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Original Research Paper

β-type and Generalized β-γ-type Multi-valued Contractive Mappings in a Menger PbM-Space

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Abstract. In the present paper, we introduce the notions of β-type and generalized β-γ-type multi-valued contractive mappings in a Menger PbM-space and obtain some new fixed point theorems for both β-admissible and (β, γ)-admissible mappings. Our results extensively generalize previous results in this framework. Furthermore, we support our results by some non-trivial examples and an application.

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1 Introduction

The category of b-metrics was defined by Bakhtin [2] and Czerwik [8]. After that, some other authors discussed on its properties defined con-

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vergent and Cauchy sequence, etc, and established several fixed point theorems for various functions in this space with their applications to nonlinear functional analysis (see [15, 18] and their references). Also, the notion of probabilistic metric spaces were defined by Menger [16]. Then, many fixed point theorems in such spaces have been introduced by many researchers, see [7, 11, 12, 14, 19] and references therein. Moreover, in 2015, Hasanvand and Khanehgir [13] defined a new version of probabilistic space which is called Menger PbM-space and established some theorems for single-valued operators.

In 1969, Nadler [17] considered the Banach contraction principle for multi-valued mapping. Then Ciric [6] extended Nadler's result. After that, many researchers proved some main fixed point theorems regarding a multi-valued mapping in various metric spaces (for example, see [1, 4, 19] and reference contained therein). In this work, by applying [8, 11, 13], we consider some nonlinear contractions for multi-valued mappings (in both case β-admissible and (β,γ)-admissible) and obtain several fixed point theorems in PbM-spaces. In Section 2, by applying the definitions of generalized β-type and generalized β-γ-type multi-valued mappings in a PbM-space, we establish some new results. In Section 3, we prepare some coupled fixed point results for generalized β-type and generalized β-γ-type mappings in a PbM-space by the similar extension of mentioned definitions in Section 2. Finally, we present an application in integral equations.

Throughout this paper, the collection of all nonempty closed and bounded subsets of a PbM-space $\mathcal{X}$ is marked by $CB(\mathcal{X})$ and the collection of all nonempty subsets of every nonempty set $\mathcal{X}$ is considered by $N(\mathcal{X})$. Also, assume that $\mathcal{T}$ is a triangular norm (t-norm) and D^+ is the collection of all Menger distance distribution functions [7].

Definition 1.1. [5] A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Φ-function if it satisfies the following conditions:

i) $\phi(t) = 0$ if and only if $t = 0$,

ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,

iii) ϕ is left-continuous in $(0, \infty)$,

iv) ϕ is continuous at 0.

Definition 1.2. [13] Let $\mathcal{X} \neq \emptyset$, $\mathcal{T}$ be a continuous t-norm and $\alpha \in (0,1]$. Also, let $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to D^+$ be a mapping with the value $\mathcal{F}_{x,y}$ at (x,y). $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PbM-space if the following properties are held:

(PbM1) $\mathcal{F}_{x,y}(t) = 1 \Leftrightarrow x = y$,

(PbM2) $\mathcal{F}_{x,y}(t) = \mathcal{F}_{y,x}(t)$,

(PbM3) $\mathcal{F}_{x,z}(t+s) \geq \mathcal{T}(\mathcal{F}_{x,y}(\alpha t), \mathcal{F}_{y,z}(\alpha s))$.

for each $x, y, z \in \mathcal{X}$ and $s, t \geq 0$.

We know that a Menger PbM-space is a Menger PM-space [7], with $\alpha = 1$. Thus the category of Menger PbM-spaces is bigger than the category of Menger PM-spaces. Further, for definitions of convergent and Cauchy sequences, completeness and examples in a Menger PbM-space, see [13].

Definition 1.3. [19] Consider a Menger PM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$. A function $H : CB(\mathcal{X}) \times CB(\mathcal{X}) \longrightarrow [0,1]$ with

$$H_{\mathcal{A},\mathcal{B}}(t) = \sup_{s \leq t} \mathcal{T}(\inf_{p \in \mathcal{A}} \mathcal{F}_{p,\mathcal{B}}(s), \inf_{q \in \mathcal{B}} \mathcal{F}_{q,\mathcal{A}}(s))$$

for all $t \geq 0$ is named a probabilistic distance between $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{F}_{x,\mathcal{A}}(t) = \sup_{p \in \mathcal{A}} \mathcal{F}_{p,x}(t)$ is the distance between a point and a set.

Definition 1.4. [19] Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PM-space.

(i) A subset $D \subset \mathcal{X}$ is called approximative if $P_{x,D}(t) = \{p \in D : \mathcal{F}_{x,D}(t) = \mathcal{F}_{p,x}(t)\}$ for every $x \in \mathcal{X}$ has nonempty value.

(ii) A multi-valued mapping $G : \mathcal{X} \longrightarrow N(\mathcal{X})$ has approximative values, AV for short, if $G(x)$ is approximative for each $x \in \mathcal{X}$.

(iii) G has the w-approximate value property if there exists $y \in G(x)$ with $\mathcal{F}_{x,y}(t) \geq H_{G(a),G(x)}(t)$ for every $a \in \mathcal{X}$, $x \in G(a)$ and for every $t > 0$.

Definitions 1.3 and 1.4 are held in PbM-space.

Lemma 1.5. [11] *Consider a Menger PbM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ with coefficient α. Assume that $\mathcal{F}_{x,y}(\alpha^k \phi(t)) \geq \mathcal{F}_{x,y}(\alpha^{k-1}\phi(\frac{t}{c}))$ for $x, y \in \mathcal{X}$, $c \in (0,1)$, $k \in \mathbb{N}$ and for each $t > 0$, where ϕ is a Φ-function. Then $\mathcal{F}_{x,y}(t) = 1$.*

In the sequel, we denote a Menger PbM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ with coefficient α by $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$ and $\mathcal{X} \times \mathcal{X}$ by $\mathcal{X}^2$.

2 Multi-valued mappings in PbM-space and fixed point results

Following the idea of Hasanvand and Khanehgir [13] and Gopal et al. [11], we introduce the following definitions in the framework of Menger PbM-space. The definitions of β-admissible and (β, γ)-admissible in the framework of Menger PbM-space are held when $f : \mathcal{X} \to CB(\mathcal{X})$ is an arbitrary multi-valued mapping.

