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Original Research Paper

Solutions of Fuzzy Time Fractional Heat Equation

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Abstract. Fuzzy fractional heat equations (FFHEs) are utilized to analyze the behaviour of the certain phenomena in various mathematical and scientific models. The main goal of this paper is to construct the solution of fuzzy fractional heat equations by taking a reliable recipe of Sumudu transformation method and homotopy analysis method into account. These method allow us to remove the difficulties and restrictions confronted in other methods. The feasibility of this method is confirmed by given numerical examples. The result presented that the proposed method is suitable, powerful and reliable for obtaining the solution of fuzzy fractional problems with FFHEs.

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1 Introduction

It is evident from the scientific studies that fractional differential equations (FDEs) have been gaining growing attention last two decades. By

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means of fuzzy quantities, crisp quantities in the FDEs can be replaced to reflect imprecision and uncertainty. This emerges fuzzy fractional differential equations (FFDEs). As a result, there are many different studies on the solutions of FFDEs [18, 22, 2, 16, 11, 24, 3]. Modelling scientific processes with uncertainty such as linear algebra, differential equations, power systems, control theory, system theory, optimization, signal processing and etc. [5, 20, 4, 21, 14, 15] by fuzzy differential equations is more suitable than ordinary or partial differential equations. Consequently, studying on the approximate solution of FFDEs is trend topic of the applied mathematics. Therefore we do research on the series solution of fuzzy heat-like equations in this paper.

In the determination of fuzzy differential equations, using fuzzy linear matrix equations plays a significant role. Therefore, establishing the solution of fuzzy linear matrix equations is very essential part of the fuzzy differential equations. There are quite a few study on fuzzy linear matrix equations by means of analytic methods such as Ezzati's method [19, 13, 9]. The Kronecker product and embedding approach are very useful to accomplish a fuzzy solution of fuzzy linear matrix equations.

The organization of this paper is as follows:
In section 2, we present fundamental definitions and concepts. Section 3 is devoted to the presentation of time fractional heat equation. Sumudu homotopy analysis transform method for fuzzy time fractional derivative equations is given in section 4. Numerical examples are presented to illustrate the implementation of this method for fuzzy time fractional equations in section 5. Finally, conclusion of this study is given in the last section.

2 Preliminaries

In this section, fundamental definitions and concepts are presented.
Fuzzy numbers $\widetilde{t}(q) : \mathbb{R}^n \to [0, 1]$ in the space E^n of n-dimensional fuzzy numbers satisfy the following conditions:

- $\widetilde{t}(q)$ is called normal, if $\exists q_0 \in \mathbb{R}^n$ for which $\widetilde{t}(q_0) = 1$,

- $\widetilde{t}\,(q)$ is called fuzzy convex, if $\forall q_1, q_2 \in \mathbb{R}^n, x \in [0,1]$, $\widetilde{t}\,(xq_1 + (1-x)\,q_2) \geq \min\{\widetilde{t}\,(q_1)\,, \widetilde{t}\,(q_2)\}$,
- The support of the $\widetilde{t}\,(q)$ is defined as $supp\widetilde{t}\,(q) = \left\{q \in \mathbb{R} : \widetilde{t}\,(q) > 0\right\}$ and its closure $cl(supp\widetilde{t}\,(q))$ is compact,
- $\widetilde{t}\,(q)$ is upper semi-continuous.

The μ-cut set of a fuzzy number $\widetilde{t}\,(q) \in E$ represented by $\left[\widetilde{t}\,(q)\right]^{\mu}$, is described as

$$\left[\widetilde{t}\,(q)\right]^{\mu} = \begin{cases} \left\{q \in \mathbb{R} : \widetilde{t}\,(q) \geq \mu\right\}, & 0 < \mu \leq 1 \\ cl\left(supp\widetilde{t}\,(q)\right), & \mu = 0 \end{cases}$$

which is a closed and bounded interval $\left[\underline{t^{\mu}}(q), \overline{t^{\mu}}(q)\right]$ where $\underline{t^{\mu}}(q)$ represents the left-hand endpoint of $\left[\widetilde{t}\,(q)\right]^{\mu}$ and $\overline{t^{\mu}}(q)$ the right-hand endpoint of $\left[\widetilde{t}\,(q)\right]^{\mu}$.

