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$\delta(2)$-Ideal Euclidean Hypersurfaces of Null L_1-2-Type

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Abstract. We say that an isometric immersion hypersurface $x : M^n \to \mathbb{E}^{n+1}$ is of null L_k-2-type if $x = x_1 + x_2$, $x_1, x_2 : M^n \to \mathbb{E}^{n+1}$ are smooth maps and $L_k x_1 = 0$, $L_k x_2 = \lambda x_2$, λ is non-zero real number, L_k is the linearized operator of the $(k+1)$th mean curvature of the hypersurface, i.e., $L_k(f) = \mathrm{tr}(P_k \circ \mathrm{Hessian} f)$ for $f \in C^\infty(M)$, where P_k is the kth Newton transformation, $L_k x = (L_k x_1, \ldots, L_k x_{n+1})$, $x = (x_1, \ldots, x_{n+1})$. In this article, we classify $\delta(2)$-ideal Euclidean hypersurfaces of null L_1-2-type.

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1 Introduction

Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometrically immersed Euclidean hypersurface. It is well-known that the Laplacian of M is the first element of n-term sequence of operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_k is the linearized operator of the first variation of the $(k+1)$th mean curvature (see

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[11, 12]). These operators are formulated by $L_k(f) = \mathrm{tr}(P_k \circ \mathrm{Hessian} f)$ for any $f \in C^\infty(M)$, where P_k is the kth Newton transformation related to the second fundamental from of M. Motivated by this consideration, S.M.B. Kashani [6] developed the idea of finite type submanifold to L_k-finite type hypersurface in the Euclidean space.

The position vector x and the $(k+1)$th mean curvature vector $\overrightarrow{H}_{k+1}$ of M^n in $\mathbb{E}^{n+1}$ are related by generalized Beltrami's formula:

$$L_k x = c_k \overrightarrow{H}_{k+1},$$

where $c_k = (n-k)\binom{n}{k}$ (see [1]). When $k = 0$, this turns into the classical Beltrami's formula $\Delta x = n\overrightarrow{H}$. This states the well-known result: M^n is a k-minimal hypersurface of $\mathbb{E}^{n+1}$ if and only if its coordinate functions are L_k-harmonic, i.e.,

$$L_k x = 0.$$

Particularly, k-minimal hypersurfaces of $\mathbb{E}^{n+1}$ are made by eigenfunctions of the operator L_k with eigenvalue zero. There are many examples of k-minimal hypersurfaces in the space forms (see for instance [14]).

From [6], we see that a right spherical cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ in $\mathbb{E}^{n+1}$ is composed of both L_k-harmonic function and eigenfunction of L_k with a single nonzero eigenvalue, say λ, when $0 < k < n-1$. Therefore, the position vector x of a right spherical cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ takes the following simple spectral decomposition

$$x = x_1 + x_2, \ L_k x_1 = 0, \ L_k x_2 = \lambda x_2, \tag{1}$$

for some non-constant smooth maps x_1 and x_2.

Baesd on L_k-finite type theory, a Euclidean hypersurface is said to be of null L_k-2-type if its position vector takes the spectral resolution (1). Similarly, a Euclidean hypersurface is said to be of L_k-1-type if its position vector satisfies the condition $L_k x = \lambda x$ (cf. [6, 7, 9, 8]). According to the genralized Takahashi's theorem [7], a L_k-1-type hypersurface of a Euclidean space is either a k-minimal Euclidean hypersurface or an open part of a hypersphere. Specially, because of the simplicity of null L_k-2-type hypersurfaces, after the classification of the L_k-1-type hypersurfaces, it seems reasonable to propose the following problem.

Problem: Classify all null L_k-2-type hypersurfaces in Euclidean spaces.

Until now, only few results have been obtained concerning this problem. In [7], the first author and Kashani obtained the first results. They proved that there exists no null L_{n-1}-2-type Euclidean hypersurface, specially, there is no null L_1-2-type Euclidean surface. When $k \neq n-1$, they also showed that every null L_k-2-type Euclidean hypersurface with at most two distinct principal curvatures is a right circular cylinder.