Definition 2.1. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PbM-space with coefficient α and $f : \mathcal{X} \to CB(\mathcal{X})$ be a given multi-valued mapping. We say that f is a generalized β-type contractive multi-valued mapping of degree $k \in \mathbb{N}$, if there exist a function $\beta : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ such that

$$\begin{aligned}\beta(x, y, \alpha^k t) H_{f(x), f(y)}(\alpha^k \phi(t)) \geq \min\{&\mathcal{F}_{x,y}(\alpha^{k-1}\phi(\frac{t}{c}), \mathcal{F}_{x,f(x)}(\alpha^{k-1}\phi(\frac{t}{c}),\\ &\mathcal{F}_{f(y),y}(\alpha^{k-1}\phi(\frac{t}{c}), \mathcal{F}_{x,f(y)}(2\alpha^{k-2}\phi(\frac{t}{c}),\\ &\mathcal{F}_{f(x),y}(2\alpha^{k-2}\phi(\frac{t}{c})\}\end{aligned}$$

for all $x, y \in \mathcal{X}$ and for all $t > 0$, where $\phi \in \Phi$ and $c \in (0, 1)$. Further, the mapping f is called a generalized β-type contractive multi-valued mapping if it is a generalized β-type contractive multi-valued mapping of degree k for each $k \in \mathbb{N}$.

Definition 2.2. Let $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ be a Menger PbM-space with coefficient α and $f : \mathcal{X} \to CB(\mathcal{X})$ be a given multi-valued mapping. We say that f is a generalized β-γ-type multi-valued contractive mapping of degree

$k \in \mathbb{N}$, if there exist two functions $\beta, \gamma : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to (0, \infty)$ such that

$$\begin{aligned}\beta(x, y, \alpha^k t) H_{f(x), f(y)}(\alpha^k \phi(t)) \geq \gamma(u, v, \alpha^{k-1} \frac{t}{c}) \min\{&\mathcal{F}_{x,y}(\alpha^{k-1} \phi(\frac{t}{c}),\\ &\mathcal{F}_{x,f(x)}(\alpha^{k-1} \phi(\frac{t}{c}), \mathcal{F}_{f(y),y}(\alpha^{k-1} \phi(\frac{t}{c}),\\ &\mathcal{F}_{x,f(y)}(2\alpha^{k-2} \phi(\frac{t}{c}), \mathcal{F}_{f(x),y}(2\alpha^{k-2} \phi(\frac{t}{c})\}\end{aligned}$$

for all $x, y \in \mathcal{X}$, $u \in f(x)$, $v \in f(y)$ and for all $t > 0$, where $\phi \in \Phi$ and $c \in (0, 1)$. Further, the mapping f is called a generalized β-γ-type multi-valued contractive mapping if it is a generalized β-γ-type multi-valued contractive mapping of degree k for each $k \in \mathbb{N}$.

Theorem 2.3. *Consider a complete Menger PbM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$. Let $\mathcal{T}(a, a) \geq a$ for each $a \in [0, 1]$, and $f : \mathcal{X} \to CB(\mathcal{X})$ be a generalized β-γ-type multi-valued contractive mapping and has the w-approximative value property. Suppose the mapping f satisfy the following properties:*

(i) f *is (β, γ)-admissible;*

(ii) *for some $x_0 \in \mathcal{X}$, there is $x_1 \in f(x_0)$ with $\beta(x_0, x_1, t) \leq 1$ and $\gamma(x_0, x_1, t) \geq 1$ for every $t > 0$;*

(iii) *for all $n \in \mathbb{N}$ and for every $t > 0$, if $\{x_n\}$ is a convergent sequence to $x \in \mathcal{X}$ with $\beta(x_{n-1}, x_n, t) \leq 1$ and $\gamma(x_n, x_{n+1}, t) \geq 1$, then $\beta(x_{n-1}, x, t) \leq 1$ and $\gamma(x_n, y, t) \geq 1$ for each $y \in f(x)$.*

Then f has a fixed point in $\mathcal{X}$.

Proof. If $x_1 = x_0$, then the proof is complete. Let $x_1 \neq x_0$, i.e., $x_0 \notin f(x_0)$. Since f has w-approximative value property, there is $x_2 \in f(x_1)$ such that $\mathcal{F}_{x_1,x_2}(t) \geq H_{f(x_0),f(x_1)}(t)$ for every $t > 0$. For $x_2 \in f(x_1)$, from (i) and (ii) we have $\beta(x_0, x_1, t) \leq 1$ and $\gamma(x_1, x_2, t) \geq 1$ for every $t > 0$. If $x_1 \in f(x_1)$, then x_1 is a fixed point of f. Assume that $x_2 \neq x_1$. Again, by the assumptions, there exists $x_3 \in f(x_2)$ such that $\mathcal{F}_{x_3,x_2}(t) \geq H_{f(x_2),f(x_1)}(t)$, $\beta(x_2, x_1, t) \leq 1$ and $\gamma(x_2, x_3, t) \geq 1$ for every $t > 0$. Continue this procedure, we get a sequence $\{x_n\}$ in $\mathcal{X}$ with $x_n \in f(x_{n-1})$ for $x_n \neq x_{n-1}$, where $\mathcal{F}_{x_n,x_{n+1}}(t) \geq H_{f(x_{n-1}),f(x_n)}(t)$,

$\beta(x_n, x_{n-1}, t) \leq 1$ and $\gamma(x_n, x_{n+1}, t) \geq 1$ for all $t > 0$. Due to the continuity of ϕ at 0, there is $r > 0$ with $t > \phi(r)$. Since f has w-approximative value property, we obtain

$$\begin{aligned}\mathcal{F}_{x_n,x_{n+1}}(t) &\geq \beta(x_{n-1}, x_n, \alpha^k r) H_{f(x_{n-1}),f(x_n)}(\alpha^k \phi(r))\\ &\geq \gamma(x_n, x_{n+1}, \alpha^{k-1}\frac{r}{c}) \min\{\mathcal{F}_{x_{n-1},f(x_{n-1})}(\alpha^{k-1}\phi(\frac{r}{c})),\\ &\qquad \mathcal{F}_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c})), \mathcal{F}_{f(x_n),x_n}(\alpha^{k-1}\phi(\frac{r}{c})),\\ &\qquad \mathcal{F}_{x_n,f(x_{n-1})}(2\alpha^{k-2}\phi(\frac{r}{c})), \mathcal{F}_{f(x_n),x_{n-1}}(2\alpha^{k-2}\phi(\frac{r}{c}))\}\\ &\geq \min\{\mathcal{F}_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c})), \mathcal{F}_{x_n,x_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))\}.\end{aligned}$$