Definition 2.1. *A pair* $\left[\underline{t^{\mu}}(q), \overline{t^{\mu}}(q)\right]$ *of functions* $\underline{t^{\mu}}(q), \overline{t^{\mu}}(q), 0 \leq \mu \leq 1$ *is said to be the parametric form of a fuzzy number* $\widetilde{t}\,(q)$. *Moreover the following conditions are satisfied:*

- $\underline{t^{\mu}}(q)$ *is an increasing left continuous function.*
- $\overline{t^{\mu}}(q)$ *is a decreasing left continuous function.*
- $\underline{t^{\mu}}\,(q) \leq \overline{t^{\mu}}(q), 0 \leq \mu \leq 1.$

Definition 2.2. *The fuzzy Sumudu transformation of a continuous fuzzy function* $\widetilde{b} : \mathbb{R} \to \mathbb{F}(\mathbb{R})$ *for which* $\widetilde{b}(uq) \odot e^{-q}$ *is improper fuzzy Riemann integrable is defined as [1]*

$$G\,(u) = \mathbb{S}\left[\widetilde{b}\,(q)\right](u) = \int_0^{\infty} \widetilde{b}(uq) \odot e^{-q} dq, u \in [-\mu_1, \mu_2],$$

where $\mu_1, \mu_2 > 0$. *The parametric form of fuzzy Sumudu transformation is denoted as:*

$$\left[\mathbb{S}\left[\widetilde{b}\,(q)\right](u)\right] = \left[\mathbb{S}\left[\underline{b}\,(q)\right](u)\,, \mathbb{S}\left[\overline{b}\,(q)\right](u)\right].$$

3 Fuzzy Time Fractional Heat Equation

This part is devoted to the presentation of time fractional heat equation is given by taking the fundemental fuzzy properties [6], [17], [7], [23] into account in a fuzzy environment. Consider the one-dimensional fuzzy fractional problem with FFHEs:

$$\frac{\partial^\alpha \widetilde{r}\,(p,t,\alpha)}{\partial^\alpha t} = D_p^2 \widetilde{r}\,(p,t)\,, 0 < p < l, t > 0 \qquad (1)$$
$$\widetilde{r}\,(p,0) = \widetilde{g}(p)$$

where $\widetilde{r}(p,t)$, $\frac{\partial^\alpha \widetilde{r}(p,t,\alpha)}{\partial^\alpha t}$ denote a fuzzy function [17] and the fuzzy time fractional derivative (FTFD) of order α respectively. Moreover the fuzzy function $\widetilde{g}(p)$ is described as follows [10]:

$$\widetilde{g}\,(p) = \widetilde{\mu} d(p) \qquad (2)$$

where $d(p), \widetilde{\mu}$ represent the crisp function of the crisp variable p and the fuzzy convex number, respectively. The fuzzification of Problem (1) for all $\beta \in [0,1]$ is as follows [10]:

$$[\widetilde{r}\,(p,t)]_\beta = [\underline{r}\,(p,t;\beta)\,, \overline{r}(p,t;\beta)], \qquad (3)$$

$$\left[\frac{\partial^\alpha \widetilde{r}\,(p,t,\alpha)}{\partial^\alpha t}\right]_\beta = \left[\frac{\partial^\alpha \underline{r}\,(p,t,\alpha;\beta)}{\partial^\alpha t}, \frac{\partial^\alpha \overline{r}\,(p,t,\alpha;\beta)}{\partial^\alpha t}\right], \qquad (4)$$

$$\left[D_p^2 \widetilde{r}\,(p,t)\right]_\beta = \left[D_p^2 \underline{r}\,(p,t;\beta)\,, D_p^2 \overline{r}\,(p,t;\beta)\right], \qquad (5)$$

$$[\widetilde{r}\,(p,0)]_\beta = [\underline{r}\,(p,0;\beta)\,, \overline{r}(p,0;\beta)], \qquad (6)$$

$$[\widetilde{g}\,(p)]_\beta = [\underline{g}\,(p;\beta)\,, \overline{g}(p;\beta)] \qquad (7)$$

where

$$[\widetilde{g}\,(p)]_\beta = \left[\underline{\mu}\,(\beta)\,, \overline{\mu}(\beta)\right] d(p). \qquad (8)$$

