1, b_2, ..., b_{n-1} \in Y$ and $t_{i1}^{n-1}, s_{j1}^{n-1} \in G_2$ where $1 \leq i \leq n-1$ and $1 \leq j \leq n$. This implies that

$$\begin{aligned} h_1'\left(k_1^{n-1}, x_1\right) &= h_1'\left(k_1^{n-1}, h_1\left(a_1, s_{11}^{n-1}\right)\right), \\ h_1'\left(k_1^{n-1}, h_1\left(a_1, t_{11}^{n-1}\right)\right) &= h_1'\left(k_1^{n-1}, h_1\left(a_2, s_{21}^{n-1}\right)\right), \\ &\vdots \\ h_1'\left(k_1^{n-1}, h_1\left(a_i, t_{i1}^{n-1}\right)\right) &= h_1'\left(k_1^{n-1}, h_1\left(a_{i+1}, s_{(i+1)1}^{n-1}\right)\right) \\ &\vdots \\ h_1'\left(k_1^{n-1}, h_1\left(a_{n-1}, t_{(n-1)1}^{n-1}\right)\right) &= h_1'\left(k_1^{n-1}, h_1\left(x_2, s_{n1}^{n-1}\right)\right). \end{aligned}$$

Thus,

$$\begin{aligned}
h_1'\left(k_1^{n-1},x_1\right) &= h_1\left(h_1'\left(k_1^{n-1},a_1\right),s_{11}^{n-1}\right),\\
h_1\left(h_1'\left(k_1^{n-1},a_1\right),t_{11}^{n-1}\right) &= h_1\left(h_1'\left(k_1^{n-1},a_2\right),s_{21}^{n-1}\right),\\
&\vdots\\
h_1\left(h_1'\left(k_1^{n-1},a_i\right),t_{i1}^{n-1}\right) &= h_1\left(h_1'\left(k_1^{n-1},a_{i+1}\right),s_{(i+1)1}^{n-1}\right),\\
&\vdots\\
h_1\left(h_1'\left(k_1^{n-1},a_{n-1}\right),t_{(n-1)1}^{n-1}\right) &= h_1\left(h_1'\left(k_1^{n-1},x_2\right),s_{n1}^{n-1}\right).
\end{aligned}$$

This implies that $h_1'\left(k_1^{n-1},x_1\right)\otimes y_1 = h_1'\left(k_1^{n-1},x_2\right)\otimes y_2$ and the map $\overline{h}_1$ is well-defined. We can see that $X\otimes Y$ is a left G_1-set by $\overline{h}_1$. In the same way, $X\otimes Y$ is a right G_3-set by the map $\overline{h}_2$. Also, for every $x\otimes y\in X\otimes Y$, $k_1^{n-1}\in G_1$ and $s_1^{n-1}\in G_3$

$$\begin{aligned}
\overline{h}_2\left(\overline{h}_1\left(k_1^{n-1},x\otimes y\right),s_1^{n-1}\right) &= \overline{h}_2\left(h_1'\left(k_1^{n-1},x\right)\otimes y,s_1^{n-1}\right)\\
&= h_1'\left(k_1^{n-1},x\right)\otimes h_2'\left(y,s_1^{n-1}\right)\\
&= \overline{h}_1\left(k_1^{n-1},\overline{h}_2\left(x\otimes y,s_1^{n-1}\right)\right).
\end{aligned}$$

Therefore, $X\otimes Y$ is a (G_1,G_3)-set. This completes the proof. □

Definition 5.6. Let X and Y be (G_1,G_2)- and (G_2,G_3)-sets, respectively. We define a map $\pi : X\times Y \longrightarrow X\otimes Y$ with $\pi(x,y)=x\otimes y$. It's easy to see that π is a bimap and is called *canonical bimap.*

Theorem 5.7. *Let X and Y be (G_1,G_2)- and (G_2,G_3)-sets, respectively. Then, $(X\otimes Y,\pi)$ is a tensor product of X and Y over G_2.*

Proof. Suppose that Z is a (G_1,G_3)-set and $\beta : X\times Y \longrightarrow Z$ is a bimap. We define $\overline{\beta} : X\otimes Y \longrightarrow Z$ by

$$\overline{\beta}(x\otimes y)=\beta(x,y),$$

where $x\in X$ and $y\in Y$. Let $x_1\otimes y_1 = x_2\otimes y_2$. Then, by Proposition