Now we show that

$$\mathcal{F}_{x_n,x_{n+1}}(\alpha^k\phi(r)) \geq \mathcal{F}_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c})). \tag{1}$$

Let $\mathcal{F}_{x_n,x_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))$ be the minimum. By Lemma 1.5, we have $x_n = x_{n+1}$, which is a contradiction with the assumption $x_n \neq x_{n+1}$. Thus, $\mathcal{F}_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum; that is, (1) is true. Now, from (1), one obtains that

$$\begin{aligned}\mathcal{F}_{x_n,x_{n+1}}(\alpha^k t) \geq \mathcal{F}_{x_n,x_{n+1}}(\alpha^k\phi(r)) &\geq \mathcal{F}_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c}))\\ &\geq \cdots \geq \mathcal{F}_{x_0,x_1}(\alpha^{k-n}\phi(\frac{r}{c^n}));\end{aligned}$$

that is, $\mathcal{F}_{x_n,x_{n+1}}(\alpha^k t) \geq \mathcal{F}_{x_0,x_1}(\alpha^{k-n}\phi(\frac{r}{c^n}))$ for every $n \in \mathbb{N}$. Next, consider $m, n \in \mathbb{N}$ with $n < m$. Then, from (PbM3) and strictly increasing of ϕ, we get

$$\begin{aligned}\mathcal{F}_{x_n,x_m}((m-n)t) &\geq \min\{\mathcal{F}_{x_n,x_{n+1}}(\alpha t), \cdots, \mathcal{F}_{x_{m-1},x_{m-2}}(\alpha^{m-n-1}t),\\ &\qquad \mathcal{F}_{x_{m-1},x_m}(\alpha^{m-n-1}t)\}\\ &\geq \min\{\mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n})), \cdots, \mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^{m-2}})),\\ &\qquad \mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^{m-1}}))\}\\ &= \mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n})).\end{aligned}$$

Since $\alpha^{1-n}\phi(\frac{r}{c^n}) \to \infty$ as $n \to \infty$, there is $n_0 \in \mathbb{N}$ so that

$$\mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n})) > 1-\epsilon$$

for fixed $\epsilon \in (0,1)$, whenever $n \geq n_0$. This implies that $\mathcal{F}_{x_n,x_m}((m-n)t) > 1-\epsilon$ for every $m > n \geq n_0$. Since $t > 0$ and $\epsilon \in (0,1)$ are optional, $\{x_n\}$ will be a Cauchy sequence in the complete PbM-Menger space $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$. Then there is $x \in \mathcal{X}$ so that $x_n \to x$ as $n \to \infty$. Now, we prove that x is a fixed point of f. From (PbM3), we get

$$\begin{aligned}\mathcal{F}_{f(x),x}(t) &\geq \mathcal{T}(\mathcal{F}_{f(x),x_{n+1}}(\alpha\phi(r)), \mathcal{F}_{x_{n+1},x}(\alpha t - \alpha\phi(r)))\\ &\geq \min\{H_{f(x),f(x_n)}(\alpha\phi(r)), \mathcal{F}_{x_{n+1},x}(\alpha t - \alpha\phi(r))\}.\end{aligned}$$

We know that if $x_n \in f(x)$ for infinitely many values of n, then $x \in f(x)$. Thus, the proof ends. Hence, suppose that $x_n \notin f(x)$ for every $n \in \mathbb{N}$. Since $x_n \to x$, we obtain $\mathcal{F}_{x_n,x}(\alpha t - \alpha\phi(r)) > 1-\epsilon$ for n large enough. So, $\mathcal{F}_{f(x),x}(t) \geq \min\{H_{f(x),f(x_n)}(\alpha\phi(r)), 1-\epsilon\}$ for every $\epsilon \in (0,1)$ and n large enough. Since $\epsilon > 0$ is optional, $\mathcal{F}_{f(x),x}(t) \geq H_{f(x),f(x_n)}(\alpha\phi(r))$. Now, for all $u \in f(x)$ and by condition (iii), we obtain

$$\begin{aligned}\mathcal{F}_{f(x),x}(t) &\geq H_{f(x),f(x_n)}(\alpha\phi(r))\\ &\geq \beta(x, x_n, \alpha r)H_{f(x),f(x_n)}(\alpha\phi(r))\\ &\geq \gamma(x_{n+1}, u, \frac{r}{c}) \min\{\mathcal{F}_{x_n,x}\phi(\frac{r}{c}), \mathcal{F}_{x_n,f(x_n)}\phi(\frac{r}{c}), \mathcal{F}_{f(x),x}\phi(\frac{r}{c}),\\ &\qquad \mathcal{F}_{x_n,f(x)}(\frac{2}{\alpha}\phi(\frac{r}{c})), \mathcal{F}_{f(x_n),x}(\frac{2}{\alpha}\phi(\frac{r}{c}))\}\\ &\geq \min\{\mathcal{F}_{x_n,x}(\phi(\frac{r}{c})), \mathcal{F}_{x_n,f(x_n)}(\phi(\frac{r}{c})), \mathcal{F}_{f(x),x}(\phi(\frac{r}{c}))\}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{F}_{f(x),x}(t) &\geq \lim_{n\to\infty} \inf H_{f(x),f(x_n)}(\alpha\phi(r))\\ &\geq \lim_{n\to\infty} \inf \min\{\mathcal{F}_{x_n,x}(\phi(\frac{r}{c})), \mathcal{F}_{x_n,x_{n+1}}(\phi(\frac{r}{c})), \mathcal{F}_{f(x),x}\phi(\frac{r}{c})\}\\ &\geq \min\{1-\epsilon, 1-\epsilon, \mathcal{F}_{f(x),x}(\phi(\frac{r}{c}))\}.\end{aligned}$$