The function described by utilizing the fuzzy extension principle [10]:

$$\begin{cases} \underline{r}\,(p,t;\beta) = \min\left\{\widetilde{r}\,(\widetilde{\mu}\,(\beta)\,, t) : \widetilde{\mu}\,(\beta) \in \widetilde{r}\,(p,t;\beta)\right\}, \\ \overline{r}\,(p,t;\beta) = \max\left\{\widetilde{r}\,(\widetilde{\mu}\,(\beta)\,, t) : \widetilde{\mu}\,(\beta) \in \widetilde{r}\,(p,t;\beta)\right\} \end{cases} \qquad (9)$$

is called the membership function.
Based on [10], after fuzzfication of Problem (1) and defuzzfication of Eqs. (2-9), Problem (1) is rewritten in the following form:
The lower bound of problem (1)

$$\begin{cases} \frac{\partial^\alpha \underline{r}(p,t,\alpha;\beta)}{\partial^\alpha t} = D_p^2 \underline{r}(p,t;\beta), \\ \underline{r}(p,0;\beta) = \underline{\mu}(\beta)\, d(p). \end{cases}$$

The upper bound of problem (1)

$$\begin{cases} \frac{\partial^\alpha \overline{r}(p,t,\alpha;\beta)}{\partial^\alpha t} = D_p^2 \overline{r}(p,t;\beta), \\ \overline{r}(p,0;\beta) = \overline{\mu}(\beta)\, d(p). \end{cases}$$

4 Sumudu homotopy analysis transform method (SHAM) for FTFD Equation

Consider the following fuzzy problem including heat-like fuzzy time-fractional differential equation

$$^{C}D_t^\alpha \widetilde{r}(p,t) = D_p^2 \widetilde{r}(p,t), 0 < p < 1, t > 0, 0 < \alpha \leq 1 \tag{10}$$

$$\widetilde{r}(p,0) = \widetilde{g}(p).$$

The initial condition can be treated homogeneously for simplicity. Based on the proposed method, the Sumudu transformation is applied to both sides of the Eq. (10):

$$\begin{aligned} &\mathbb{S}\left[{}^{C}D_t^\alpha \widetilde{r}(p,t)\right] = \mathbb{S}\left[D_p^2 \widetilde{r}(p,t)\right]. \\ &w^{-\alpha}\mathbb{S}\left[\widetilde{r}(p,t)\right] - w^{-\alpha}\widetilde{r}(p,0) = \mathbb{S}\left[D_p^2 \widetilde{r}(p,t)\right]. \\ &\mathbb{S}\left[\widetilde{r}(p,t)\right] - w^{\alpha}\mathbb{S}\left[D_p^2 \widetilde{r}(p,t)\right] - \widetilde{r}(p,0) = 0. \end{aligned} \tag{11}$$

Eq. (11) is rewritten in terms of nonlinear operator as follows:

$$N\left[\widetilde{r}(p,t)\right] = 0,$$

where $\widetilde{r}(p,t), \widetilde{r}_0(p,t)$ and $h \neq 0$ denote unknown function, initial approximation and an auxiliary parameter, respectively. The nonlinear operator can be defined in terms of embedding parameter $e \in [0,1]$ as follows:

$$N\left[\widetilde{\phi}\,(p,t;e)\right]=\mathbb{S}\left[\widetilde{\phi}\,(p,t;e)\right]-w^{\alpha}\mathbb{S}\left[D_p^2\widetilde{\phi}\,(p,t;e)\right]-\widetilde{\phi}\,(p,0;e)=0.$$

We construct such a homotopy [12], [19]

$$(1-e)\,\mathbb{S}\left[\widetilde{\phi}\,(p,t;e)-\widetilde{r}_0\,(p,t)\right]=ehH(p,t)N\left[\widetilde{\phi}\,(p,t;e)\right]$$

is zeroth-order deformation equation. Here, $H\,(p,t)\neq 0$. The zero-order deformation equations are obtained by taking $e=0$ and $e=1$, as follows:

$$\widetilde{\phi}\,(p,t;0)=\widetilde{r}_0\,(p,t)\,,\widetilde{\phi}\,(p,t;1)=\widetilde{r}\,(p,t)\,.$$