5.4, we have

$$\begin{aligned}\beta(x_1,y_1)=\beta(h_1\left(a_1,s_{11}^{n-1}\right),y_1) &= \beta(a_1,h_2\left(s_{11}^{n-1},y_1\right))\\ &= \beta(a_1,h_2\left(t_{11}^{n-1},b_1\right)\\ &= \beta(h_1(a_1,t_{11}^{n-1}),b_1)\\ &\vdots\\ &= \beta(x_2,h_2(s_{1n}^{n-1},b_{n-1}))\\ &= \beta(x_2,y_2).\end{aligned}$$

Hence, $\overline{\beta}(x_1\otimes y_1)=\overline{\beta}(x_2\otimes y_2)$. This implies that $\overline{\beta}$ is well-defined. It is now routine to establish that $\overline{\beta}$ is bimap and $\overline{\beta}\circ\pi=\beta$. Moreover, $\overline{\beta}$ is unique with respect to these properties. □

Proposition 5.8. *Let X and Y be left G_1- and right G_2- sets, respectively. Then, the tensor product of them is unique up to isomorphism*

Proof. Suppose that (P,ψ) and $(P^{'},\psi^{'})$ are tensor product of X and Y. Then, we find a unique $\widehat{\psi}^{'}:P\longrightarrow P^{'}$ and $\widehat{\psi}:P^{'}\longrightarrow P$ such that $\psi\circ\widehat{\psi}^{'}=\psi^{'}$ and $\psi^{'}\circ\widehat{\psi}=\psi$. Hence, $\psi\circ\widehat{\psi}^{'}\circ\widehat{\psi}=\psi$ and by the uniqueness property, we have $\widehat{\psi}^{'}\circ\widehat{\psi}=Id$. In the same way, $\widehat{\psi}\circ\widehat{\psi}^{'}=Id$ and so $P\cong P^{'}$. This completes the proof. □

Let G_1 and G_2 be n-ary semihypergroups. Then, a map $\varphi:G_1\longrightarrow G_2$ is called *morphism*, when

$$\varphi(f(g_1,g_2,...,g_n))=f(\varphi(g_1),\varphi(g_2),...,\varphi(g_n)),$$

where $g_1^n\in G_1$. When G_1 and G_2 are n- ary semihypergroups with identities elements, $\varphi(e_1)=e_2$.

Definition 5.9. Let H be an n-ary sub semihypergroup of G and $g\in G$. Then, we say that H *dominates* g, when for any n-ary subsemigroup T and all morphisms $\varphi_1,\varphi_2:G\longrightarrow T$, the following implication holds

$$\forall\ h\in H,\ \varphi_1(h)=\varphi_2(h)\Longrightarrow\varphi_1(g)=\varphi_2(g).$$

More informally, H dominates g if any two morphisms of G that coincide on elements of H, coincide also on g. The set of elements dominated by H is called *dominion* of H in G and is written $Dom_H(G)$.

It is clear that $H \subseteq Dom_H(G)$. When G is a n-ary semigroup, $Dom_H(G)$ is an n-ary subsemigroup of G. Indeed, suppose that $g_1, g_2, ..., g_n \in Dom_H(G)$ and morphisms $\varphi_1, \varphi_2 : G \longrightarrow T$ such that $\varphi_1(h) = \varphi_2(h)$, for all $h \in H$. Hence, $\varphi_1(g_i) = \varphi_2(g_i)$, for $1 \leq i \leq n$. Hence,

$$\begin{aligned}\varphi_1(f(g_1, g_2, ..., g_n)) &= f(\varphi_1(g_1), \varphi_1(g_2)...\varphi_1(g_n))\\ &= f(\varphi_2(g_1), \varphi_2(g_2)...\varphi_2(g_n))\\ &= \varphi_2(f(g_1, g_2, ..., g_n)).\end{aligned}$$

Theorem 5.10. *Let H be an n-ary subsemihypergroup of G, $g \in G$, which G has identity element and $g \otimes e = e \otimes g$. Then, $g \in Dom_H(G)$.*

