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An Extended Biconservativity Condition on Hypersurfaces of the Minkowski Spacetime

F. Pashaie
University of Maragheh

Abstract. Isoparametric hypersurfaces of Lorentz-Minkowski spaces, which has been classified by M.A. Magid in 1985, have motivated some researchers to study biconservative hypersurfaces. A biconservative hypersurface has conservative stress-energy with respect to the bienergy functional. A timelike (Lorentzian) hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$, isometrically immersed into the Lorentz-Minkowski space $\mathbb{E}_1^{n+1}$, is said to be biconservative if the tangent component of vector field $\Delta^2\mathbf{x}$ on M_1^n is identically zero. In this paper, we study the L_k-extension of biconservativity condition. The map L_k on a hypersurface (as the kth extension of Laplace operator $L_0 = \Delta$) is the linearized operator arisen from the first variation of $(k+1)$th mean curvature of hypersurface. After illustrating some examples, we prove that an L_k-biconservative timlike hypersurface of $\mathbb{E}_1^{n+1}$, with at most two distinct principal curvatures and some additional conditions, is isoparametric.

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Keywords and Phrases: L_k-biconservative, Timelike hypersurface, Minkowski space, Isoparametric hypersurface.

1 Introduction

The study of biconservative maps, with conservative stress-energy with respect to the bienergy functional, is a natural extension of the theory

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of conservative maps. The matter has been motivated by their several physical and geometric applications. For instance, according to the role of biharmonic surfaces in elastics and fluid mechanics ([1, 13]) and also in computational geometry, some researchers are interested to generalize the subject of biconservative hypersurfaces. In [9], I. Dimitrić has generalized the subject to the submanifolds of higher dimensional Euclidean spaces which belong to one of several families containing regular curves, submanifolds with constant mean curvatures, hypersurfaces with at most two distinct principal curvatures, pseudo-umbilical submanifolds of dimension $n \neq 4$ and finite type submanifolds. The subject of biconservative hypersurfaces will be complicated when we consider them in the Minkowski space. The biconservativity condition plays a main role in the classification of biharmonic hypersurfaces. Several examples of biharmonic spacelike surfaces in $\mathbb{E}^4_s$ is presented in [7], which are failed to be minimal. However, biharmonicity implies minimality in some special cases. In [8], Chen and Munteanu gave a classification of biharmonic timelike surfaces in $\mathbb{E}^4_s$ with constant nonzero mean curvature and flat normal connection. In [5], it is proved that any biharmonic timelike hypersurface in $\mathbb{E}^4_1$ is minimal. On the other hand, the family of finite type submanifolds was interested by many researchers (see Chen's book [6]). In [10], Kashani has introduced the notion of L_k-finite type hypersurfaces in the Euclidean spaces, where, L_k is the linearized operator of the first variation of the $(k+1)$th mean curvature of a hypersurface, defined by $L_k(f) = tr(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, and P_k denotes the k-th Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. Note that, the L_k-operator is a natural generalization of the Laplace operator $L_0 = \Delta$. Recently, many people ([2, 3, 11, 14, 16, 17]) have used the L_k-operators to study some hypersurfaces of the Riemannian or Lorentzian space forms. Therefore, it is natural to advance Chen's conjecture for hypersurfaces of the Lorentz-Minkowski spacetime, replacing Δ by L_k (see, for instance, [2, 17]). This operator is defined by $L_k(f) = tr(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_k denotes the kth Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f. It is interesting to generalize the definition of biconservative hypersurface by replacing Δ by

L_k. In this paper, we show that every L_k-biconservative hypersurfaces in the Minkowski space $\mathbb{E}_1^{n+1}$, with constant ordinary mean curvature and at most two distinct real principal curvatures, is isoparametric. On the other hand, Martin A. Magid (in [12]) has proved that a timelike isoparametric hypersurface in $\mathbb{E}_1^{n+1}$, whose principal curvatures are all real numbers, has at most one non-zero principal curvature.

The organization of paper is as follow. In section 2, we remember some preliminary concepts and notations and in the rest of section we present some examples of L_k-biconservative Lorentzian hypersurfaces in $\mathbb{E}_1^{n+1}$. Section 3 is dedicated to L_k-biconservative Lorentzian hypersurfaces of $\mathbb{E}_1^{n+1}$. First, in theorems 3.2, 3.3 and 3.4 we discuss on L_k-biconservative Lorentzian hypersurfaces of $\mathbb{E}_1^{n+1}$ with diagonalizable shape operator. Other cases that the shape operator of hypersurface is non-diagonalizable will be seen in theorems 3.5, 3.6 and 3.7.

2 Preliminaries

In this section, we recall preliminaries from [2, 11, 12, 15]. The Lorentz-Minkowski space $\mathbb{E}_1^m$ is the m-dimensional vector space $\mathbb{R}^m$ endowed with the Lorentz scalar product $< x, y >:= -x_1y_1 + \Sigma_{i=2}^m x_iy_i$, for $x, y \in \mathbb{R}^m$. In $\mathbb{E}_1^{n+1}$, any n-dimensional submanifold with induced metric of index p is called a *spacelike hypersurface* when $p = 0$ and a *timelike hypersurface* when $p = 1$. For a hypersurface $\mathbf{x} : M_p^n \to \mathbb{E}_1^{n+1}$, the symbols ∇ and ∇^0 denote the Levi-Civita connections of M_p^n and $\mathbb{E}_1^{n+1}$, respectively, and the Weingarten formula is $\nabla_X^0 Y = \nabla_X Y + < SX, Y > N$, for every $X, Y \in \chi(M)$, where, $\mathbf{N}$ is a (locally) unit normal vector field on M and S is the shape operator of M relative to $\mathbf{N}$.

Definition 2.1. ([12]) (i) For a Lorentzian vector space V_1^n, a basis $\mathcal{B} := \{e_1, ..., e_n\}$ is said to be *orthonormal* if it satisfies $< e_i, e_j >= \epsilon_i\delta_i^j$ for $i, j = 1, ..., n$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, ..., n$. As usual, δ_i^j stands for the Kronecker function.

(ii) A basis $\mathcal{B} := \{e_1, ..., e_n\}$ for V_1^n is called *pseudo-orthonormal* if it satisfies $< e_1, e_1 >=< e_2, e_2 >= 0$, $< e_1, e_2 >= -1$ and $< e_i, e_j >= \delta_i^j$, for $i = 1, ..., n$ and $j = 3, ..., n$.

Remark 2.2. For any pseudo-orthonormal basis $\mathcal{B} := \{e_1, ..., e_n\}$, taking $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$, we obtain an orthonormal basis denoted by $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, ..., e_n\}$.

From [12, 18], it is well-known that any self-adjoint linear operator $T : V_1^n \to V_1^n$ (i.e. $< Tv, w >=< v, Tw >$ for every $v, w \in V_1^n$) has four possible matrix forms named I, II, III and IV with respect to suitable bases of V_1^n. Precisely, in cases I and IV the considered basis is orthonormal and in cases II and III the basis is pseudo-orthonormal. In three first cases the eigenvalues are real, while in case IV there exist two complex eigenvalues $\kappa \pm i\lambda$. So, denoting the matrix form of T in cases I and IV (where the basis is orthonormal) with B_1 and B_2, respectively, we have

$$B_1 = diag[\lambda_1, ..., \lambda_n], \qquad B_4 = \begin{pmatrix} \kappa & \lambda & & & \\ -\lambda & \kappa & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{pmatrix}, \ (\lambda \neq 0)$$

Also, in cases II and III (where, the chosen basis is pseudo - orthonormal), we denote the matrix form of T with B_2 and B_3, respectively, as follow.

$$B_2 = \begin{pmatrix} \kappa & 0 & & & \\ 1 & \kappa & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \kappa & 0 & 0 & & & \\ 0 & \kappa & 1 & & & \\ -1 & 0 & \kappa & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-3} \end{pmatrix}.$$

Remark 2.3. In two cases II and III (where, the chosen basis is pseudo-orthonormal and the matrix form of T is denoted by B_2 and B_3, respectively), we introduce another representation of T by changing the pseudo-orthonormal basis of V_1^n to an orthonormal one. pseudo-orthonormal basis to an orthonormal one, by transformation $\mathcal{B} \longrightarrow \tilde{\mathcal{B}}$ as Remark 2.2. Therefore, we obtain new matrix forms $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) for T as follow.

$$\tilde{B_2} = \begin{pmatrix} \kappa + \frac{1}{2} & \frac{1}{2} & & & \\ -\frac{1}{2} & \kappa - \frac{1}{2} & & & \\ & & \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_{n-2} \end{pmatrix},$$

$$\tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} & & & \\ 0 & \kappa & -\sqrt{2}/2 & & & \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{n-3} \end{pmatrix}.$$

Also, to unify the notations we put $\tilde{B_1} := B_1$ and $\tilde{B_4} := B_4$.