$\widetilde{\phi}_i\,(p,t;e)$ can be obtained in the power series form in e by the help of Taylor's theorem as follows:

$$\widetilde{\phi}\,(p,t;e)=\widetilde{r}_0\,(p,t)+\sum_{l=1}^{\infty}\widetilde{f}_l\,(p,t)e^l \tag{12}$$

where

$$\widetilde{r}_l\,(p,t)=\left.\frac{1}{l!}\frac{\partial^l\widetilde{\phi}\,(p,t;e)}{\partial e^l}\right|_{e=0}.$$

The parameter h is utilized to make (12) convergent. The series (12) converges at $e=1$ for properly chosen the auxiliary linear operator, the initial guess, the auxiliary function and the auxiliary parameter h. Hence

$$\widetilde{r}\,(p,t)=\widetilde{r}_0\,(p,t)+\sum_{l=1}^{\infty}\widetilde{r}_l(p,t)$$

is the obtained solution of the original nonlinear equations. It is seen from the above expression that exact solution $\widetilde{r}(p,t)$ and the initial guess $\widetilde{r}_0(p,t)$ have a relationship in terms of $\widetilde{r}_l(p,t)(l=1,2,3,\ldots)$.
In order to determine them, the following vectors are defined

$$\vec{\widetilde{r}}=\{\widetilde{r}_0\,(p,t)\,,\widetilde{r}_1\,(p,t)\,,\widetilde{r}_2\,(p,t)\,,\ldots,\widetilde{r}_l(p,t)\}\,.$$

The l^{th}-order deformation equation is obtained in the following form

$$\mathbb{S}\left[\widetilde{r}_l\,(p,t)-\chi_l\widetilde{r}_{l-1}\,(p,t)\right]=hH\,(p,t)\,R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right). \tag{13}$$

If both sides of Eq. (13) is operated the inverse Sumudu transform, then the following expression is obtained:

$$\widetilde{r}_l\,(p,t) = \chi_l \widetilde{r}_{l-1}\,(p,t) + \mathbb{S}^{-1}\left[hH\,(p,t)\,R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right)\right]$$

where

$$R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right) = \frac{1}{(l-1)!}\frac{\partial^{l-1}N\left[\widetilde{\phi}\,(p,t;e)\right]}{\partial e^{l-1}}\Bigg|_{e=0}$$

and

$$\chi_l = \begin{cases} 0, & l \leq 1 \\ 1, & l > 1. \end{cases}$$

In our case

$$R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right) = {}^{C}D_t^{\alpha}\widetilde{r}\,(p,t) - D_p^2\widetilde{r}\,(p,t)\,.$$

As a result $\widetilde{r}_l\,(p,t)$ for $l \geq 1$, at M^{th} order is obtained without any difficulty. Therefore an approximate solution of the Eq. (10) is constructed as

$$\widetilde{r}(p,t) = \sum_{l=0}^{M}\widetilde{r}_l(p,t) \tag{14}$$

where $M \to \infty$.

Theorem 4.1. *If the series (14) converges as $M \to \infty$, then, the limit must be the exact solution Eq. (10).*

Proof. Assume that the series (14) is convergent. Hence

$$\sum_{l=0}^{\infty}\widetilde{r}_l(p,t) = \widetilde{r}_0\,(p,t) + \sum_{l=1}^{\infty}\widetilde{r}_l(p,t) = \widetilde{K}(p,t).$$

As a result $\lim_{M\to\infty} \widetilde{r}_l(p,t) = 0$. Hence taking the Eq. (13) into account the following is obtained

$$\begin{aligned}\lim_{M\to\infty}\left[hH(p,t)\sum_{l=1}^{M} R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right)\right] &= \lim_{M\to\infty}\left(\sum_{l=1}^{M}\mathbb{S}\left[\widetilde{r}_l(p,t)-\chi_l\widetilde{r}_{l-1}(p,t)\right]\right)\\ &= \mathbb{S}\left[\lim_{M\to\infty}\sum_{l=1}^{M}\left[\widetilde{r}_l(p,t)-\chi_l\widetilde{f}_{l-1}(p,t)\right]\right]\\ &= \mathbb{S}\left[\lim_{M\to\infty}\widetilde{r}_l(p,t)\right]\\ &= 0.\end{aligned}$$