Now, assume that M is a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_pM$, $p \in M$. The scalar curvature τ at p is defined by $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$, where $e_1, \dots, e_n$ is an orthonormal basis of T_pM. By choosing $e_1, \dots, e_r$, a r-orthonormal basis of the r-plane section L^r, the scalar curvature $\tau(L^r)$ is defined by

$$\tau(L^r) = \sum_{i<j} K(e_i \wedge e_j), \qquad 1 \leq i, j \leq r.$$

For an integer $r \in [2, n-1]$, the δ-invariant $\delta(r)$ of M is defined by

$$\delta(r)(p) = \tau(p) - \inf\{\tau(L^r)\},$$

where L^r runs into all r-plane sections of T_pM.

For any n-dimensional submanifold M in $\mathbb{E}^m$, Chen [3] proved that the δ-invariant $\delta(2)$ satisfies the following inequality

$$\delta(2) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2. \tag{2}$$

The equality case of (2) is called the $\delta(2)$-equality. The classification of submanifolds in the Euclidean space $\mathbb{E}^m$ which satisfy the $\delta(2)$-equality condition is an interesting and important subject to research (see [4]). A submanifold M^n in $\mathbb{E}^m$ is called $\delta(2)$-ideal if it satisfies the $\delta(2)$-equality.

Inspired by the above observation, it was proved in [5] that a null 2-type hypersurfaces in $\mathbb{E}^{n+1}$ is an open part of a spherical cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ if and only if it is $\delta(2)$-ideal.

The main purpose of this paper is to extend this classification result to null L_1-2-type hypersurfaces as follows.

Theorem 1.1. *A null L_1-2-type hypersurface in the Euclidean space $\mathbb{E}^{n+1}$ with $n \geq 3$, is an open part of a spherical cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ if and only if it is $\delta(2)$-ideal.*

Note that from the before mentioned, if $n = 2$, there is no null L_1-2-type surface in $\mathbb{E}^3$.

2 Null L_1-2-type Hypersurfaces

Let $x : M^n \to \mathbb{E}^{n+1}$ be an isometric immersion, with Gauss map N. Denote by ∇ and $\overline{\nabla}$ the Levi-Cevita connections on M^n and $\mathbb{E}^{n+1}$, respectively. The formulas of Gauss, Weingarten and Codazzi are given respectively by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + \langle SX, Y\rangle N, \\ SX &= -\overline{\nabla}_X N, \\ (\nabla_X S)Y &= (\nabla_Y S)X, \end{aligned}$$

for $X, Y \in \mathfrak{X}(M^n)$, where $S : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ is the shape operator of M^n arises from the Gauss map N.
The equation of Gauss is given by

$$R(X, Y)Z = \langle SY, Z\rangle SX - \langle SX, Z\rangle SY,$$

for $X, Y, Z \in \mathfrak{X}(M^n)$ where R is the Riemann curvature tensor. The eigenvalues of S are called the principal curvatures of M^n.
Let $\{k_1, \ldots, k_n\}$ be the n principal curvatures of M^n. Associated to the principal curvatures, the 2th mean curvature H_2 of the hypersurface is defined by

$$\binom{n}{2} H_2 = \sum_{i_1<i_2}^{n} \kappa_{i_1}\kappa_{i_2}.$$

H_2 defines an intrinsic invariant which is relevant to the scalar curvature of M^n.

Related to the shape operator S, the classical Newton transformation $P_1 : \mathfrak{X}(M^n) \to \mathfrak{X}(M^n)$ is defined by

$$P_1 = \binom{n}{2} H_2 I - S. \tag{3}$$

Now, consider the second-order linear differential operator

$$L_1 : C^\infty(M) \to C^\infty(M),$$

which is given by

$$L_1(f) = \operatorname{tr}(P_1 \circ \operatorname{Hessian} f).$$

For isometric immersion $x : M^n \to \mathbb{E}^{n+1}$, it is well-known (see [1]),

$$L_1 x = n(n-1)\overrightarrow{H_2}, \tag{4}$$

where $\overrightarrow{H_2} = H_2 N$ defines the 2th mean curvature vector field. By formula in [1] page 122, we find

$$\begin{aligned} L_1\overrightarrow{H_2} = &-\binom{n}{2} H_2 \nabla H_2 - 2(S \circ P_1)(\nabla H_2) \\ &- (H_2 \operatorname{tr}(S^2 \circ P_1) - L_1 H_2)N. \end{aligned} \tag{5}$$