Now, Let $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ be an isometric immersion of a connected Lorentzian hypersurface into (n+1)-dimensional Lorentz-Minkowski space with a chosen spacelike unit normal vector field $\mathbf{N}$ and the shape operator S. At each $p \in M$, the operator S has (locally) matrix of the form $\tilde{B_i}$ ($1 \leq i \leq 4$).

Notation: According to four possible matrix forms of the shaper operator S, at each point $p \in M$, we define the principal curvatures κ_i's of M, as follow. When $S_p = \tilde{B_1}$, we put $\kappa_i(p) = \lambda_i$, for $i = 1, ..., n$. In the second and third cases where $S_p = \tilde{B_l}$ (for $l = 2, 3$), we take $\kappa_i(p) := \kappa$ for $i = 1, ..., l-1$, and $\kappa_i(p) = \lambda_{i-l+1}$ for $i = l, ..., n$. Finally, in the case $S_p = \tilde{B_4}$, we put $\kappa_1(p) = \kappa + \lambda\mathbf{i}$, $\kappa_2(p) = \kappa - \lambda\mathbf{i}$, and $\kappa_i(p) = \lambda_{i-2}$, for $i = 3, ..., n$.

The characteristic polynomial of S_p is of the form

$$Q_p(t) = \prod_{i=1}^{n}(t - \kappa_i(p)) = \sum_{j=0}^{n}(-1)^j s_j(p) t^{n-j},$$

where, $s_j(p) = \sum_{1\leq i_1 < ... < i_j \leq n} \kappa_{i_1}(p)...\kappa_{i_j}(p)$. For $j = 1, ..., n$, the *j-th mean curvature* H_j of M is defined by $H_j = \frac{1}{\binom{n}{j}} s_j$. When H_{j+1} is identically null, M_1^n is said to be *j-minimal*.

Definition 2.4. (i) A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$, with diagonalizable shape operator, is said to be *isoparametric* if all of it's principal curvatures are constant on M_1^n.
(ii) A Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial of it's shape operator is constant on M_1^n.

Remark 2.5. Here we remember Theorem 4.10 from [12], which assures us that there is no isoparametric Lorentzian hypersurface of $\mathbb{E}_1^{n+1}$ with complex principal curvatures.

The well-known Newton transformations on the hypersurface, $P_j : \chi(M) \to \chi(M)$, is defined by

$$P_0 = I, \quad P_j = s_j I - S \circ P_{j-1}, \qquad (j = 1, ..., n).$$

Using its explicit formula, $P_j = \sum_{i=0}^{j} (-1)^i s_{j-i} S^i$ (where $s_0 = 1$ and $S^0 = I$ is the identity map), it can be seen that, P_j is self-adjoint and commutative with S (see [2, 17]).

Now, we define the general notation

$$\mu_{j_1,j_2,...,j_t;k} := \sum_{i_1<...<i_k,\ i_j \notin \{j_1,j_2,...,j_t\}} \kappa_{i_1} ... \kappa_{i_k},$$

where the positive integers j_l's are mutually distinct, $1 \leq k < n$ and $t \leq n - k$. Specially, we use the formula

$$\mu_{j;k} = \sum_{l=0}^{k} (-1)^l \binom{n}{k-l} H_{k-l} \kappa_j^l. \qquad (1 \leq j \leq n,\ 1 \leq k < n)$$

Corresponding to the four possible forms $\tilde{B}_i$ (for $1 \leq i \leq 4$) of S, the Newton transformation P_j has different representations. In the case I, where $S_p = \tilde{B}_1$, we have $P_j(p) = diag[\mu_{1;j}(p), ..., \mu_{n;j}(p)]$, for $j = 1, ..., n-1$.

When $S_p = B_2$ (in the case II), we have

$$P_j(p) = \begin{pmatrix} \mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1} & -\frac{1}{2}\mu_{1,2;j-1} & & & \\ \frac{1}{2}\mu_{1,2;j-1} & \mu_{1,2;j} + (\kappa + \frac{1}{2})\mu_{1,2;j-1} & & & \\ & & \mu_{3;j}(p) & & \\ & & & \ddots & \\ & & & & \mu_{n;j}(p) \end{pmatrix}$$

and for $j = 1, ..., n-1$,

$$s_j = \mu_{1,2;j} + 2\kappa\mu_{1,2;j-1} + \kappa^2\mu_{1,2;j-2}.$$

In the case III, we have $S_p = B_3$, and

$$P_j(p) = \begin{pmatrix} \Lambda & & & \\ & \mu_{4;j}(p) & & \\ & & \ddots & \\ & & & \mu_{n;j}(p) \end{pmatrix}$$

where

$$\Lambda = \begin{pmatrix} u_j + 2\kappa u_{j-1} + (\kappa^2 - \frac{1}{2})u_{j-2} & -\frac{1}{2}u_{j-2} & -\frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) \\ \frac{1}{2}u_{j-2} & u_j + 2\kappa u_{j-1} + (\kappa^2 + \frac{1}{2})u_{j-2} & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) \\ \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & \frac{\sqrt{2}}{2}(u_{j-1} + \kappa u_{j-2}) & u_j + 2\kappa u_{j-1} + \kappa^2 u_{j-2}, \end{pmatrix}$$

and

$$s_j = u_j + 3\kappa u_{j-1} + 3\kappa^2 u_{j-2} + \kappa^3 u_{j-3}$$

for $j = 1, ..., n-1$, where $u_l = \mu_{1,2,3;l}$ for every $l \in \{1, ..., n-3\}$. In the case IV, $S_p = B_4$,

$$P_j(p) = \begin{pmatrix} \kappa\mu_{1,2;j-1} + \mu_{1,2;j} & -\lambda\mu_{1,2;j-1} & & & \\ \lambda\mu_{1,2;j-1} & \kappa\mu_{1,2;j-1} + \mu_{1,2;j} & & & \\ & & \mu_{3;j}(p) & & \\ & & & \ddots & \\ & & & & \mu_{n;j}(p) \end{pmatrix},$$

and

$$s_j = \mu_{1,2;j} + 2\kappa\mu_{1,2;j-1} + (\kappa^2 + \lambda^2)\mu_{1,2;j-2},$$

for $j = 1, ..., n-1$.

Fortunately, in all cases we have the following important identities for $j = 1, ..., n-1$, similar to those in [2, 3, 17].

$$s_{j+1} = \kappa_i\mu_{i;j} + \mu_{i;j+1}, \qquad (1 \leq i \leq n)$$

$$\mu_{i;j+1} = \kappa_l\mu_{i,l;j} + \mu_{i,l;j+1}, \quad (1 \leq i, l \leq n, \ i \neq l)$$

$$tr(P_j) = (n-j)s_j = c_j H_j,$$

$$tr(P_j \circ S) = (n-(n-j-1))s_{j+1} = c_j H_{j+1},$$

$$tr(P_j \circ S^2) = \binom{n}{j+1}[nH_1H_{j+1} - (n-j-1)H_{j+2}],$$

where $c_j = (n-j)\binom{n}{j} = (j+1)\binom{n}{j+1}$.