Proof. Suppose that $g \otimes e = e \otimes g$ and we have an n-ary semihypergroup T such that $\varphi_1, \varphi_2 : G \longrightarrow T$ are morphisms. Let $\varphi_1(h) = \varphi_2(h)$, for every $h \in H$. Then, T is an (H, H)-set if we define

$$\begin{aligned} h : H^{n-1} \times T &\longrightarrow T\\ (h_1^{n-1}, t) &\longmapsto f_2(\varphi_1(f_1(h_1^{n-1}, e)), t, \underbrace{e, e, ..., e}_{n-2}),\end{aligned}$$

$$\begin{aligned} h' : T \times H^{n-1} &\longrightarrow T\\ (t, h_1^{n-1}) &\longmapsto f_2(t, \varphi_1(f_1(h_1^{n-1}, e)), \underbrace{e, e, ..., e}_{n-2}).\end{aligned}$$

We define $\psi : G \times G \longrightarrow T$ as follows

$$(g_1, g_2) \longmapsto f_2(\varphi_1(g_1), \varphi_2(g_2), \underbrace{e, e, ..., e}_{n-2}).$$

Hence, ψ is an (H, H)-map and is even bimap. Indeed,

$$\begin{aligned}\psi(f_1(g_1, h_1^{n-1}), g_2) &= f_2(\varphi_1(f_1(g_1, h_1^{n-1})), \varphi_2(g_2), \underbrace{e, e, ..., e}_{n-2})\\ &= f_2(f_2(\varphi_1(g_1), \varphi_1(h_1), \cdots, \varphi_1(h_{n-1})), \varphi_2(g_2), \underbrace{e, e, ..., e}_{n-2})\\ &= f_2(f_2(\varphi_1(g_1), \varphi_2(h_1), \cdots, \varphi_2(h_{n-1})), \varphi_2(g_2), \underbrace{e, e, ..., e}_{n-2})\\ &= f_2(\varphi_1(g_1), f_2(\varphi_2(h_1), \cdots, \varphi_2(h_{n-1}), \varphi_2(g_2)), \underbrace{e, e, ..., e}_{n-2})\\ &= \psi(g_1, f_2(h_1^{n-1}, g_2)).\end{aligned}$$

It follows that there is a map $\overline{\psi} : G \otimes G \longrightarrow T$

$$(g_1 \otimes g_2) \longmapsto \psi(g_1, g_2),$$

for every $g_1 \otimes g_2 \in G \otimes G$. Since $g \otimes e = e \otimes g$, we have

$$\begin{aligned} \varphi_1(g) = f_2(\varphi_1(g), \varphi_2(e), e, e, .., e) = \overline{\psi}(g \otimes e) \quad &= \overline{\psi}(e \otimes g) \\ &= f_2(\varphi_1(e), \varphi_2(g), e, e, ..., e) \\ &= \varphi_2(g). \end{aligned}$$

This completes the proof. □

Let X_1, X_2 and X_3 be left G-sets, $\varphi_1 : X_1 \longrightarrow X_2$, $\varphi_2 : X_1 \longrightarrow X_3$, $\psi_1 : X_2 \longrightarrow X$ and $\psi_2 : X_3 \longrightarrow X$ be morphisms such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$. If there exist a left G-set $X^{'}$ and morphisms $\psi_1^{'} : X_2 \longrightarrow X^{'}$ and $\psi_2^{'} : X_3 \longrightarrow X^{'}$ such that $\psi_1^{'} \circ \varphi_1 = \psi_2^{'} \circ \varphi_2$, then there exists a unique morphism $\omega : X \longrightarrow X^{'}$ such that

$$\omega \circ \psi_1 = \psi_1^{'}, \quad \omega \circ \psi_2 = \psi_2^{'}.$$

Hence, we say that $[X_i, \varphi_j, \psi_r]$, $1 \leq i \leq 3, 1 \leq j \leq 2, 1 \leq r \leq 2$ is a *push out system*. We note that $X = [\bigcup_{i=1}^{3} X_i : \rho^*]$ of the disjoint of X_1, X_2, X_3, where ρ^* is the congruence relation generated by the following relation:

$$x_1 \ \rho \ x_2 \Longleftrightarrow x_1 \in X_1 \ \text{ and } \ x_2 = \varphi_1(x_1) \ \text{ or } \ x_2 = \varphi_2(x_1).$$