Suppose that M is of null L_1-2-type hypersurface, from (4) we get

$$L_1\overrightarrow{H_2} = \lambda\overrightarrow{H_2}. \tag{6}$$

By combining (5) and (6) we obtain

$$(S \circ P_1)\nabla H_2 = \frac{n(1-n)}{4} H_2 \nabla H_2, \tag{7}$$

$$L_1 H_2 - H_2 \operatorname{tr}(S^2 \circ P_1) = \lambda H_2. \tag{8}$$

Consequently, ∇H_2 is an eigenvector of $S \circ P_1$ whenever $\nabla H_2 \neq 0$.

For the proof of our theorem we need the following lemma from [10].

Lemma 2.1. *Let M be a hypersurface of a Euclidean space. Then the 2th mean curvature vector field $\overrightarrow{H_2}$ satisfies the condition $L_1\overrightarrow{H_2} = \lambda\overrightarrow{H_2}$ for some constant λ if and only if M is one of the following hypersurfaces*

(a) a L_1-biharmonic hypersurface,
(b) a L_1-1-type hypersurface,
(c) a null L_1-2-type hypersurface.

3 Proof of Theorem 1.1

Suppose that M is a null L_1-2-type hypersurface of $\mathbb{E}^{n+1}$, which is $\delta(2)$-ideal. Because M is a $\delta(2)$-ideal, there is an orthonormal frame $\{e_1, ..., e_n\}$ such that the shape operator with respect to this frame takes the following form [Lemma 3.2 of [2]]

$$S = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \beta & 0 & \cdots & 0 \\ 0 & 0 & \alpha+\beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha+\beta \end{pmatrix}, \tag{9}$$

for some functions α and β on M.

If H_2 is constant, then M is non 1-minimal from lemma 2.1. So, (8) implies that $\mathrm{tr}(S^2 \circ P_1)$ is constant. Since H_2 and $\mathrm{tr}(S^2 \circ P_1)$ are constant, it follows from (9) that M is isoparametric. According to a well-known result of Segre [13], any isoparametric hypersurface of $\mathbb{E}^{n+1}$ has l distinct principal curvatures with $l \leq 2$. If $l = 0$, then M is 1-minimal which is impossible by lemma 2.1. (9) shows easily that case $l = 1$ does not occur. So, $l = 2$. Therefore, from (9) we conclude that one of the principal curvatures is simple. Thus, M is locally isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$ by Theorem 3.12 of [7].

From now on we assume that H_2 is non-constant. First, from (3) and (9) we see that the classical Newton transformation P_1 satisfies

$$P_1 = \begin{pmatrix} (n-2)\alpha+(n-1)\beta & 0 & 0 & \cdots & 0 \\ 0 & (n-2)\beta+(n-1)\alpha & 0 & \cdots & 0 \\ 0 & 0 & (n-2)(\alpha+\beta) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)(\alpha+\beta) \end{pmatrix}.$$

Therefore, we have

$$S \circ P_1 = \begin{pmatrix} \alpha((n-2)\alpha+(n-1)\beta) & 0 & 0 & \cdots & 0 \\ 0 & \beta((n-2)\beta+(n-1)\alpha) & 0 & \cdots & 0 \\ 0 & 0 & (n-2)(\alpha+\beta)^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)(\alpha+\beta)^2 \end{pmatrix}, \tag{10}$$

and

$$S^2 \circ P_1 = \begin{pmatrix} \alpha^2((n-2)\alpha+(n-1)\beta) & 0 & 0 & \cdots & 0 \\ 0 & \beta^2((n-2)\beta+(n-1)\alpha) & 0 & \cdots & 0 \\ 0 & 0 & (n-2)(\alpha+\beta)^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n-2)(\alpha+\beta)^3 \end{pmatrix}. \tag{11}$$