The *linearized operator* of the $(j+1)$th mean curvature of M, $L_j : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_j(f) := tr(P_j \circ \nabla^2 f)$, where, $< \nabla^2 f(X), Y > = < \nabla_X \nabla f, Y >$ for every $X, Y \in \chi(M)$.

For a Lorentzian hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$, with a chosen (locally) unit normal vector field $\mathbf{N}$, for an arbitrary vector $\mathbf{a} \in \mathbb{E}_1^{n+1}$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$ where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae from [2, 17, 4].

$$(i) \ \ \nabla < \mathbf{x}, \mathbf{a} > = \mathbf{a}^T, \qquad (ii) \ \ \nabla < \mathbf{N}, \mathbf{a} > = -S\mathbf{a}^T. \tag{1}$$

and, then

$$
\begin{aligned}
&(i)\quad L_k\mathbf{x} = c_k H_{k+1}\mathbf{N}, \qquad (k=1,...,n-1)\\
&(ii)\quad L_k\mathbf{N}\\
&= -\binom{n}{k+1}\nabla(H_{k+1}) - \binom{n}{k+1}[nH_1H_{k+1} - (n-k-1)H_{k+2}]\mathbf{N},
\end{aligned}
\tag{2}
$$

and

$$
\begin{aligned}
L_k^2\mathbf{x} = &-c_k[3\binom{n}{k+1}H_{k+1}\nabla H_{k+1} - 2P_{k+1}\nabla H_{k+1}]\\
&- c_k[n\binom{n}{k+1}H_1H_{k+1}^2 + c_{k+1}H_{k+1}H_{k+2} - L_kH_{k+1}]\mathbf{N}.
\end{aligned}
$$

Assume that a hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ satisfies the condition $L_k^2\mathbf{x} = 0$, For an integer k (where, $0 \leq k < n$), then it is said to be L_k-*biharmonic*. By (2), one clearly obtain a condition equivalent to L_k-biharmonicity, as $L_k(H_{k+1}\mathbf{N}) = 0$. Clearly, k-minimal immersions are L_k-biharmonic. By elementary calculations (as in [4]), one obtains equivalent conditions for M_1^n to be L_k-biharmonic in $\mathbb{E}_1^{n+1}$, namely

$$
\begin{aligned}
&(i)\quad L_kH_{k+1} = \binom{n}{k+1}H_{k+1}(nH_1H_{k+1} - (n-k-1)H_{k+2}),\\
&(ii)\quad P_{k+1}\nabla H_{k+1} = \frac{3}{2}\binom{n}{k+1}H_{k+1}\nabla H_{k+1}.
\end{aligned}
\tag{3}
$$

A timelike hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ is said to be L_k-*bicoservative* if its $(k+1)$th mean curvature satisfies the condition (3)(ii).

The structure equations of $\mathbb{E}_1^{n+1}$ are given by

$$
d\omega_i = \sum_{j=1}^{n+1}\omega_{i,j}\wedge\omega_j, \qquad \omega_{i,j} + \omega_{j,i} = 0, \tag{4}
$$

$$
d\omega_{i,j} = \sum_{l=1}^{n+1}\omega_{i,l}\wedge\omega_{l,j}. \tag{5}
$$

When restricted to M, we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_{i=1}^{n} \omega_{n+1,i} \wedge \omega_i. \tag{6}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{n+1,i} = \sum_{j=1}^{n} h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{7}$$

This gives the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$. The mean curvature H is defined by $H = \frac{1}{n}\sum_{i} h_{ii}$. From (4) - (7) we obtain the structure equations of M.

$$d\omega_i = \sum_{j=1}^{n} \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{n} R_{ijkl}\omega_k \wedge \omega_l,$$

and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}.$$

Thus, by exterior differentiation of (7), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}.$$

Now we recall the definition of an L_k-finite type hypersurface from [10], which is a basic notion in this paper.

Definition 2.6. An isometrically immersed hypersurface $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ is said to be of L_k-finite type if $\mathbf{x}$ has a finite decomposition $\mathbf{x} = \sum_{i=0}^m \mathbf{x}_i$, for some positive integer m, satisfying the condition $L_k\mathbf{x}_i = \tau_i\mathbf{x}_i$, where , $\tau_i \in \mathbb{R}$ and $\mathbf{x}_i : M^n \to \mathbb{E}_1^{n+1}$ is smooth maps, for $i = 1, 2, \cdots, m$, and $\mathbf{x}_0$ is constant. If all τ_i's are mutually different, M^n is said to be of L_k-m-type. An L_k-m-type hypersurface is said to be null if for some i $(1 \leq i \leq m)$, $\tau_i = 0$.

Now, we see two families of examples of L_k-biconservative Lorentzian hypersurfaces in $\mathbb{E}_1^{n+1}$, some of them are not L_k-biharmonic.

Example 2.7. Assume that $\mathcal{M}_1(r)$ be the product $\mathbb{S}_1^m(r) \times \mathbb{E}^{n-m} \subset \mathbb{E}_1^{n+1}$ where $r > 0$ is a real number and $m = 2, 3, \cdots, n-1$. It has another representation as

$$\mathcal{M}_1(r) = \{(y_1, ..., y_{n+1}) \in \mathbb{R}_1^{n+1} | -y_1^2 + y_2^2 + ... + y_{m+1}^2 = r^2\},$$

having the spacelike normal vector field $\mathbf{N}(y) = -\frac{1}{r}(y_1, ..., y_{m+1}, 0, ..., 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = ... = \kappa_m = \frac{1}{r}$, $\kappa_{m+1} = ... = \kappa_n = 0$, and the constant higher order mean curvatures $H_k = \frac{m!(n-k)!}{n!(m-k)!r^k}$ for $k \leq m$ and $H_k = 0$ for $k > m$. Also, one can see that for $k > m$ we have $L_k^2 x = 0$ and otherwise $L_k^2 x \neq 0$.

Example 2.8. Let $\mathcal{M}_2(r)$ be the product $\mathbb{E}_1^m \times \mathbb{S}^{n-m}(r) \subset \mathbb{E}_1^{n+1}$ where $r > 0$ is a real number and $m = 2, 3, \cdots, n-1$. It can be represented as

$$\mathcal{M}_2(r) = \{(y_1, ..., y_{n+1}) \in \mathbb{R}_1^{n+1} | y_{m+1}^2 + ... + y_{n+1}^2 = r^2\},$$

with the Gauss map $\mathbf{N}(y) = -\frac{1}{r}(0, ...0, y_{m+1}, ..., y_{n+1})$. It has two distinct principal curvatures $\kappa_1 = ... = \kappa_m = 0$, $\kappa_{m+1} = ... = \kappa_n = \frac{1}{r}$, and the constant higher order mean curvatures $H_k = \frac{(n-m)!(n-k)!}{n!(n-m-k)!r^k}$ for $k \leq n-m$, and $H_k = 0$ for $k > n-m$. So, Also, one can see that $L_k^2 x \neq 0$ for $k \leq n-m$, we have $L_k^2 x = 0$ for $k > n-m$.