The map $\psi_1 : X_2 \longrightarrow X$ and $\psi_2 : X_3 \longrightarrow X$,

$$\psi_1(x_2) = \rho^*(x_2), \quad \psi_2(x_3) = \rho^*(x_3).$$

Let $x_2 \in X_2$ and $x_3 \in X_3$ and $\psi_1(x_2) = \psi_2(x_3)$. Then, $x_2 \in Im\varphi_1$. Indeed, We have $\rho^*(x_2) = \rho^*(x_3)$. This implies that there are $b_1, b_2, ..., b_n$ such that $b_1 = x_2$ and $b_2 = x_3$ and $(b_i, b_{i+1}) \in \rho$. Such a sequence cannot even unless $x_2 \in Im\varphi_1$.

Definition 5.11. Let H be an n-ary subsemihypergroup of n-ary semihypergroup G with identity. We say that H has the *extension property* in G if for every left H-set X and right H-set Y the map $X \times Y \longrightarrow X \otimes G \otimes Y$ defined by $x \otimes y \longrightarrow x \otimes e \otimes y$ is one to one.

Theorem 5.12. *Let H be an n-ary subsemihypergroup of an n-ary semihypergroup G with identity and H has the extension property in G and $\varphi : X \longrightarrow Y$ be a morphism and $y \otimes e = \varphi(x) \otimes g$ in $Y \otimes G$. Then, $y \in Im\varphi$.*

Proof. Suppose that $[X, X, Y, Y, P]$ is a push out system, where $\varphi_1, \varphi_2 : X \longrightarrow Y$, $\psi_1, \psi_2 : Y \longrightarrow P$. Hence $[X \otimes G, X \otimes G, Y \otimes G, Y \otimes G, P \otimes G]$, where $\varphi_1 \otimes I : X \otimes G \longrightarrow Y \otimes G$, $\varphi_2 \otimes I : X \otimes G \longrightarrow Y \otimes G$, $\psi_1 \otimes I : Y \otimes G \longrightarrow P \otimes G$ and $\psi_2 \otimes I : Y \otimes G \longrightarrow P \otimes G$ is a push out system. Let $y \otimes e = \varphi(x) \otimes g$ in $Y \otimes G$. Then,

$$\begin{aligned}
\psi_1(y) \otimes e = (\psi_1 \otimes I)(y \otimes e) = (\psi_1 \otimes I)(\varphi(x) \otimes g) \quad &= \psi_1\varphi(x) \otimes g \\
&= \psi_2\varphi(x) \otimes g \\
&= (\psi_2 \otimes I)(\varphi(x) \otimes g) \\
&= (\psi_2 \otimes I)(y \otimes e) \\
&= \psi_2(y) \otimes e.
\end{aligned}$$

By the extension property the map $y \longrightarrow y \otimes e$ from Y to $Y \otimes H$ is one to one. Hence $\psi_1(y) = \psi_2(y)$. This implies that $y \in Im\varphi$. $\square$

Theorem 5.13. *Let X, Y and Z be left G-sets. Then, $X \otimes Y$ and $Mor(Y, Z)$ are left G-sets and*

$$Mor(X \otimes Y, Z) \cong Mor(X, Mor(Y, Z)).$$

Proof. Suppose that $g_1, g_2, ..., g_{n-1} \in G$, $h_1 : G^{n-1} \times X \longrightarrow X$, $h_2 : G^{n-1} \times Y \longrightarrow Y$, $h_3 : G^{n-1} \times Z \longrightarrow Z$ and $\alpha \in Mor(Y, Z)$. We define $h : G^{n-1} \times Mor(Y, Z) \longrightarrow Mor(Y, Z)$ by

$$h(g_1^{n-1}, \alpha)(x) = h_3\left(g_1^{n-1}, \alpha(x)\right),$$

where $x \in Y$. Hence,

$$\begin{aligned}
h(f(g_1^n), g_{n+1}^{2n-2}, \alpha)(x) \quad &= h_3\left(f(g_1^n), g_{n+1}^{2n-2}, \alpha(x)\right) \\
&\vdots \\
&= h_3(g_1^{n-1}, h_3(g_n^{2n-2}, \alpha(x))) \\
&= h(g_1^{n-1}, h(g_n^{2n-2}, \alpha))(x),
\end{aligned}$$

for every $x \in Y$. This implies that $Mor(Y, Z)$ is a left G-set. By Proposition 5.5, $X \otimes Y$ is a left G-set.