Since $h \neq 0, H(p,t) \neq 0$, therefore, $\sum_{l=1}^{\infty} R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right) = 0$. From (3.15)

$$\sum_{l=1}^{\infty} R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right) = \sum_{l=1}^{\infty} {}_0^C D_t^{\alpha}\widetilde{r}_{l-1}(p,t) - \sum_{l=1}^{\infty} D_p^2\widetilde{r}_{l-1}(p,t).$$

$$\sum_{l=1}^{\infty} R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right) = {}_0^C D_t^{\alpha}\sum_{l=1}^{\infty}\widetilde{r}_{l-1}(p,t) - D_p^2\sum_{l=1}^{\infty}\widetilde{r}_{l-1}(p,t).$$

$$\sum_{l=1}^{\infty} R_l\left(\vec{\widetilde{r}}_{l-1}(p,t)\right) = {}_0^C D_t^{\alpha}\sum_{l=0}^{\infty}\widetilde{r}_{l}(p,t) - D_p^2\sum_{l=0}^{\infty}\widetilde{r}_{l}(p,t).$$

$${}_0^C D_t^{\alpha}\widetilde{K}(p,t) - D_p^2\widetilde{K}(p,t) = 0. \tag{15}$$

Above equation (15) shows that, $\widetilde{K}(p,t)$ satisfies the original problem (10). □

5 Numerical Illustrations

This section is devoted to illustrated examples for demonstrating the efficacy of SHAM.

Example 5.1. Consider the following fuzzy problem [19]:

$$\begin{cases} {}^C D_t^{\alpha}\widetilde{r}(p,t) = D_p^2\widetilde{r}(p,t), & 0<p<1, t>0, 0<\alpha\leq 1\\ \widetilde{r}(p,0) = \widetilde{g}(p) = \widetilde{k}\sin(\pi p), & 0<p<1.\end{cases}$$

The general definitions of the fuzzy problems

$$\begin{cases} {}^{C}D_t^{\alpha}\overline{r}\left(p,t\right)=D_p^2\overline{r}\left(p,t\right), & 0<p<1,\ t>0,\ 0<\alpha\leq 1 \\ \overline{r}\left(p,0\right)=\overline{g}\left(p\right)=\overline{k}\left(\beta\right)\sin\left(\pi p\right), & 0<p<1 \end{cases} \tag{16}$$

$$\begin{cases} {}^{C}D_t^{\alpha}\underline{r}\left(p,t\right)=D_p^2\underline{r}\left(p,t\right), 0<p<1,\ t>0,\ 0<\alpha\leq 1 \\ \underline{r}\left(p,0\right)=\underline{g}\left(p\right)=\underline{k}\left(\beta\right)\sin\left(\pi p\right), \qquad 0<p<1 \end{cases} \tag{17}$$

Application of the Sumudu transform to both sides of Problem (16) yields

$$\mathbb{S}\left[\overline{r}\left(p,t\right)\right]-w^{\alpha}\mathbb{S}\left[D_p^2\overline{r}\left(p,t\right)\right]-\overline{r}\left(p,0\right)=0.$$

The operator N becomes

$$N\left[\overline{\phi}\left(p,t;e\right)\right]=\mathbb{S}\left[\overline{\phi}\left(p,t;e\right)\right]-w^{\alpha}\mathbb{S}\left[D_p^2\overline{\phi}\left(p,t;e\right)\right]=0,t>0,0\leq e\leq 1$$

and thus, we have

$$R_l\left(\vec{\overline{r}}_{l-1}(p,t)\right)=\mathbb{S}\left[\overline{r}_{l-1}\left(p,t\right)\right]-w^{\alpha}\mathbb{S}\left[D_p^2\overline{r}_{l-1}\left(p,t\right)\right]=0,t>0$$

The deformation equation of order l becomes

$$\mathbb{S}\left[\overline{f}_l\left(p,t\right)-\chi_l\overline{r}_{l-1}\left(p,t\right)\right]=hH\left(p,t\right)R_l\left(\vec{\overline{r}}_{l-1}(p,t)\right).$$

Applying the inverse Sumudu transform yields

$$\overline{r}_l\left(p,t\right)=\chi_l\overline{r}_{l-1}\left(p,t\right)+\mathbb{S}^{-1}\left[hH\left(p,t\right)R_l\left(\vec{\overline{r}}_{l-1}(p,t)\right)\right].$$