3 Results on timelike hypersurfaces

From now on, Let $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ be an isometrically immersion from a connected timelike hypersurface M_1^n into the Minkowski space $\mathbb{E}_1^{n+1}$, with the Gauss map $\mathbf{N}$. We have six theorems on the L_k-biconservative connected orientable timelike hypersurface in $\mathbb{E}_1^{n+1}$ with constant ordinary mean curvature. Theorems 3.2, 3.3 and 3.4 are appropriated to the case that the shape operator on hypersurface is diagonalizable. Theorems 3.5, 3.6 and 3.7 are related to the cases that the shape operator on hypersurface is of type II, III and IV, respectively. First we see a common lemma.

Lemma 3.1. *Let M_1^n be a connected orientable timelike hypersurface in the Minkowski space $\mathbb{E}_1^{n+1}$ with non-zero $(k+1)$th mean curvature, which satisfies the L_k-biconservativity condition for some integer $k \in \{0, 1, ..., n-1\}$. Let $\{e_1, ..., e_n\}$ be a local orthonormal tangent frame on M_1^n. Then, we have*

$$\sum_{i=1}^{n} \epsilon_i < \nabla H_{k+1}, e_i >< P_{k+1} e_i, e_j >= \frac{3}{2}\binom{n}{k+1} H_{k+1} < \nabla H_{k+1}, e_j >,$$

for $j = 1, 2, ..., n$, where $\epsilon_1 := -1$ and $\epsilon_i := 1$ for $i \geq 2$.

Proof. Using the polar decomposition of the gradient vector field ∇H_{k+1} in terms of the orthonormal basis $\{e_1, ..., e_n\}$, and the linearity of P_{k+1}, we have $P_{k+1}\nabla H_{k+1} = \sum_{i=1}^{n} \epsilon_i < \nabla H_{k+1}, e_i > P_{k+1} e_i$, which, by comparing with the equation (3)(ii), gives the result. □

3.1 Timelike hypersurfaces with diagonalizable shape operator

First, we remember the Theorem 2.2 from [12] on a Lorentzian isoparametric hypersurface of $\mathbb{E}_1^{n+1}$ with diagonalizable shape operator and exactly l distinct constant principal curvatures $\lambda_1, \cdots, \lambda_l$ (respectively) of multiplicities $m_1, \cdots, m_l$, which says that on such a hypersurface we have equalities

$$\sum_{j \in \{1, \cdots, l\} - \{i\}} \frac{m_j \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0$$

for $i = 1, \cdots, l$. An easy consequence of this fact is that, if Lorentzian hypersurface of $\mathbb{E}_1^{n+1}$ with diagonalizable shape operator has exactly two distinct constant principal curvatures λ_1 and λ_2, then we have $\lambda_1 \lambda_2 = 0$ which gives $\lambda_1 = 0$ or $\lambda_2 = 0$.

In Theorem 3.2, the only principal curvature is not assumed to be constant.

Theorem 3.2. *Let $x : M_1^n$ be a connected orientable timelike hypersurface in the Minkowski space $\mathbb{E}_1^{n+1}$ with diagonalizable shape operator, which has one principal curvature of multiplicity n. If M_1^n satisfies the L_k-biconservativity condition (3)(ii) for an integer $k \in \{1, ..., n-1\}$, then it has to be isoparametric.*

Proof. Let $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ be the position vector field of M_1^n in $\mathbb{E}_1^{n+1}$ which satisfies assumed conditions. Defining the open subset $\mathcal{U}$ of M as $\mathcal{U} := \{p \in M_1^n : \nabla H_{k+1}(p) \neq 0\}$, we prove that $\mathcal{U}$ is empty. Assuming $\mathcal{U} \neq \varnothing$, we consider $\{e_1, \cdots, e_n\}$ as a (local) orthonormal frame of principal directions of S on $\mathcal{U}$ such that for $i = 1, \cdots, n$ we have $Se_i = \lambda e_i$ and

$$\mu_{i,k+1} = \binom{n-1}{k+1}\lambda^{k+1}, \quad H_{k+1} = \lambda^{k+1}. \tag{8}$$

By condition (3)(ii), we have $P_{k+1}(\nabla H_{k+1}) = \frac{3}{2}\binom{n}{k+1}H_{k+1}\nabla H_{k+1}$, which, using the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{n} \epsilon_i < \nabla H_{k+1}, e_i > e_i$, gives $\epsilon_i < \nabla H_{k+1}, e_i > (\mu_{i,k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1}) = 0$ on $\mathcal{U}$ for $i = 1, \cdots, n$. Hence, if for some i we assume $< \nabla H_{k+1}, e_i > \neq 0$ on $\mathcal{U}$, then we get

$$\mu_{i,k+1} = \frac{3}{2}\binom{n}{k+1}H_{k+1},$$

which, using equalities (8), gives $\lambda^{k+1} = 0$ and then $H_{k+1} = 0$ on $\mathcal{U}$, which is a contradiction. Hence $\mathcal{U}$ is empty and H_{k+1} is constant on M_1^n and then, λ is constant and M_1^n is isoparametric. □

Theorem 3.3. *Let M_1^n (for an integer number $n \geq 3$) be a timelike hypersurface of $\mathbb{E}_1^{n+1}$ with diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions η and λ of multiplicities 1 and $n-1$, respectively. If M_1^n satisfies the L_k-biconservativity condition (3)(ii) for an integer $k \in \{1, ..., n-1\}$, then it has to be isoparametric and at least one of η and λ is identically zero.*

Proof. Let $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ be the position vector field of M_1^n in $\mathbb{E}_1^{n+1}$ which satisfies assumed conditions. Taking the open subset $\mathcal{V}$ of

M_1^n as $\mathcal{V} := \{p \in M_1^n : \nabla H_{k+1}^2(p) \neq 0\}$, we prove that $\mathcal{V}$ is empty. Assuming $\mathcal{V} \neq \varnothing$, we consider $\{e_1, \cdots, e_n\}$ as a local orthonormal frame of principal directions of A on $\mathcal{V}$ such that $Se_i = \lambda e_i$ for $i = 1, \cdots, n-1$ and $Ae_n = \eta e_n$. Therefore, we obtain

$$\begin{aligned}
&\mu_{1,k+1} = \cdots = \mu_{n-1,k+1} = \binom{n-2}{k+1}\lambda^{k+1} + \binom{n-2}{k}\lambda^k\eta,\\
&\mu_{n,k+1} = \binom{n-1}{k+1}\lambda^{k+1},\\
&nH_1 = (n-1)\lambda + \eta, \quad n(n-1)H_2 = (n-1)(n-2)\lambda^2 + 2(n-1)\lambda\eta,\\
&\binom{n}{k+1}H_{k+1} = \binom{n-1}{k+1}\lambda^{k+1} + \binom{n-1}{k}\lambda^k\eta.
\end{aligned} \tag{9}$$

Using the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^n \epsilon_i < \nabla H_{k+1}, e_i > e_i$, from equality (3)($ii$) we have $\epsilon_i < \nabla H_{k+1}, e_i > (\mu_{i,k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1}) = 0$ on $\mathcal{V}$ for $i = 1, \cdots, n$. Since, by definition of the subset $\mathcal{V}$, we have $< \nabla H_{k+1}, e_i > \neq 0$ on $\mathcal{V}$ for some i, then we get

$$\mu_{i,k+1} = \frac{3}{2}\binom{n}{k+1}H_{k+1}, \tag{10}$$

for some i, which gives one of the following states:

State 1. $< \nabla H_{k+1}, e_i > \neq 0$, for some $i \in \{1, \cdots, n-1\}$. Using formulae (9), from equality (10) we obtain $(n+2k+1)(n-k-1)\lambda^{k+1} + (n+2k-l)(k+1)\lambda^k\eta = 0$. If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\eta = -\frac{(n+2k+1)(n-k-1)}{(n+2k-1)(k+1)}\lambda$, which, using $nH_1 = (n-1)\lambda + \eta$, gives $\lambda = \frac{n(k+1)(n+2k-1)}{nk(n+2k-1)-2(n-k-1)}H_1$ and then H_{k+1} is constant on $\mathcal{V}$. Therefore, we obtain a contradiction which implies that $\mathcal{V} = \emptyset$.