Put $\nabla H_2 = \sum_{i=1}^{n} \lambda_i e_i$ for some functions $\lambda_1, ..., \lambda_n$ on M, then from (10) we have

$$\begin{aligned}(S \circ P_1)\nabla H_2 =& \sum_{i=1}^{n} \lambda_i (S \circ P_1) e_i \\ =& \alpha(\beta + (n-2)(\alpha+\beta))\lambda_1 e_1 + \beta(\alpha + (n-2)(\alpha+\beta))\lambda_2 e_2 \\ &+ \sum_{i=3}^{n} (n-2)(\alpha+\beta)^2 \lambda_i e_i \\ =& ((3-n)\alpha\beta + (2-n)\beta^2)\lambda_1 e_1 + ((3-n)\alpha\beta \\ &+ (2-n)\alpha^2)\lambda_2 e_2 + (n-2)(\alpha+\beta)^2 \nabla H_2.\end{aligned}$$

Thus, Eq. (7) yields

$$\begin{aligned}((n-2)(\alpha+\beta)^2 + \frac{1}{4}n(n-1)H_2)\nabla H_2 =& ((n-3)\alpha\beta + (n-2)\beta^2)\lambda_1 e_1 \\ &+ ((n-3)\alpha\beta + (n-2)\alpha^2)\lambda_2 e_2.\end{aligned}$$

Hence, $\lambda_3 = \cdots = \lambda_n = 0$ and

$$\begin{aligned}&[(n-2)(\alpha+\beta)^2 + \frac{1}{4}n(n-1)H_2 + (3-n)\alpha\beta + (2-n)\beta^2]\lambda_1 = 0, \quad (12)\\ &[(n-2)(\alpha+\beta)^2 + \frac{1}{4}n(n-1)H_2 + (3-n)\alpha\beta + (2-n)\alpha^2]\lambda_2 = 0.\end{aligned}$$

Since ∇H_2 is a nonzero, which implies at least one of λ_1 and λ_2 does not vanish. If both λ_1 and λ_2 do not vanish, then we find either $\alpha = \beta$ or $\alpha = -\beta$. If $\alpha = \beta$, then M has at most two distinct principal curvatures, so from [7] we know that any null L_1-2-type hypersurfaces with at most two distinct principal curvatures have constant 2-th mean curvature, this is a contradiction. If $\alpha = -\beta$, then $H_2 = \frac{4\beta^2}{n(n-1)}$. Hence, we get

$$n(n-1)H_2 = \mathrm{tr}(S \circ P_1) = -2\beta^2 = -\frac{n(n-1)}{2}H_2,$$

which implies $H_2 = 0$, but this is impossible.

Therefore, we have either

(a) $\lambda_1 \neq 0$ and $\lambda_2 = 0$, or

(b) $\lambda_2 \neq 0$ and $\lambda_1 = 0$.

We only need to consider the case (a), case (b) can be done in a similar arguments as case (a).

First, from relation (12) we obtain that

$$H_2 = \frac{4\alpha\left((\alpha+\beta)n - 2\,\alpha - \beta\right)}{n\,(1-n)}. \tag{13}$$

On the other hand, since $\operatorname{tr}(S \circ P_1) = n(n-1)H_2$, by using (10) we can write

$$\begin{aligned} H_2 = \frac{1}{n(n-1)}[\alpha\left((n-2)\,\alpha + \beta\,(n-1)\right) + \beta\left((n-2)\,\beta\right. \\ \left. + \,\alpha\,(n-1) + (n-2)^2\,(\alpha+\beta)^2\right]. \end{aligned} \tag{14}$$

Comparing (13) and (14), then after a straightforward computation, we find that there exist real numbers in terms of n, say a_1, a_2, such that $\{\alpha, \beta\} = \{a_1\sqrt{H_2}, a_2\sqrt{H_2}\}$. Note that since ∇H_2 is a nonzero without loss of generality, we may assume that $H_2 > 0$.