Let $f \in Mor(X \otimes Y, Z)$. Then, f is a G-map. Hence,

$$f\left(h^{'}\left(g_1^{n-1}, x \otimes y\right)\right) = h_3\left(g_1^{n-1}, f(x \otimes y)\right).$$

for every $g_1^{n-1} \in G$ and $x \otimes y \in X \otimes Y$.
For every $x \in X$, we define $f_x(y) = f(x \otimes y)$. Hence $f_x \in Mor(Y, Z)$. Indeed, for every $g_1^{n-1} \in G$ we have

$$\begin{aligned} f_x(h_2\left(g_1^{n-1}, y\right)) &= f(x \otimes h_2(g_1^{n-1}, y)) \\ &= f(h_1(g_1^{n-1}, x) \otimes y) \\ &= f\left(h^{'}\left(g_1^{n-1}, x \otimes y\right)\right) \\ &= h_3\left(g_1^{n-1}, f(x \otimes y)\right). \end{aligned}$$

We define

$$\psi : Mor(X \otimes Y, Z) \longrightarrow Mor(X, Mor(Y, Z))$$

$$f \longmapsto T_f,$$

where $T_f : X \longrightarrow Mor(Y, Z$ by $T_f(x) = f_x$. Hence ψ is a morphism. Indeed, for every $g_1^{n-1} \in G$, $x \in X, y \in Y$ and $f \in Mor(X \otimes Y, Z)$,

$$\begin{aligned} \psi\left(h(g_1^{n-1}, f)\right)(x \otimes y) = T_{h(g_1^{n-1}, f)}(x \otimes y) &= h(g_1^{n-1}, f)_x(y) \\ &= h_3\left(g_1^{n-1}, f(x \otimes y)\right) \\ &= h_3\left(g_1^{n-1}, f_x(y)\right) \\ &= h\left(g_1^{n-1}, \psi(f)\right)(x \otimes y). \end{aligned}$$

Let $f \in Mor(X, Mor(Y, Z))$ and $T : G^{n-1} \times Mor(Y, Z) \longrightarrow Mor(Y, Z)$. Then, for every $x \in X$, $f(x)$ is a morphism. We define

$$\begin{aligned} \varphi : Mor(X, Mor(Y, Z)) &\longrightarrow Mor(X \otimes Y, Z) \\ f &\longmapsto \overline{f}, \end{aligned}$$

where $\overline{f}(x \otimes y) = (f(x))(y)$. We have

$$\varphi\left(h(g_1^{n-1}, f\right)(x)) = \overline{h}\left(g_1^{n-1}, f\right),$$

where

$$\begin{aligned}\overline{h}\left(g_1^{n-1}, f\right)(x\otimes y) = (h\left(g_1^{n-1}, f\right)(x))(y) &= T\left(g_1^{n-1}, f(x)\right)(y)\\ &= T\left(g_1^{n-1}, (f(x))(y)\right)\\ &= T(g_1^{n-1}, (\varphi(f)(x))(y),\end{aligned}$$

for every $x \in X$ and $y \in Y$. Hence φ is a morphism. Moreover, for every $f \in Mor(X \otimes Y, Z)$ and $x \otimes y \in X \otimes Y$,

$$(\varphi\circ\psi)(f)(x\otimes y) = \varphi(\psi(f)(x\otimes y)) = \varphi(T_f(x)(y)) = f_x(y) = f(x\otimes y)$$

Hence $\varphi\circ\psi = I_{Mor(X\otimes Y,Z)}$. On the other hand, for every $f \in Mor(X, Mor(Y,Z))$,

$$(\psi\circ\varphi)(f) = \psi(\varphi(f)) = \psi(\overline{f}) = T_{\overline{f}}$$

such that $T_{\overline{f}} \in Mor(X\otimes Y, Z)$ and for every $x\otimes y \in X\otimes Y$, we have

$$T_{\overline{f}}(x\otimes y) = \overline{f}(x\otimes y) = (f(x))(y).$$

Hence,

$$\psi\circ\varphi = I_{Mor(X,Mor(Y,Z))}.$$

This completes the proof. $\square$

Theorem 5.14. *Let $(X_i, \alpha_{ij})_{i,j\in I}$ be a direct system of left (G,H)-sets and X be a right G-set. Then, $(X_i\otimes X, \widehat{\alpha}_{ij})_{i,j\in I}$ is a direct system and $lim_{i\in I}(X_i\otimes X) = (lim_{i\in I}X_i)\otimes X$.*