State 2. $< \nabla H_{k+1}, e_i >= 0$, for all $i \in \{1, \cdots, n-1\}$ and $< \nabla H_{k+1}, e_n > \neq 0$ and then,

$$\mu_{n,k+1} = \frac{3}{2}\binom{n}{k+1}H_{k+1}.$$

Similar to State 1, by equalities (9), we obtain $\lambda = 0$ or $\eta = -\frac{n-k-1}{3(k+1)}\lambda$. If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\lambda = \frac{3n(k+1)}{(n-1)(3k+2)+k}H_1$ and then H_{k+1} is constant on $\mathcal{V}$. Therefore, we obtain a contradiction which implies that $\mathcal{V} = \emptyset$.

Therefore, H_{k+1} is constant on M_1^n. Since H_1 is constant on M_1^n, we obtain that, λ and η are constant on M_1^n. Hence, M_1^n is isoparametric. So, by Theorem 2.2 in [12], we get $\lambda\eta = 0$. □

Theorem 3.4. *Let M_1^n be a timelike hypersurface of $\mathbb{E}_1^{n+1}$ with diagonalizable shape operator, constant ordinary mean curvature and exactly two distinct principal curvature functions η and λ of multiplicities t and $n-t$, respectively, where $2 \le t \le n-2$. If M_1^n satisfies the L_k-biconservativity condition (3)(ii) for an integer $k \in \{1, ..., n-1\}$, then it has to be isoparametric. Furthermore, when $n \neq 2t$, we get $\lambda\eta = 0$.*

Proof. Let $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ be the position vector field of M_1^n in $\mathbb{E}_1^{n+1}$ which satisfies assumed conditions. Defining the open subset $\mathcal{V}$ of M_1^n as $\mathcal{V} := \{p \in M_1^n : \nabla H_{n+1}^2(p) \neq 0\}$, we prove that $\mathcal{V}$ is empty. Assuming $\mathcal{V} \neq \varnothing$, we consider $\{e_1, \cdots, e_n\}$ as a local orthonormal frame of principal directions of S on $\mathcal{V}$ such that $Se_i = \lambda e_i$ for $i = 1, \cdots, n-t$ and $Se_i = \eta e_i$ for $i = n-t+1, \cdots, n$. Therefore, we obtain

$$\begin{aligned}
&(i) \quad \mu_{1,k+1} = \cdots = \mu_{n-t,k+1} = \sum_{s=0}^{k+1} \binom{n-t-1}{s} \binom{t}{k+1-s} \lambda^s \mu^{k+1-s},\\
&(ii) \ \ \mu_{n-t+1,2} = \cdots = \mu_{n,2} = \sum_{s=0}^{k+1} \binom{n-t}{s} \binom{t-1}{k+1-s} \lambda^s \mu^{k+1-s},\\
&(iii) \ nH_1 = (n-k)\lambda + k\eta,\\
&(iv) \ \binom{n}{k+1} H_{k+1} = \sum_{s=0}^{k+1} \binom{n-t}{s} \binom{t}{k+1-s} \lambda^s \mu^{k+1-s}.
\end{aligned} \tag{11}$$

Using the definition of P_{k+1} and equation (3)(ii), we obtain

$$P_{k+1}(\nabla H_{k+1}) = \frac{3}{2} \binom{n}{k+1} H_{k+1} \nabla H_{k+1}$$

on $\mathcal{U}$. Therefore, applying $\nabla H_{k+1} = \sum_{i=1}^n \epsilon_i < \nabla H_{k+1}, e_i > e_i$, we get

$$< \nabla H_{k+1}, e_i > (\mu_{i,k+1} - \frac{3}{2} \binom{n}{k+1} H_{k+1}) = 0,$$

and then, $<\nabla H_{k+1}, e_i>=0$ or

$$\mu_{1,k+1} = \cdots = \mu_{q,k+1} = \frac{3}{2}\binom{n}{k+1} H_{k+1}. \tag{12}$$

on $\mathcal{U}$ for every $i = 1, \ldots, n$. This gives one or both of the following states.

State 1. $<\nabla H_{k+1}, e_i>\neq 0$, for some $i \in \{1, \cdots, n-t\}$. Then, by (12) and (11)(i), we obtain

$$\sum_{s=0}^{k+1}[\binom{n-t-1}{s}\binom{t}{k+1-s} - \frac{3}{2}\binom{n-t}{s}\binom{t}{k+1-s}]\lambda^s\mu^{k+1-s} = 0,$$

which, using (11)(iii), gives a polynomial equation in terms of λ. This implies that λ and the η and H_{k+1} are constant on $\mathcal{U}$. Therefore, H_{k+1} is constant on M_1^n.

State 2. $<\nabla H_{k+1}, e_i>=0$, for all $i \in \{1, \cdots, n-t\}$ and $<\nabla H_{k+1}, e_j>\neq 0$ for some $j \in \{n-t+1, \cdots, n\}$. By (12) and (11)($ii$), we obtain

$$\sum_{s=0}^{k+1}[\binom{n-t}{s}\binom{t-1}{k+1-s} - \frac{3}{2}\binom{n-t}{s}\binom{t}{k+1-s}]\lambda^s\mu^{k+1-s} = 0,$$

which, using (11)(iii), gives a polynomial equation in terms of λ. This implies that λ and the η and H_{k+1} are constant on $\mathcal{U}$. Therefore, H_{k+1} is constant on M_1^n.

Since H_1 is also constant on M_1^n, we obtain that, λ and η are constant on M_1^n and M_1^n is isoparametric. So, in the case $n \neq 2t$, by Theorem 2.2 in [12], we get $\lambda\eta = 0$. $\square$

3.2 Hypersurfaces with non-diagonalizable shape operator

This subsection is appropriated to cases that the Lorentzian hypersurfaces of $\mathbb{E}^{n+1}$ have shape operator of type II, III or IV. First, on the type II, we will use Theorem 2.4 from [12], which says that each isoparametric timelike hypersurface M_1^n of $\mathbb{E}_1^{n+1}$ with shape operator of type II (with minimal polynomial as $m(x) = (x-\lambda_1)^2(x-\lambda_2)\cdots(x-\lambda_l)$)

and exactly l distinct constant principal curvatures $\lambda_1, \cdots, \lambda_l$ (respectively) of multiplicities $m_1, \cdots, m_l$, satisfies $\sum_{j\in\{1,\cdots,l\}-\{i\}} \frac{m_j\lambda_i\lambda_j}{\lambda_i-\lambda_j} = 0$ for $i = 1, \cdots, l$. As a consequence of this fact, if a Lorentzian hypersurface of $\mathbb{E}^{n+1}_1$ with shape operator of type II has exactly two distinct constant principal curvatures λ_1 and λ_2, then we have $\lambda_1\lambda_2 = 0$ which gives $\lambda_1 = 0$ or $\lambda_2 = 0$.