Ultimately, since $\epsilon \in (0,1)$ is arbitrary, $\mathcal{F}_{f(x),x}(t) \geq \mathcal{F}_{f(x),x}(\alpha t) \geq \mathcal{F}_{f(x),x}(\alpha\phi(r)) \geq \mathcal{F}_{x,f(x)}(\phi(\frac{r}{c}))$. From Lemma 1.5, we deduce that $x \in f(x)$. Here, the proof ends. $\square$

Example 2.4. Let $\mathcal{X} = \mathbb{R}^+$ and $\mathcal{F} : X \times X \to D^+$ be defined by

$$\mathcal{F}_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|^2}, & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

for every $x, y \in X$ and $\mathcal{T}(x, y) = \min\{x, y\}$. Then $(\mathcal{X}, \mathcal{F}, \mathcal{T})$ is a Menger PbM-space with $\alpha = \frac{1}{2}$. Define $f : \mathcal{X} \to CB(\mathcal{X})$ by $f(x) = [0, \frac{x}{4}]$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = t$. Also, let $\beta(x, y, t) = \gamma(x, y, t) = 1$ for every $x, y \in \mathcal{X}$ and $c = \frac{1}{2}$. Note that $\beta(x, y, \alpha^k t)H_{f(x),f(y)}(\alpha^k \phi(t)) = H_{f(x),f(y)}(\alpha^k \phi(t))$. Using the definition of the probabilistic Hausdorff metric in Definition 1.3, we have

$$\begin{aligned}
\beta(x, y, \alpha^k t)H_{f(x),f(y)}(\alpha^k \phi(t)) &= \frac{\frac{1}{2^k}t}{\frac{1}{2^k}t + |\frac{x}{4} - \frac{y}{4}|^2} \\
&= \frac{t}{t + 2^{k-4}|x-y|^2} \\
&\geq \frac{t}{t + 2^{k-2}|x-y|^2} \\
&= \mathcal{F}_{x,y}(\alpha^{k-1}\phi(\frac{t}{c})) \\
&\geq \min\{\mathcal{F}_{x,y}(\alpha^{k-1}\phi(\frac{t}{c})), \\
&\quad \mathcal{F}_{x,f(x)}(\alpha^{k-1}\phi(\frac{t}{c})), \mathcal{F}_{f(y),y}(\alpha^{k-1}\phi(\frac{t}{c})), \\
&\quad \mathcal{F}_{x,f(y)}(2\alpha^{k-2}\phi(\frac{t}{c})), \mathcal{F}_{f(x),y}(2\alpha^{k-2}\phi(\frac{t}{c}))\} \\
&= \gamma(u, v, \alpha^{k-1}\frac{t}{c}) \min\{\mathcal{F}_{x,y}(\alpha^{k-1}\phi(\frac{t}{c})), \\
&\quad \mathcal{F}_{x,f(x)}(\alpha^{k-1}\phi(\frac{t}{c})), \mathcal{F}_{f(y),y}(\alpha^{k-1}\phi(\frac{t}{c})), \\
&\mathcal{F}_{x,f(y)}(2\alpha^{k-2}\phi(\frac{t}{c})), \mathcal{F}_{f(x),y}(2\alpha^{k-2}\phi(\frac{t}{c}))\}.
\end{aligned}$$

Hence f is a β-γ-type multi-valued contractive mapping. Since f has the compact values it has the w-approximative value property. On the others hands, by definitions of f, β and γ, all of the assumptions of Theorem 2.3 (i)-(iii) are hold and so f has a fixed point.

Corollary 2.5. *Consider $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$ as a complete Menger PbM-space. Let $\mathcal{T}(a, a) \geq a$ for each $a \in [0, 1]$, and $f : \mathcal{X} \to CB(\mathcal{X})$ be a generalized β-type contractive multi-valued mapping and has the w-approximative value property. Suppose the mapping f satisfy the following properties:*

(i) f is β-admissible;

(ii) for some $x_0 \in \mathcal{X}$, there is $x_1 \in f(x_0)$ with $\beta(x_0, x_1, t) \leq 1$ for every $t > 0$;

(iii) for all $n \in \mathbb{N}$ and every $t > 0$, if $\{x_n\}$ is a convergent sequence to $x \in \mathcal{X}$ with $\beta(x_{n-1}, x_n, t) \leq 1$, then $\beta(x_{n-1}, x, t) \leq 1$.

Then f has a fixed point in $\mathcal{X}$.

Theorem 2.6. *Consider a complete Menger PbM-space $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$. Let $\mathcal{T}(a, a) \geq a$ for each $a \in [0, 1]$, and $f : \mathcal{X} \to CB(\mathcal{X})$ be a generalized β-γ-type multi-valued contractive mapping and has w-approximative value property. Suppose the mapping f satisfy the properties* (i)-(iii) *of Theorem 2.3 and the following property:*

(iv) For every $u, v \in \mathcal{F}ix(f)$ and for every $t > 0$, suppose that $\beta(z, c, t) \leq 1$ by $\beta(u, z, t) \leq 1$ and $\beta(v, z, t) \leq 1$, and $\gamma(z, c, t) \geq 1$ by $\gamma(u, z, t) \geq 1$ and $\gamma(v, z, t) \geq 1$ for a $z \in \mathcal{X}$ and for all $c \in f(z)$.

Then f has a unique fixed point in $\mathcal{X}$.

Proof. Consider $u, v \in \mathcal{X}$ with $u \in fu$ and $v \in f(v)$. By (iv), there is $z \in \mathcal{X}$ so that $\beta(z, c, t) \leq 1$ by $\beta(u, z, t) \leq 1$ and $\beta(v, z, t) \leq 1$, and $\gamma(z, c, t) \geq 1$ by $\gamma(u, z, t) \geq 1$ and $\gamma(v, z, t) \geq 1$ for all $c \in f(z)$. Since f has w-approximative value property, then there is $a \in f(z)$ with $\mathcal{F}_{a,z}(t) \geq H_{f(a),f(z)}(t)$ for all $t > 0$. Without generally suppose $a = c$. Since f is (β, γ)-admissible, then

$$\begin{aligned} &\beta(c, z_1, t) \leq 1, \quad \beta(u, c, t) \leq 1, \quad \beta(v, c, t) \leq 1, \\ &\gamma(c, z_1, t) \geq 1, \quad \gamma(u, c, t) \geq 1, \quad \gamma(v, c, t) \geq 1, \end{aligned}$$