Choosing $H\left(p,t\right)=1$ in above equation yields

$$\overline{r}_l\left(p,t\right)=\chi_l\overline{r}_{l-1}\left(p,t\right)+\mathbb{S}^{-1}\left[h\left[\mathbb{S}\left[\overline{r}_{l-1}\left(p,t\right)\right]-w^{\alpha}\mathbb{S}\left[D_p^2\overline{r}_{l-1}\left(p,t\right)\right]\right]\right],$$

$$\overline{r}_1\left(p,t\right)=h\pi^2\overline{k}\left(\beta\right)\sin\left(\pi p\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$\overline{r}_2\left(p,t\right)=h\pi^2\overline{k}\left(\beta\right)\sin\left(\pi p\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+h^2\pi^2\overline{k}\left(\beta\right)\sin\left(\pi p\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$+h^2\pi^4\overline{k}\left(\beta\right)\sin\left(\pi p\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$\begin{aligned}\overline{r}_3(p,t) &= h\pi^2\overline{k}(\beta)\sin(\pi p)\frac{t^\alpha}{\Gamma(\alpha+1)} + h^2\pi^2\overline{k}(\beta)\sin(\pi p)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ h^2\pi^4\overline{k}(\beta)\sin(\pi p)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h^2\pi^2\overline{k}(\beta)\sin(\pi p)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ h^3\pi^2\overline{k}(\beta)\sin(\pi p)\frac{t^\alpha}{\Gamma(\alpha+1)} + h^3\pi^4\overline{k}(\beta)\sin(\pi p)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ h^2\pi^4\overline{k}(\beta)\sin(\pi p)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h^3\pi^4\overline{k}(\beta)\sin(\pi p)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ h^3\pi^6\overline{k}(\beta)\sin(\pi p)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.\end{aligned}$$

Therefore the series solution is determined as

$$\overline{r}(p,t;\beta) = \overline{r}_0(p,t;\beta) + \sum_{l=1}^{\infty}\overline{r}_l(p,t;\beta).$$

The following approximate solution is obtained at $h=-1$

$$\begin{aligned}\overline{r}(p,t;\beta) &= \overline{k}(\beta)\sin(\pi p) - \pi^2\overline{k}(\beta)\sin(\pi p)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ &+ \pi^4\overline{k}(\beta)\sin(\pi p)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \pi^6\overline{k}(\beta)\sin(\pi p)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \\ &= \overline{k}(\beta)\sin(\pi p)\sum_{j=0}^{\infty}\frac{(-1)^j\pi^{2j}t^{j\alpha}}{\Gamma(j\alpha+1)}.\end{aligned}$$

Similarly, the solution for the Problem (17) is determined as

$$\underline{r}(p,t;\beta) = \underline{k}(\beta)\sin(\pi p)\sum_{j=0}^{\infty}\frac{(-1)^j\pi^{2j}t^{j\alpha}}{\Gamma(j\alpha+1)}.$$

The approximate solutions SHAM of order 11 are compared and plotted for $\alpha = 0.9,\ 0.95,\ 1, p = 0.25$ and $t = 0.25$ and $\beta = (0.75+0.25\beta; 1.25-0.25\beta)$ in Figure 1.

Example 5.2. Consider the following fuzzy problem [19]:

$$\begin{cases} {}^{C}D_t^\alpha\widetilde{r}(p,t) = \frac{1}{2}p^2D_p^2\widetilde{r}(p,t), & 0<p<1, t>0, 0<\alpha\le 1 \\ \widetilde{r}(p,0) = \widetilde{g}(p) = \widetilde{k}p^2, & 0<p<1. \end{cases}$$