Theorem 3.5. *Let M^n_1 be a timelike hypersurface of $\mathbb{E}^{n+1}_1$ with shape operator of type II, constant ordinary mean curvature and exactly two distinct principal curvatures. If M^n_1 satisfies the L_k-biconservativity condition (3)(ii) for an integer $k \in \{1, ..., n-1\}$, then it has to be isoparametric and at least one of it's principal curvatures is identically zero.*

Proof. Assume that, an isometric immersion $\mathbf{x} : M^n_1 \to \mathbb{E}^{n+1}_1$ satisfies all conditions of the theorem. Taking the open subset $\mathcal{U} = \{p \in M^n_1 : \nabla H_{k+1}(p) \neq 0\}$, we show that $\mathcal{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_n\}$ on M^n_1, the shape operator S has the matrix form $\tilde{B}_2$, such that $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$ and $Se_i = \lambda e_i$ for $i = 3, \cdots, n$. Then, we have the following equalities.

$$
\begin{aligned}
&(i)\quad nH_1 = 2\kappa + (n-2)\lambda,\\
&(ii)\quad \binom{n}{k+1}H_{k+1} = \binom{n-2}{k+1}\lambda^{k+1} + 2\binom{n-2}{k}\kappa\lambda^k + \binom{n-2}{k-1}\kappa^2\lambda^{k-1}\\
&(iii)\quad P_{k+1}e_1 = [\binom{n-2}{k+1}\lambda^{k+1} + \binom{n-2}{k}(\kappa - \frac{1}{2})\lambda^k]e_1 + \frac{1}{2}\binom{n-2}{k}\lambda^k e_2,\\
&(iv)\quad P_{k+1}e_2 = -\frac{1}{2}\binom{n-2}{k}\lambda^k e_1 + [\binom{n-2}{k+1}\lambda^{k+1} + \binom{n-2}{k}(\kappa + \frac{1}{2})\lambda^k]e_2,\\
&(v)\quad P_{k+1}e_i = [\binom{n-3}{k-1}\kappa^2\lambda^{k-1} + 2\binom{n-3}{k}\kappa\lambda^k + \binom{n-3}{k+1}\lambda^{k+1}]e_i\\
&\qquad (i = 3, \cdots, n).
\end{aligned}
\tag{13}
$$

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{n} \epsilon_i e_i(H_2) e_i$, from condition (3)$(ii)$ we get

$$\begin{aligned} &(i) \quad (A-C)\epsilon_1 e_1(H_{k+1}) = C\epsilon_2 e_2(H_{k+1}),\\ &(ii) \quad (A+C)\epsilon_2 e_2(H_{k+1}) = -C\epsilon_1 e_1(H_{k+1}),\\ &(iii) \quad D\epsilon_i e_i(H_{k+1}) = 0, \quad (i = 3, \cdots, n), \end{aligned} \tag{14}$$

where $A := \binom{n-2}{k+1}\lambda^{k+1} + \binom{n-2}{k}\kappa\lambda^k - \frac{3}{2}\binom{n}{k+1}H_{k+1}$, $C := \frac{1}{2}\binom{n-2}{k}\lambda^k$ and

$$D = \binom{n-3}{k-1}\kappa^2\lambda^{k-1} + 2\binom{n-3}{k}\kappa\lambda^k + \binom{n-3}{k+1}\lambda^{k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1}.$$

Now, we prove the main claim.

Claim: $e_i(H_{k+1}) = 0$ for $i = 1, \cdots, n$.

If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (14)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$(i) \;\; A - C = Cu, \qquad (ii) \;\; (A+C)u = -C, \tag{15}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (15)$(i)$ in (15)(ii), we obtain $\lambda^k(1+u)^2 = 0$, then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then, from (15)(i) and (13)(i) we obtain that H_{k+1} is constant. Otherwise, we have $u = -1$, which gives $A = 0$, then we obtain

$$\binom{n-2}{k+1}\lambda^{k+1} + \binom{n-2}{k}\kappa\lambda^k - \frac{3}{2}\binom{n}{k+1}H_{k+1} = 0.$$

Since $nH_1 = 2\kappa + (n-2)\lambda$ is assumed to be constant on M, by substituting which in the last equality, we get a polynomial equation which means κ and then H_{k+1} is constant on M_1^n. So, we got a contradiction and therefore, the first part of the claim is proved.

If $e_2(H_2) \neq 0$, then by dividing both sides of equalities (14)(i, ii) by $\epsilon_2 e_2(H_2)$ we get

$$(i) \;\; (A-C)v = C, \qquad (ii) \;\; A + C = -Cv, \tag{16}$$

where $v := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$. By substituting the equation (16)(ii) in (16)(i), we obtain $\lambda^k(1+v)^2 = 0$. If $\lambda = 0$, then, from (13)(i) and (16)(ii), we obtain that H_{k+1} is constant. Otherwise, we have $v = -1$, which gives $A = 0$, then similar to the first part, we obtain that H_{k+1} is constant on M_1^n. So, we got a contradiction and therefore, the second part of the claim is proved.

Finally, each of assumptions $e_i(H_2) \neq 0$ for $i = 3, \cdots, n$, gives $D = 0$, which by simplification gives the polynomial equation

$$[\frac{n+2k}{2(n-k-1)}\lambda^2 + \frac{(k+1)(n+2k-2)}{(n-k-3)(n-k-2)}\kappa\lambda$$
$$+\frac{k(k+1)(n+2k-4)}{2(n-k-1)(n-k-2)(n-k-3)}\kappa^2]\lambda^{k-1} = 0.$$

Similar to two first cases, using formula $nH_1 = 2\kappa + (n-2)\lambda$, from the last equation we obtain a polynomial equation in terms of λ, which gives that H_{k+1} is constant on M. Since H_1 is also constant on M_1^n, we obtain that, λ and κ are constant on M_1^n and M_1^n is isoparametric. So, by Theorem 2.4 from [12], we get $\lambda\kappa = 0$. □

Now, for the type III, we recall Theorem 2.6 from [12] which says that each isoparametric timelike hypersurface M_1^n of $\mathbb{E}_1^{n+1}$ with shape operator of type III (with minimal polynomial as $m(x) = (x-\lambda_1)^3(x-\lambda_2)\cdots(x-\lambda_l)$) and exactly l distinct constant principal curvatures $\lambda_1, \cdots, \lambda_l$ (respectively) of multiplicities $m_1, \cdots, m_l$, satisfies the equalities $\sum_{j\in\{1,\cdots,l\}-\{i\}} \frac{m_j\lambda_i\lambda_j}{\lambda_i-\lambda_j} = 0$ for $i = 1, \cdots, l$. As a consequence of this fact, if an isoparametric Lorentzian hypersurface of $\mathbb{E}_1^{n+1}$ with shape operator of type III has exactly two distinct constant principal curvatures λ_1 and λ_2, then we have $\lambda_1\lambda_2 = 0$ which gives $\lambda_1 = 0$ or $\lambda_2 = 0$.

Theorem 3.6. *Let M_1^n be a timelike hypersurface of $\mathbb{E}_1^{n+1}$ with shape operator of type III, constant ordinary mean curvature and exactly two distinct principal curvatures. If M_1^n satisfies the L_k-biconservativity condition (3)(ii) for an integer $k \in \{1, ..., n-1\}$, then it is isoparametric and at least one of it's principal curvatures is identically zero.*

Proof. Assume that, an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ satisfies all conditions of the theorem. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_n\}$ on M_1^n, the shape operator S has the matrix form $\tilde{B}_3$, such that $Se_1 = \kappa e_1 - \frac{\sqrt{2}}{2}e_3$,