for each $z_1 \in fc$. So, we have $\mathcal{F}_{c,z_1}(t) \geq H_{f(c),f(z_10}(t)$. By induction, we derive

$$\begin{array}{lll} \beta(z_n, z_{n+1}, t) \leq 1, & \beta(u, z_n, t) \leq 1, & \beta(v, z_n, t) \leq 1, \\ \gamma(z_n, z_{n+1}, t) \geq 1, & \gamma(u, z_n, t) \geq 1, & \gamma(v, z_n, t) \geq 1, \end{array}$$

for every $t > 0$. Thus, we obtain $\mathcal{F}_{z_n,z_{n+1}}(t) \geq H_{f(z_n),f(z_{n+1})}(t)$, where $z_n \in f(z_{n-1})$ $(n \in \mathbb{N})$. By the continuity of ϕ at 0, there is $r > 0$ with $t > \phi(r)$. Since f has w-approximative value property, we have

$$\begin{aligned} \mathcal{F}_{u,z_{n+1}}(t) \geq \mathcal{F}_{u,z_{n+1}}(\alpha^k\phi(r)) &\geq \beta(u, z_n, \alpha^k r) H_{f(u),f(z_n)}(\alpha^k\phi(r)) \\ &\geq \gamma(u, z_{n+1}, \alpha^{k-1}\frac{r}{c}) \min\{\mathcal{F}_{u,f(u)}(\alpha^{k-1}\phi(\frac{r}{c})), \mathcal{F}_{u,z_n}(\alpha^{k-1}\phi(\frac{r}{c})), \\ &\qquad \mathcal{F}_{f(z_n),z_n}(\alpha^{k-1}\phi(\frac{r}{c})), \mathcal{F}_{u,f(z_n)}(2\alpha^{k-2}\phi(\frac{r}{c})), \\ &\qquad \mathcal{F}_{z_n,f(u)}(2\alpha^{k-2}\phi(\frac{r}{c}))\} \\ &\geq \min\{\mathcal{F}_{u,z_n}(\alpha^{k-1}\phi(\frac{r}{c})), \mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))\} \end{aligned}$$

with $k \in \mathbb{N}$. Now, we have the following Cases:
Case I. Let $\mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))$ be the minimum. It follows from (PbM1) and (PbM3) that

$$\begin{aligned} \mathcal{F}_{u,z_{n+1}}(\alpha^k\phi(r)) &\geq \mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c})) \\ &\geq \beta(z_{n-1}, z_n, \alpha^{k-1}r) H_{f(z_{n-1}),f(z_n)}(\alpha^{k-1}\phi(\frac{r}{c})) \\ &\geq \gamma(z_n, z_{n+1}, \alpha^{k-2}\frac{r}{c^2}) \min\{\mathcal{F}_{z_{n-1},z_n}(\alpha^{k-2}\phi(\frac{r}{c^2})), \\ &\qquad \mathcal{F}_{z_{n-1},f(z_{n-1})}(\alpha^{k-2}\phi(\frac{r}{c^2})), \mathcal{F}_{f(z_n),z_n}(\alpha^{k-2}\phi(\frac{r}{c^2})), \\ &\qquad \mathcal{F}_{z_{n-1},f(z_n)}(2\alpha^{k-3}\phi(\frac{r}{c^2})), \mathcal{F}_{z_n,f(z_{n-1})}(2\alpha^{k-3}\phi(\frac{r}{c^2}))\} \\ &\geq \min\{\mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-2}\phi(\frac{r}{c})), \mathcal{F}_{z_{n-1},z_n}(\alpha^{k-2}\phi(\frac{r}{c}))\}. \end{aligned}$$

Now, if $\mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-2}\phi(\frac{r}{c}))$ is the minimum for a $n \in \mathbb{N}$, then $z_n = z_{n+1}$ (see Lemma 1.5). Since $\mathcal{F}_{u,z_{n+1}}(\alpha^k\phi(r)) \geq \mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c})) = 1$, we conclude $\beta(v, u, t) \leq 1$ and $\gamma(v, u, t) \geq 1$ for all $t > 0$. Thus, by (PbM1) and (PbM3), we have

$$\begin{aligned}\mathcal{F}_{u,v}(\alpha^k\phi(t)) &\geq \beta(u,v,\alpha^k t)H_{f(u),f(v)}(\alpha^k\phi(t))\\ &\geq \gamma(u,v,\alpha^{k-1}\frac{t}{c})\min\{\mathcal{F}_{u,v}(\alpha^{k-1}\phi(\frac{t}{c})),\mathcal{F}_{v,f(v)}(\alpha^{k-1}\phi(\frac{t}{c})),\\ &\qquad \mathcal{F}_{u,f(u)}(\alpha^{k-1}\phi(\frac{t}{c})),\mathcal{F}_{v,f(u)}(2\alpha^{k-2}\phi(\frac{t}{c})),\\ &\qquad \mathcal{F}_{u,f(v)}(2\alpha^{k-2}\phi(\frac{r}{c}))\}\\ &\geq \mathcal{F}_{v,u}(\alpha^{k-1}\phi(\frac{t}{c})).\end{aligned}$$

Once more, by applying Lemma 1.5, we obtain $u=v$. Moreover, if $\mathcal{F}_{z_{n-1},z_n}(\alpha^{k-2}\phi(\frac{r}{c}))$ is the minimum, then

$$\begin{aligned}\mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c})) \geq \mathcal{F}_{z_{n-1},z_n}(\alpha^{k-2}\phi(\frac{r}{c^2})) &\geq\\ \cdots \geq \mathcal{F}_{z_0,z_1}(\alpha^{k-(n+1)}\phi(\frac{r}{c^{n+1}})),&\end{aligned}$$

which by letting $n\to\infty$, we have $\mathcal{F}_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))\to 1$. Therefore, $\lim\limits_{n\to\infty}\mathcal{F}_{u,z_{n+1}}(t)=1$. So $z_{n+1}\to u$ as $n\to\infty$. Similarly, $z_{n+1}\to v$ as $n\to\infty$. Since the limit is unique, we get $u=v$.
Case II. Let $\mathcal{F}_{u,z_n}(\alpha^{k-1}\phi(\frac{r}{c}))$ be the minimum. Then we obtain

$$\mathcal{F}_{u,z_{n+1}}(\alpha^k\phi(r)) \geq \mathcal{F}_{u,z_n}(\alpha^{k-1}\phi(\frac{r}{c})) \geq \cdots \geq \mathcal{F}_{u,z_0}(\alpha^{k-(n+1)}\phi(\frac{r}{c^{n+1}})).$$