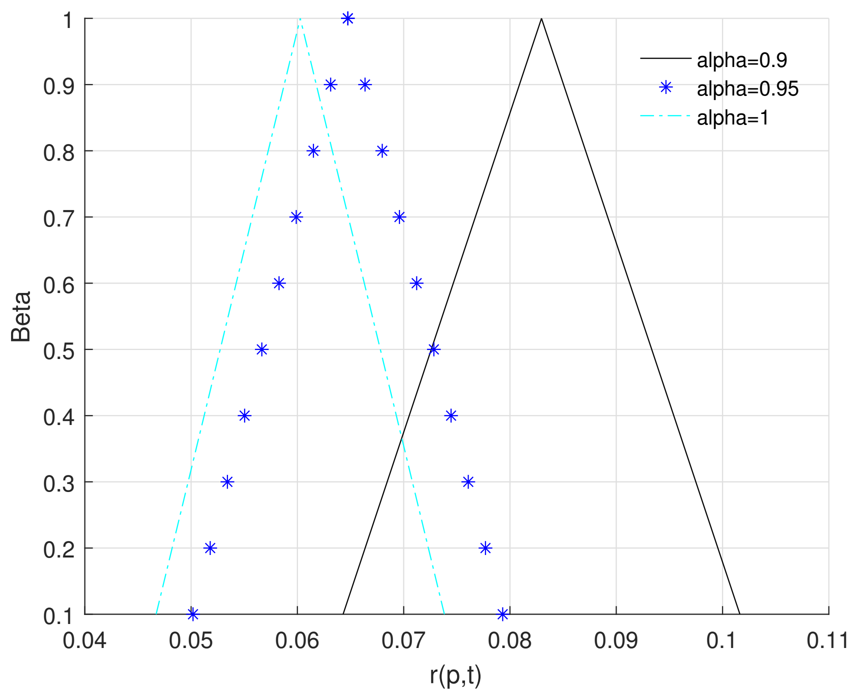

Figure 1: The approximate solutions for Example 1

The general definitions of the fuzzy problems

$$\begin{cases} {}^{C}D_t^{\alpha}\overline{r}\,(p,t) = D_p^2\overline{r}\,(p,t)\,, & 0<p<1,\ t>0,\ 0<\alpha\leq 1 \\ \overline{r}\,(p,0) = \overline{g}\,(p) = \overline{k}\,(\beta)\,p^2, & 0<p<1. \end{cases} \tag{18}$$

$$\begin{cases} {}^{C}D_t^{\alpha}\underline{r}\,(p,t) = D_p^2\underline{r}\,(p,t)\,, & 0<p<1,\ t>0,\ 0<\alpha\leq 1 \\ \underline{r}\,(p,0) = \underline{g}\,(p) = \underline{k}\,(\beta)\,p^2, 0<p<1. \end{cases} \tag{19}$$

Application of the Sumudu transform to both sides problem (18) yields

$$\mathbb{S}\left[\overline{r}\,(p,t)\right] - w^{\alpha}\mathbb{S}\left[\frac{1}{2}p^2D_p^2\overline{r}\,(p,t)\right] - \overline{r}\,(p,0) = 0.$$

The operator N is

$$N\left[\overline{\phi}\,(p,t;e)\right] = \mathbb{S}\left[\overline{\phi}\,(p,t;e)\right] - w^{\alpha}\mathbb{S}\left[\frac{1}{2}p^2D_p^2\overline{\phi}\,(p,t;e)\right] = 0, t>0, 0\leq e\leq 1$$

and thus

$$R_l\left(\vec{\overline{r}}_{l-1}(p,t)\right) = \mathbb{S}\left[\overline{r}_{l-1}\,(p,t)\right] - w^{\alpha}\mathbb{S}\left[\frac{1}{2}p^2D_p^2\overline{r}_{l-1}\,(p,t)\right] = 0, t>0$$

The deformation equation of order l becomes

$$\mathbb{S}\left[\overline{r}_l\,(p,t) - \chi_l\overline{r}_{l-1}\,(p,t)\right] = hH\,(p,t)\,R_l\left(\vec{\overline{r}}_{l-1}(p,t)\right).$$

Applying the inverse Sumudu transform yields

$$\overline{r}_l\,(p,t) = \chi_l\overline{r}_{l-1}\,(p,t) + \mathbb{S}^{-1}\left[hH\,(p,t)\,R_l\left(\vec{\overline{r}}_{l-1}(p,t)\right)\right].$$

Taking $H\,(p,t) = 1$ in above equation yields

$$\overline{r}_l\,(p,t) = \chi_l\overline{r}_{l-1}\,(p,t) + \mathbb{S}^{-1}\left[h\left[\mathbb{S}\left[\overline{r}_{l-1}\,(p,t)\right] - w^{\alpha}\mathbb{S}\left[\frac{1}{2}p^2D_p^2\overline{r}_{l-1}\,(p,t)\right]\right]\right],$$

$$\overline{r}_1\,(p,t) = -hp^2\overline{k}\,(\beta)\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$\overline{r}_2\,(p,t) = -hp^2\overline{k}\,(\beta)\,\frac{t^{\alpha}}{\Gamma\,(\alpha+1)} - h^2p^2\overline{k}\,(\beta)\,\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2p^2\overline{k}\,(\beta)\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$\begin{aligned}\overline{r}_3(p,t) = &-hp^2\overline{k}(\beta)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - h^2p^2\overline{k}(\beta)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^2p^2\overline{k}(\beta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &- h^2p^2\overline{k}(\beta)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - h^3p^2\overline{k}(\beta)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h^3p^2\overline{k}(\beta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &+ h^2p^2\overline{k}(\beta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h^3p^2\overline{k}(\beta)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - h^3p^2\overline{k}(\beta)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.\end{aligned}$$