$Se_2 = \kappa e_2 - \frac{\sqrt{2}}{2}e_3$, $Se_3 = \frac{\sqrt{2}}{2}e_1 - \frac{\sqrt{2}}{2}e_2 + \kappa e_3$ and $Se_i = \lambda e_i$ for $i = 4, \cdots, n$. Then, we have

$$
\begin{aligned}
&(i)\quad nH_1 = 3\kappa + (n-3)\lambda,\\
&(ii)\quad \binom{n}{k+1}H_{k+1} = u_{k+1} + 3\kappa u_k + 3\kappa^2 u_{k-1} + \kappa^3 u_{k-2},\\
&(iii)\quad P_{k+1}e_1 = [u_{k+1} + 2\kappa u_k + (\kappa^2 - \frac{1}{2})u_{k-1}]e_1 + \frac{1}{2}u_{k-1}e_2\\
&\qquad\qquad + \frac{\sqrt{2}}{2}[u_k + \kappa u_{k-1}]e_3,\\
&(iv)\quad P_{k+1}e_2 = -\frac{1}{2}u_{k-1}e_1 + [u_{k+1} + 2\kappa u_k + (\kappa^2 + \frac{1}{2})u_{k-1}]e_2\\
&\qquad\qquad + \frac{\sqrt{2}}{2}[u_k + \kappa u_{k-1}]e_3,\\
&(v)\quad P_{k+1}e_3 = \frac{-\sqrt{2}}{2}[u_k + \kappa u_{k-1}]e_1 + \frac{\sqrt{2}}{2}[u_k + \kappa u_{k-1}]e_2\\
&\qquad\qquad + [u_{k+1} + 2\kappa u_k + \kappa^2 u_{k-1}]e_3,\\
&(vi)\quad P_{k+1}e_i = [\binom{n-4}{k-2}\kappa^3\lambda^{k-2} + 3\binom{n-4}{k-1}\kappa^2\lambda^{k-1}\\
&\qquad\qquad + 3\binom{n-4}{k}\kappa\lambda^k + \binom{n-4}{k+1}\lambda^{k+1}]e_i,
\end{aligned}
\tag{17}
$$

for $i = 4, \cdots, n$. Where, $u_l = \mu_{1,2,3;l}$ for every $l \in \{1, ..., n-3\}$.

Similar to proof of Theorem 3.5, we assume that H_{k+1} is non-constant and considering the open subset $\mathcal{U} = \{p \in M_1^n : \nabla H_{k+1}(p) \neq 0\}$, we prove that $\mathcal{U} = \emptyset$. Using polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{n} \epsilon_i e_i(H_{k+1})e_i$, from condition (3)(ii) we get the following system of conditions:

$$
\begin{aligned}
&(i)\quad [\tilde{A} - \tilde{C}]\epsilon_1 e_1(H_{k+1}) - \tilde{C}\epsilon_2 e_2(H_{k+1}) - \tilde{D}\epsilon_3 e_3(H_{k+1}) = 0\\
&(ii)\quad \tilde{C}\epsilon_1 e_1(H_{k+1}) + [\tilde{A} + \tilde{C}]\epsilon_2 e_2(H_{k+1}) + \tilde{D}\epsilon_3 e_3(H_{k+1}) = 0\\
&(iii)\ \tilde{D}(\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)) + \tilde{A}\epsilon_3 e_3(H_2) = 0,\\
&(iv)\quad [\kappa^3\mu_{1,2,3,i;k-2} + 3\kappa^2\mu_{1,2,3,i;k-1} + 3\kappa\mu_{1,2,3,i;k} + \mu_{1,2,3,i;k+1}\\
&\qquad - \frac{3}{2}\binom{n}{k+1}H_{k+1})]\epsilon_i e_i(H_{k+1}) = 0.\quad (i = 4, \cdots, n).
\end{aligned}
\tag{18}
$$

where $\tilde{A} := [u_{k+1} + 2\kappa u_k + \kappa^2 u_{k-1}] - \frac{3}{2}\binom{n}{k+1}H_{k+1}$, $\tilde{C} := \frac{1}{2}u_{k-1}$ and $\tilde{D} = \frac{\sqrt{2}}{2}[u_k + \kappa u_{k-1}]$.

Now, we prove that H_{k+1} is constant .

Claim: $e_i(H_{k+1}) = 0$ for $i = 1, \cdots, n$.

If $e_1(H_{k+1}) \neq 0$, then by dividing both sides of equalities (18)(i, ii, iii) by $\epsilon_1 e_1(H_{k+1})$, and using the identity (17)(ii) and notations $\nu_1 := \frac{\epsilon_2 e_2(H_{k+1})}{\epsilon_1 e_1(H_{k+1})}$ and $\nu_2 := \frac{\epsilon_3 e_3(H_{k+1})}{\epsilon_1 e_1(H_{k+1})}$, we get

$$\begin{aligned} &(i)\ \tilde{A} - \tilde{C} - \tilde{C}\nu_1 - \tilde{D}\nu_2 = 0,\\ &(ii)\ \tilde{C} + (\tilde{A} + \tilde{C})\nu_1 + \tilde{D}\nu_2 = 0,\\ &(iii)\ \tilde{D}(1 + \nu_1) + \tilde{A}\nu_2 = 0, \end{aligned} \tag{19}$$

From summation of equations (19)(i) and (19)(ii), we obtain $\tilde{A}(1+\nu_1) = 0$.

Assuming $\tilde{A} \neq 0$, from the last equality we get $\nu_1 = -1$ and then, by (19)(iii), we obtain $\nu_2 = 0$. From these results, by (19)(ii), we get $\tilde{A} = 0$. So, we get a contradiction, which implies that $\tilde{A} = 0$.

The equality $\tilde{A} = 0$ gives a polynomial equation in terms λ and κ. Since $nH_1 = 3\kappa + (n-3)\lambda$ is assumed to be constant, so we obtain a polynomial equation in terms λ, which implies that λ and then κ and H_{k+1} are constant on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_1(H_{k+1}) \equiv 0$ is proved.

If $e_2(H_2) \neq 0$, then by dividing both sides of equalities (18)(i, ii, iii) by $\epsilon_2 e_2(H_2)$, and using the identity (17)(ii) and notations $v_1 := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$ and $v_3 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_2 e_2(H_2)}$, we get

$$\begin{aligned} &(i)\ (\tilde{A} - \tilde{C})v_1 - \tilde{C} - \tilde{D}v_2 = 0,\\ &(ii)\ \tilde{C}v_1 + \tilde{A} + \tilde{C} + \tilde{D}v_2 = 0,\\ &(iii)\ \tilde{D}(v_1 + 1) + \tilde{A}v_2 = 0, \end{aligned} \tag{20}$$

From summation of equations (20)(i) and (20)(ii), we obtain $\tilde{A}(1+v_1) = 0$.

Assuming $\tilde{A} \neq 0$, from the last equality we get $v_1 = -1$ and then, by (20)(iii), we obtain $v_2 = 0$. From these results, by (20)(ii), we get $\tilde{A} = 0$. So, we get a contradiction, which implies that $\tilde{A} = 0$.

The equality $\tilde{A} = 0$ gives a polynomial equation in terms λ and κ. Since $nH_1 = 3\kappa + (n-3)\lambda$ is assumed to be constant, so we obtain a polynomial equation in terms λ, which implies that λ and then κ and H_{k+1} are constant on $\mathcal{U}$. This is a contradiction and implies that, the first claim $e_1(H_{k+1}) \equiv 0$ is proved.

If $e_3(H_2) \neq 0$, then by dividing both sides of equalities (18)(i, ii, iii) by $\epsilon_3 e_3(H_2)$, and using the identity (17)(ii) and notations $w_1 := \frac{\epsilon_1 e_1(H_2)}{\epsilon_3 e_3(H_2)}$ and $w_2 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_3 e_3(H_2)}$, we get

$$\begin{aligned} &(i)\ (\tilde{A} - \tilde{C})w_1 - \tilde{C}w_2 - \tilde{D} = 0, \\ &(ii)\ \tilde{C}w_1 + (\tilde{A} + \tilde{C})w_2 + \tilde{D} = 0, \\ &(iii)\ \tilde{D}(W_1 + w_2) + \tilde{A} = 0, \end{aligned} \tag{21}$$

From equations (21)(i) and (21)(ii), we obtain $\tilde{A}(w_1 + w_2) = 0$.

Assuming $\tilde{A} \neq 0$, from the last equality we get $w_2 = -w_1$ and then, by (21)(iii), we obtain $\tilde{A} = 0$.