Now, let $n\to\infty$. Then we obtain $\lim_{n\to\infty}\mathcal{F}_{u,z_{n+1}}(\alpha^k\phi(r))=1$, which implies that $z_{n+1}\to u$ as $n\to\infty$. Similarly, we can prove $z_{n+1}\to v$ as $n\to\infty$. Here, by the uniqueness of the limit, we have $u=v$. So, the proof ends. ☐

Corollary 2.7. *Consider $(\mathcal{X},\mathcal{F},\mathcal{T},\alpha)$ as a complete Menger PbM-space. Let $\mathcal{T}(a,a)\geq a$ for each $a\in[0,1]$, and $f:\mathcal{X}\to CB(\mathcal{X})$ be a generalized β-type contractive multi-valued mapping and has w-approximative value property. Suppose the mapping f satisfy the properties* (i)-(iii) *of Corollary 2.5 and the following property:*

(iv) for every $u,v\in\mathcal{F}ix(f)$ and for every $t>0$, suppose that $\beta(z,c,t)\leq 1$ by $\beta(u,z,t)\leq 1$ and $\beta(v,z,t)\leq 1$ for a $z\in\mathcal{X}$ and for all $c\in f(z)$.

Then f has a unique fixed point in $\mathcal{X}$.

3 Coupled fixed point theorems in PbM-spaces

In the last decade, the entity a coupled fixed point for a mapping $f : \mathcal{X}^2 \to \mathcal{X}$ has been extended by many authors, see [3, 9, 10, 20]. Following the idea of Hasanvand and Khanehgir [13] and Gopal et al. [11] we have the following definitions for the mapping f. The definitions of β-admissible and (β,γ)-admissible in the framework of a $(\mathcal{X},\mathcal{F},\mathcal{T},\alpha)$ are held when $f : \mathcal{X}^2 \to \mathcal{X}$ is an arbitrary mapping and a usual point has been changed to a couple point.

Definition 3.1. Let $(\mathcal{X},\mathcal{F},\mathcal{T})$ be a Menger PbM-space with coefficient α and $f : \mathcal{X}^2 \to \mathcal{X}$ be a given mapping. We say that f is a generalized β-type contractive mapping of degree $k \in \mathbb{N}$, if there exists a function $\beta : \mathcal{X}^2 \times \mathcal{X}^2 \times (0,\infty) \to (0,\infty)$ such that

$$\beta((x,y),(u,v),\alpha^k t)\mathcal{F}_{f(x,y),f(u,v)}(\alpha^k\phi(t)) \geq \min\{\mathcal{F}_{x,u}(\alpha^{k-1}\phi(\frac{t}{c}),\mathcal{F}_{y,v}(\alpha^{k-1}\phi(\frac{t}{c})\}$$

for all $(x,y),(u,v) \in \mathcal{X}^2$ and for all $t > 0$, where $\phi \in \Phi$ and $c \in (0,1)$. Further, the mapping f is called a generalized β-type contractive mapping if it is a generalized β-type contractive mapping of degree k for all $k \in \mathbb{N}$.

Definition 3.2. Let $(\mathcal{X},\mathcal{F},\mathcal{T})$ be a Menger PbM-space with coefficient α and $f : \mathcal{X}^2 \to \mathcal{X}$ be a given mapping. We say that f is a generalized β-γ-type contractive mapping of degree $k \in \mathbb{N}$, if there exist two functions $\beta : \mathcal{X}^2 \times \mathcal{X}^2 \times (0,\infty) \to (0,\infty)$ and $\gamma : \mathcal{X}^2 \times \mathcal{X}^2 \times (0,\infty) \to (0,\infty)$ such that

$$\begin{aligned}\beta((x,y),(u,v),\alpha^k t)\mathcal{F}_{f(x,y),f(u,v)}(\alpha^k\phi(t)) \geq \\ \gamma((f(x,y),f(y,x)),(f(u,v),f(v,u)),\alpha^{k-1}\frac{t}{c}) \\ \min\{\mathcal{F}_{x,u}(\alpha^{k-1}\phi(\frac{t}{c}),\mathcal{F}_{y,v}(\alpha^{k-1}\phi(\frac{t}{c})\}\end{aligned}$$

for all $(x,y),(u,v) \in \mathcal{X}^2$ and for all $t > 0$, where $\phi \in \Phi$ and $c \in (0,1)$. Further, the mapping f is called a generalized β-γ-type contractive mapping if it is a generalized β-γ-type contractive mapping of degree k for all $k \in \mathbb{N}$.

Theorem 3.3. *Consider a complete Menger PbM-space* $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$ *and* $\mathcal{T}(a, a) \geq a$ *for each* $a \in [0, 1]$. *Suppose that a generalized* β-γ-*type contractive mapping* $f : \mathcal{X}^2 \to \mathcal{X}$ *satisfies the following properties:*

(i) f *is* (β, γ)-*admissible;*

(ii) *we have*
$$\beta((x_0, y_0), (f(x_0, y_0), f(y_0, x_0)), t) \leq 1$$
and
$$\gamma((x_0, y_0), (f(x_0, y_0), f(y_0, x_0)), t) \geq 1$$
for some $(x_0, y_0) \in \mathcal{X}^2$ *and for all* $t > 0$;

(iii) *for every* $n \in \mathbb{N}$ *and* $t > 0$, *if* $(x_n, y_n) \in \mathcal{X}^2$ *with*
$$\beta((x_{n-1}, y_{n-1}), (x_n, y_n), t) \leq 1$$
and
$$\gamma((x_n, y_n), (x_{n+1}, y_{n+1}), t) \geq 1$$
and $(x_n, y_n) \to (x, y)$ *as* $n \to \infty$, *then* $\beta((x_{n-1}, y_{n-1}), (x, y), t) \leq 1$ *and* $\gamma((x_n, y_n), (f(x, y), f(y, x)), t) \geq 1$;

(iv) f *is continuous.*

Then f *has a coupled fixed point in* $\mathcal{X}^2$.