Therefore the series solution is determined as

$$\overline{r}(p,t;\beta) = \overline{r}_0(p,t;\beta) + \sum_{l=1}^{\infty}\overline{r}_l(p,t;\beta).$$

The following approximate solution is obtained at $h=-1$

$$\begin{aligned}\overline{r}(p,t;\beta) &= \overline{k}(\beta)p^2 + \overline{k}(\beta)p^2\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \overline{k}(\beta)p^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\ &\quad + \overline{k}(\beta)p^2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\\ &= \overline{k}(\beta)p^2\sum_{j=0}^{\infty}\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\\ &= \overline{k}(\beta)p^2E_{\alpha}(t).\end{aligned}$$

Similarly, the solution of Problem (19) in terms of one parameter Mittag-Leffler function $E_{\alpha}(t)$ is given by

$$\underline{r}(p,t;\beta) = \underline{k}(\beta)p^2\sum_{j=0}^{\infty}\frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = \underline{k}(\beta)p^2E_{\alpha}(t).$$

The approximate solutions SHAM of order 11 are compared and plotted for $\alpha = 0.9,\ 0.95,\ 1, p = 1$ and $t = 0.25$ and $\beta = (0.75 + 0.25\beta; 1.25 - 0.25\beta)$ in Figure 2.

6 Conclusion

In this research, the approximate analytical solutions of the fuzzy problems including fuzzy fractional heat-like equation is constructed by im-

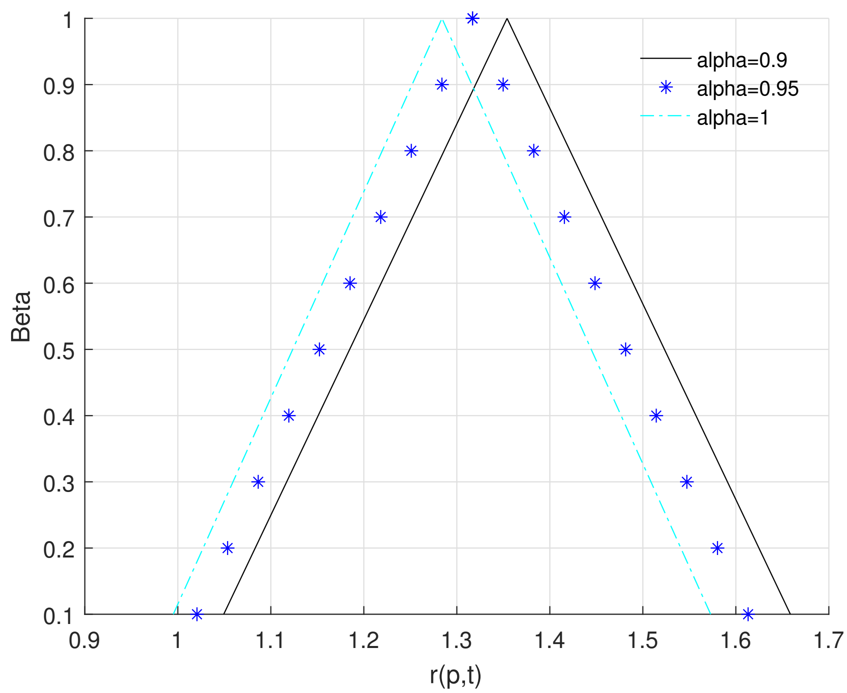

Figure 2: The approximate solutions for Example 2

plementing the proposed Sumudu homotopy analysis method. The advantages of this method are requiring less computational work and implementing without any difficulty as well as being effective and powerful. The numerical examples illustrated that the convergence and accuracy of the solution is very high.

In the future work, this study is extended to determining the solutions of fuzzy space fractional and fuzzy time space fractional differential equations.

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