The equality $\tilde{A} = 0$, by (21)(i, ii, iii), gives $\tilde{D} = 0$. So, we get $\tilde{A} - \tilde{D} = u_{k+1} - \kappa u_k = 0$, which is a polynomial equation in terms λ and κ. Since $nH_1 = 3\kappa + (n-3)\lambda$ is assumed to be constant, so we obtain a polynomial equation in terms λ, which implies that λ and then κ and H_{k+1} are constant on $\mathcal{U}$. This is a contradiction and implies that, the 3rd claim $e_1(H_{k+1}) \equiv 0$ is proved.

The forth stage is assumption $e_i(H_{k+1}) \neq 0$ for some $i \geq 4$. By a same manner, from (18)(iv) we get

$$\binom{n-4}{k-2}\frac{-n-2k+7}{2(n-k-1)}\kappa^3\lambda^{k-2} + \binom{n-4}{k-1}\frac{3(-n-2k+5)}{2(n-k-2)}\kappa^2\lambda^{k-1}$$
$$+\binom{n-4}{k}\frac{3(-n-2k+3)}{2(n-k-3)}\kappa\lambda^k + \binom{n-4}{k+1}\frac{-n-2k+1}{2(n-k-4)}\lambda^{k+1} = 0.$$

Since $nH_1 = 3\kappa + (n-3)\lambda$ is assumed to be constant, so we obtain a polynomial equation in terms λ, which implies that λ and then κ and H_{k+1} are constant on $\mathcal{U}$. Hence, M_1^n is isoparametric and by Theorem 2.6 in [12], we get $\lambda\kappa = 0$. $\square$

Finally, about the case that shape operator is of type IV, we restate Corollary 2.9 from [12], which says that each isoparametric timelike hypersurface M_1^n of $\mathbb{E}_1^{n+1}$ with shape operator of type IV and complex principal curvatures $\kappa \pm \lambda \mathbf{i}$ (where $\lambda \neq 0$) has at most one non-zero real principal curvature.

Theorem 3.7. *Let M_1^n be a timelike hypersurface of $\mathbb{E}_1^{n+1}$ with shape operator of type IV, two complex principal curvatures $\kappa \pm \lambda \boldsymbol{i}$ with constant $\lambda^2 + \kappa^2 = \kappa_0$, a real principal curvature η and constant ordinary mean curvature H_1. If M_1^n satisfies the L_k-biconservativity condition (3)(ii) for an integer $k \in \{1, ..., n-1\}$, then it has constant $k+1th$ mean curvature.*

Proof. Assume that, an isometric immersion $\mathbf{x} : M_1^n \to \mathbb{E}_1^{n+1}$ satisfies all conditions of the theorem. By the assumption, with respect to a suitable (local) orthnormal tangent frame $\{e_1, \cdots, e_n\}$ on M_1^n, the shape operator S has the matrix form $\tilde{B}_4$. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_{k+1}(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By assumption, the shape operator S of M_1^4 is of type IV with at most three distinct eigenvalue functions. Then, we have $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_i = \eta e_i$ for $i = 3, \cdots, n$. Then, we have

$$\binom{n}{k+1} H_{k+1} = \mu_{1,2;k+1} + 2\kappa\mu_{1,2;k} + (\kappa^2 + \lambda^2)\mu_{1,2;k-1},$$

$$P_{k+1}e_1 = (\kappa\mu_{1,2;k} + \mu_{1,2;k+1})e_1 + \lambda\mu_{1,2;k}e_2,$$

$$P_{k+1}e_2 = -\lambda\mu_{1,2;k}e_1 + (\kappa\mu_{1,2;k} + \mu_{1,2;k+1})e_2,$$

and $P_{k+1}e_i = \mu_{i,k+1}e_i$ for $i = 3, \cdots, n$.

Using the polar decomposition $\nabla H_{k+1} = \sum_{i=1}^{n} \epsilon_i e_i(H_{k+1})e_i$, from condition (3)$(ii)$ we get

$$\begin{aligned}
&(i) \quad (\kappa\mu_{1,2;k} + \mu_{1,2;k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1})\epsilon_1 e_1(H_{k+1}) \\
&\qquad - \lambda\mu_{1,2;k}\epsilon_2 e_2(H_{k+1}) = 0, \\
&(ii) \quad \lambda\mu_{1,2;k}\epsilon_1 e_1(H_{k+1}) + (\kappa\mu_{1,2;k} + \mu_{1,2;k+1} \\
&\qquad - \frac{3}{2}\binom{n}{k+1}H_{k+1})\epsilon_2 e_2(H_{k+1}) = 0, \\
&(iii) \ (\mu_{i;k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1})\epsilon_i e_i(H_{k+1}) = 0, \qquad (i = 3, \cdots, n).
\end{aligned} \tag{22}$$

Claim: $e_i(H_{k+1}) = 0$ for $i = 1, \cdots, n$.
If $e_1(H_{k+1}) \neq 0$, then by dividing both sides of equalities (22)(i, ii) by $\epsilon_1 e_1(H_{k+1})$, and using the notations $\zeta := \frac{\epsilon_2 e_2(H_{k+1})}{\epsilon_1 e_1(H_{k+1})}$, we get a system of equations as

$$\begin{aligned} &(i) \quad \kappa\mu_{1,2;k} + \mu_{1,2;k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1} = \lambda\mu_{1,2;k}\zeta, \\ &(ii) \;\; \lambda\mu_{1,2;k} + (\kappa\mu_{1,2;k} + \mu_{1,2;k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1})\zeta = 0, \end{aligned} \tag{23}$$

which gives $\lambda\mu_{1,2;k}(1+\zeta^2) = 0$ and then $\lambda\eta^k = 0$. Since λ is assumed non-zero, we get $\eta = 0$ and by (23)(i), $H_{k+1} = 0$ on $\mathcal{U}$. This is a contradiction which proves that $e_1(H_{k+1}) = 0$.

If $e_2(H_{k+1}) \neq 0$, then by dividing both sides of equalities (22)(i, ii) by $\epsilon_1 e_1(H_{k+1})$, and using the notations $\bar{\zeta} := \frac{\epsilon_1 e_1(H_{k+1})}{\epsilon_2 e_2(H_{k+1})}$, we get a system of equations as

$$\begin{aligned} &(i) \quad (\kappa\mu_{1,2;k} + \mu_{1,2;k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1})\bar{\zeta} - \lambda\mu_{1,2;k} = 0, \\ &(ii) \;\; \lambda\mu_{1,2;k}\bar{\zeta} + \kappa\mu_{1,2;k} + \mu_{1,2;k+1} - \frac{3}{2}\binom{n}{k+1}H_{k+1} = 0, \end{aligned}$$

which gives $\lambda\mu_{1,2;k}(1+\bar{\zeta}^2) = 0$ and then $\lambda\eta^k = 0$. Then, similar to the first case, we get $e_2(H_{k+1}) = 0$.

The third stage is assumption $e_i(H_{k+1}) \neq 0$ for some $i \geq 3$. By a same manner, from (22)(iii) we get

$$\begin{aligned} &\binom{n-3}{k-1}\frac{-n-2k+4}{2(n-k-1)}\kappa_0\eta^{k-1} + \binom{n-3}{k}\frac{-n-2k+2}{n-k-2}\kappa\eta^k \\ &+\binom{n-3}{k+1}\frac{-n-2k}{2(n-k-3)}\eta^{k+1} = 0. \end{aligned}$$

Since $nH_1 = 2\kappa + (n-2)\eta$ is assumed to be constant, so we obtain a polynomial equation in terms λ, which implies that λ and then κ and H_{k+1} are constant on $\mathcal{U}$. $\square$

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Firooz Pashaie
Department of Mathematics
Associate Professor of Mathematics
Faculty of Sciences, University of Maragheh, P.O.Box 55181-83111
Maragheh, Iran.
E-mail: f_pashaie@maragheh.ac.ir