Proof. By (ii), there is $(x_0, y_0) \in \mathcal{X}^2$ with
$$\beta((x_0, y_0), (f(x_0, y_0), f(y_0, x_0)), t) \leq 1$$
and
$$\gamma((x_0, y_0), (f(x_0, y_0), f(y_0, x_0)), t) \geq 1$$
for all $t > 0$. We consider $(x_1, y_1) \in \mathcal{X}^2$ as $x_1 = f(x_0, y_0)$ and $y_1 = f(y_0, x_0)$. Let $x_2 = f(x_1, y_1)$ and $y_2 = f(y_1, x_1)$. Then $f^2(x_0, y_0) = f(x_1, y_1) = x_2$ and $f^2(y_0, x_0) = f(y_1, x_1) = y_2$. Continuing the above procedure, we obtain $(x_{n+1}, y_{n+1}) = (f^{n+1}(x_0, y_0), f^{n+1}(y_0, x_0))$ for each $n \geq 0$. If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$ for all $n = 1, 2, 3, \cdots$, then we have nothing to prove. Let $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n = 1, 2, 3, \cdots$.

By (i), (ii), and induction, we have $\beta((x_n,y_n),(x_{n-1},y_{n-1}),t)\leq 1$ and $\gamma((x_n,y_n),(x_{n+1},y_{n+1}),t)\geq 1$ for every $t>0$. By the continuity of ϕ at 0, there is $r>0$ with $t>\phi(r)$. Thus, we get

$$\begin{aligned}\mathcal{F}_{x_n,x_{n+1}}(t) &\geq \beta((x_{n-1},y_{n-1}),(x_n,y_n),\alpha^k r)\mathcal{F}_{f(x_{n-1},y_{n-1}),f(x_n,y_n)}(\alpha^k\phi(r))\\ &\geq \gamma((x_n,y_n),(x_{n+1},y_{n+1}),\alpha^{k-1}\frac{r}{c})\min\{\mathcal{F}_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{t}{c})),\\ &\qquad \mathcal{F}_{y_{n-1},y_n}(\alpha^{k-1}\phi(\frac{t}{c}))\}.\end{aligned}$$

Now, by simplification compute, one obtains that

$$\mathcal{F}_{x_n,x_{n+1}}(\alpha^k t)\geq \mathcal{F}_{y_0,y_1}(\alpha^{k-n}\phi(\frac{t}{c^n}))$$

for every $n\in\mathbb{N}$. Now, let $m,n\in\mathbb{N}$ with $m>n$. From (PbM3) and strictly increasing of ϕ, we get

$$\begin{aligned}\mathcal{F}_{x_n,x_m}((m-n)t) &\geq \min\{\mathcal{F}_{x_n,x_{n+1}}(\alpha t),\cdots,\mathcal{F}_{x_{m-1},x_{m-2}}(\alpha^{m-n-1}t),\\ &\qquad \mathcal{F}_{x_{m-1},x_m}(\alpha^{m-n-1}t)\}\\ &\geq \min\{\min\{\mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n})),\\ &\qquad \mathcal{F}_{y_0,y_1}(\alpha^{1-n}\phi(\frac{r}{c^n}))\},\cdots,\\ &\qquad \min\{\mathcal{F}_{x_0,x_1}(\alpha^{-n}\phi(\frac{r}{c^{m-1}})),\\ &\qquad \mathcal{F}_{y_0,y_1}(\alpha^{-n}\phi(\frac{r}{c^{m-1}}))\}\}\\ &= \min\{\mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n})),\mathcal{F}_{y_0,y_1}(\alpha^{1-n}\phi(\frac{r}{c^n}))\}.\end{aligned}$$

Since $\alpha^{1-n}\phi(\frac{r}{c^n})\to\infty$ as $n\to\infty$, there exists $n_0\in\mathbb{N}$ such that $\mathcal{F}_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n}))>1-\epsilon$ and $\mathcal{F}_{y_0,y_1}(\alpha^{1-n}\phi(\frac{r}{c^n}))>1-\epsilon$ for fixed $\epsilon\in(0,1)$ whenever $n\geq n_0$. This implies that $\mathcal{F}_{x_n,x_m}((m-n)t)>1-\epsilon$ for every $m>n\geq n_0$. Due to $t>0$ and $\epsilon\in(0,1)$ are optional, $\{x_n\}$ will be a Cauchy sequence. Similarly, $\{y_n\}$ is a Cauchy sequence. Now, by the completeness of $\mathcal{X}$, there exist $x,y\in\mathcal{X}$ with $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$. Now, we prove that f has a coupled fixed point in $\mathcal{X}^2$. Due to $x_{n+1}=f(x_n,y_n)$, f is continuous and by computing the limit as $n\to\infty$, we obtain $f(x,y)=x$. In a similar way, $f(y,x)=y$. So, (x,y) will be a coupled fixed point of f. □

Example 3.4. Consider $\mathcal{X}$, $\mathcal{F}$, $\mathcal{T}$, ϕ, c, β and γ as in Example 2.4. Define the mapping $f : \mathcal{X}^2 \to \mathcal{X}$ by $f(x,y) = \frac{x}{4}$. Note that

$$\beta((x,y),(u,v),\alpha^k t)\mathcal{F}_{f(x,y),f(u,v)}(\alpha^k \phi(t)) = \mathcal{F}_{f(x,y),f(u,v)}(\alpha^k \phi(t))$$

(by the definition β). Using the definition of $\mathcal{F}$, we obtain

$$\begin{aligned}
\mathcal{F}_{f(x,y),f(u,v)}(\alpha^k \phi(t)) &= \frac{\frac{1}{2^k}t}{\frac{1}{2^k}t + |\frac{x}{4} - \frac{u}{4}|^2} = \frac{\frac{1}{2^k}t}{\frac{1}{2^k}t + \frac{1}{2^4}|x-u|^2} \\
&= \frac{t}{t + \frac{2^k}{2^4}|x-u|^2} = \frac{t}{t + 2^{k-4}|x-u|^2} \\
&\geq \frac{t}{t + 2^{k-2}|x-u|^2} = \mathcal{F}_{x,u}(\alpha^{k-1}\phi(\frac{t}{c})) \\
&\geq \min\{\mathcal{F}_{x,u}(\alpha^{k-1}\phi(\frac{t}{c})), \mathcal{F}_{y,v}(\alpha^{k-1}\phi(\frac{t}{c}))\} \\
&= \gamma((f(x,y), f(y,x)), (f(u,v), f(v,u)), \alpha^{k-1}\frac{t}{c}) \\
&\qquad \min\{\mathcal{F}_{x,u}(\alpha^{k-1}\phi(\frac{t}{c})), \mathcal{F}_{y,v}(\alpha^{k-1}\phi(\frac{t}{c}))\}.
\end{aligned}$$

This means that f is a β-γ-type contractive mapping. On the other hand, by definitions of f, β and γ, all of the assumptions of Theorem 3.3 hold. Consequently, Theorem 3.3 implies that f has a coupled fixed point in $\mathbb{R}^{+2}$.