Proof. Suppose that $\alpha_{ij} : X_i \longrightarrow X_j$ and $\overline{\alpha}_{ij} : X_i\times X \longrightarrow X_i\otimes X$ defined by $\overline{\alpha}_{ij}(x_i, x) = \alpha_{ij}(x_i)\otimes x$. Hence,

$$\overline{\alpha}_{ij}(h(x_i, t_1^{n-1}), x) = h(x_i, t_1^{n-1})\otimes x = x_i\otimes h(t_1^{n-1}, x) = \overline{\alpha_{ij}}(x_i, h(t_1^{n-1}, x)).$$

Then, $\overline{\alpha}_{ij}$ is a bimap. Thus, there is $\widehat{\alpha}_{ij} : X_i\otimes X \longrightarrow X_j\otimes X$ such that $\widehat{\alpha}_{ij}(x_i\otimes x) = \alpha_{ij}(x_i)\otimes x$, where $x_i \in X_i$ and $x\in X$. Also,

$$\widehat{\alpha}_{jk}\circ\widehat{\alpha}_{ij}(x_i\otimes x) = \alpha_{jk}\circ\alpha_{ij}(x_i)\otimes x = \alpha_{ik}(x_i)\otimes x = \widehat{\alpha}_{ik}((x_i)\otimes x).$$

and

$$\widehat{\alpha}_{ii}(x_i\otimes x) = \alpha_{ij}(x_i)\otimes x = x_i\otimes x.$$

Let $\beta_i : X_i \longrightarrow lim_{i\in I}X_i$ and $\overline{\beta}_i : X_i \times X \longrightarrow lim_{i\in I}X_i \otimes X$ defined by $\overline{\beta}_i(t, x) = \beta_i(t) \otimes x$. Then, $\overline{\beta}_i$ is a bimap. Thus, there exists $\widehat{\beta}_i : X_i \otimes X \longrightarrow lim_{i\in I}X_i \otimes X$ such that $\widehat{\beta}_i(t \otimes x) = \beta_i(t) \otimes x$.

$$\widehat{\beta}_j \circ \widehat{\alpha}_{ij}(x_i \otimes x) = \beta_j(\alpha_{ij}(x_i) \otimes x) = \beta_i(x_i) \otimes x = \widehat{\beta}_i(x_i \otimes x).$$

Let $x \in X$ be a fixed element and $\sigma_i : X_i \longrightarrow X_i \otimes X$ defined by $\sigma_i(x_i) = x_i \otimes x$ and Y be a (G, H)-set and $\gamma_i : X_i \otimes X \longrightarrow Y$ such that $\widehat{\gamma}_j \circ \widehat{\alpha}_{ij} = \widehat{\gamma}_i$. Then, $\gamma_j \circ \sigma_j \circ \alpha_{ij} = \gamma_i \circ \sigma_i$. Thus, there exists $\delta : Y \longrightarrow lim_{i\in I}X_i$ such that $\delta \circ \gamma_i \circ \sigma_i = \beta_i$. Therefore, $\gamma_j \circ \widehat{\alpha}_{ij} = \gamma_i$ and $lim_{i\in I}(X_i \otimes X) = (lim_{i\in I}X_i) \otimes X$. □

6 Conclusion

The study of homological concepts in the context of hypergroups theory is a new research theory. This generalizes the existing research of these concepts on hyperstructures, done especially in from a different point of view [13, 18]. In the present paper, we have introduced and studied left(right) G-sets on n- ary hypergroups and resent some examples. Also, the various properties of these concept are emphasized. Moreover, we have introduced and studied direct limit and tensor product of left(right) G-sets on n-ary semihypergroups. A possible future study could be devoted to the introduction and analysis of flat left(right) G-sets and "Tor" functor.

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