Corollary 3.5. *Consider a complete Menger PbM-space* $(\mathcal{X}, \mathcal{F}, \mathcal{T}, \alpha)$ *and* $\mathcal{T}(a,a) \geq a$ *for each* $a \in [0,1]$. *Suppose that a generalized* β*-type contractive mapping* $f : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *satisfy the following properties:*

- *(i)* f *is* β*-admissible;*
- *(ii) we have* $\beta((x_0,y_0),(f(x_0,y_0), f(y_0,x_0)), t) \leq 1$ *for some* $(x_0,y_0) \in \mathcal{X}^2$ *and for all* $t > 0$,
- *(iii) for every* $n \in \mathbb{N}$ *and for every* $t > 0$, *if* $(x_n, y_n) \in \mathcal{X}^2$ *with* $\beta((x_{n-1},y_{n-1}),(x_n,y_n),t) \leq 1$ *and* $(x_n,y_n) \to (x,y)$ *as* $n \to \infty$, *then* $\beta((x_{n-1},y_{n-1}),(x,y),t) \leq 1$.
- *(iv)* f *is continuous.*

Then f *has a coupled fixed point in* $\mathcal{X}^2$.

4 Application

Consider the system

$$\begin{cases} x(t) = \int_a^b G(t,s)K(s,x(s),y(s))ds, \\ y(t) = \int_a^b G(t,s)K(s,y(s),x(s))ds. \end{cases} \tag{2}$$

for every $t \in I = [a,b]$, $G \in C(I \times I, [0,\infty))$ and $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. Let $C(I,\mathbb{R})$ be the Banach space of whole real continuous functions considered on I with the maximum norm $||x||_\infty$ for $x \in C(I,\mathbb{R})$ and $C(I \times I \times C(I,\mathbb{R}), \mathbb{R})$ be the space of whole continuous functions considered on $I \times I \times C(I,\mathbb{R})$ and derived $b-$metric $d(x,y) = ||x-y||^2$ for every $x, y \in C(I,\mathbb{R})$. Note that d with $s = 2$ is a complete b-metric. Consider $\mathcal{F} : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to D^+$ by $\mathcal{F}_{x,y}(t) = \chi(t - d(x,y))$ for $x, y \in C(I,\mathbb{R})$ and $t > 0$, where

$$\chi(t) = \begin{cases} 0 & if \quad t \leq 0 \\ 1 & if \quad t > 0. \end{cases}$$

Then $(C(I,\mathbb{R}), \mathcal{F}, \mathcal{T})$ with $\mathcal{T}(a,b) = min\{a,b\}$ and $\alpha = \frac{1}{2}$ is a complete Menger PbM-space, see [13].

Theorem 4.1. *Consider the Menger PbM-space* $(C(I,\mathbb{R}), \mathcal{F}, \mathcal{T})$. *Let* $f : C(I,\mathbb{R}) \times C(I,\mathbb{R}) \to C(I,\mathbb{R})$ *be an operator with*

$$f(x,y)t = \int_a^b G(t,s)K(s,x(s),y(s))ds,$$

where $G \in C(I \times I, [0,\infty))$ *and* $K \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *be two operators satisfying the following properties:*

(i) $||K||_\infty = \sup_{s \in I,\, x,y \in C(I,\mathbb{R})} |K(s,x(s),y(s))| < \infty$;

(ii) for every $x, y \in C(I,\mathbb{R})$ *and every* $t, s \in I$,

$$||K(s,x(s),y(s)) - K(s,u(s),v(s))|| \leq \max\{\|x(s) - u(s)\|^2, \|y(s) - v(s)\|^2\};$$

(iii) $\sup_{t\in I}\int_a^b G(t,s)ds < 1.$

Then (2) *has a solution in* $C(I,\mathbb{R})\times C(I,\mathbb{R})$.

Proof. Consider $d(x,y) = \max_{t\in I}(|x(t)-y(t)|^2)$ for all $x, y \in C(I,\mathbb{R})$. For every $x, y \in C(I,\mathbb{R})$, we obtain

$$\begin{aligned} d(f(x,y), f(u,v)) &\le \max_{t\in I}\int_a^b G(t,s)|K(s,x(s)y(s)) - K(s,u(s),v(s))|ds \\ &\le \max\{\|x(s)-u(s)\|^2, \|y(s)-v(s)\|^2\}\max_{t\in I}\int_a^b G(t,s)ds. \end{aligned}$$

Let $c = \max_{t\in I}\int_a^b G(t,s)ds$. Then, for every $r>0$ and $k\in\mathbb{N}$, we have

$$\begin{aligned} \mathcal{F}_{f(x,y),f(u,v)}(\frac{r}{2^k}) &= \chi(\frac{r}{2^k} - d(f(x,y), f(u,v)) \\ &\ge \chi(\frac{r}{2^k} - \frac{c}{2}\max\{\|x(s)-u(s)\|^2, \|y(s)-v(s)\|^2\}) \\ &= \chi(\frac{r}{2^{k-1}c} - \max\{\|x(s)-u(s)\|^2, \|y(s)-v(s)\|^2\}) \\ &= \min\{\mathcal{F}_{x,u}(\frac{r}{2^{k-1}c}), \mathcal{F}_{y,v}(\frac{r}{2^{k-1}c})\} \end{aligned}$$

for every $x, y \in C(I,\mathbb{R})$. Now, by applying Theorem 3.3 with $\phi(r) = r$ for every $r>0$ and $\beta(x,y,t) = \gamma(x,y,t) = 1$ for every $x, y \in C(I,\mathbb{R})$ and all $t>0$, f has a coupled fixed point which is the solution of system (2). □

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