Next, by taking e_1 in the direction of ∇H_2, the shape operator satisfies

$$S = \begin{pmatrix} a_1\sqrt{H_2} & 0 & 0 & \cdots & 0 \\ 0 & a_2\sqrt{H_2} & 0 & \cdots & 0 \\ 0 & 0 & a_3\sqrt{H_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_3\sqrt{H_2} \end{pmatrix},$$

where $a_3 = a_1 + a_2$. Moreover, we have

$$e_1(H_2) \neq 0, \qquad e_k(H_2) = 0, \qquad \forall k > 1. \tag{15}$$

We put $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k$, then using the equation of Codazzi for $X = e_i$ and $Y = e_j$ we get

$$(\nabla_{ei} S)e_j = \frac{a_j}{2}\frac{e_i(H_2)}{\sqrt{H_2}} e_j + \sqrt{H_2}\sum_k (a_j - a_k)\omega_{ij}^k e_k.$$

Then, we consider the special cases of i and j.

For $i = 1, j = 2$, one obtains

$$\frac{a_2}{2}e_1(H_2)e_2 + H_2\sum_k (a_2 - a_k)\omega_{12}^k e_k = H_2\sum_k (a_1 - a_k)\omega_{21}^k e_k.$$

Under the identification the coefficients corresponding to $\{e_1, ..., e_n\}$, we have the following

$$\omega_{12}^1 = 0, \tag{16}$$

$$e_1(H_2) + 2(1 - \frac{a_1}{a_2})H_2\omega_{21}^2 = 0, \tag{17}$$

$$a_1\omega_{12}^k = a_2\omega_{21}^k, \qquad k \geq 3. \tag{18}$$

Similarly, for $i = 1, j \geq 3$, we obtain the following

$$\omega_{1j}^1 = 0, \qquad j \geq 3, \tag{19}$$

$$\omega_{1j}^2 = (1 - \frac{a_2}{a_1})\omega_{j1}^2, \qquad j \geq 3, \tag{20}$$

$$a_j e_1(H_2)\delta_{jk} + 2a_2H_2\omega_{j1}^k = 0, \qquad j, k \geq 3.$$

Finally, for $i = 2, j \geq 3$ we get

$$\omega_{2j}^1 = (1 - \frac{a_1}{a_2})\omega_{j2}^1, \qquad j \geq 3, \tag{21}$$

$$\omega_{2j}^2 = 0, \qquad j \geq 3,$$

$$\omega_{j2}^k = 0, \qquad j, k \geq 3.$$

From (15), we see easily $[e_2, e_j](H_2) = 0$. So, we have

$$\sum_k (\omega_{2j}^k - \omega_{j2}^k)e_k(H_2) = 0.$$

Again, using (15), we get $\omega_{2j}^1 = \omega_{j2}^1$, for $j \geq 3$. Combining this with (21) yields

$$\omega_{2j}^1 = \omega_{j2}^1 = 0. \tag{22}$$

Since $\{e_k\}_{k=1}^n$ is an orthonormal basis, we have

$$0 = e_i\langle e_j, e_k\rangle = \langle\nabla_{e_i}e_j, e_k\rangle + \langle e_j, \nabla_{e_i}e_k\rangle = \omega_{ij}^k + \omega_{ik}^j, \quad \forall i, j, k = 1, ..., n. \tag{23}$$

By using (23), we derive that

$$\omega_{11}^1 = 0, \quad \omega_{12}^2 = 0, \quad \omega_{1j}^j = 0, \quad j \geq 3, \tag{24}$$

$$\omega_{21}^1 = 0, \quad \omega_{22}^2 = 0, \quad \omega_{2j}^j = 0, \quad j \geq 3, \tag{25}$$

$$\omega_{k1}^1 = 0, \quad \omega_{k2}^2 = 0, \quad \omega_{kj}^j = 0, \quad j, k \geq 3. \tag{26}$$

Combining (23) with (16), (17) and (22) we find that

$$\omega_{11}^2 = 0, \quad \omega_{22}^1 = (\frac{a_2}{2(a_2 - a_1)})\frac{e_1(H_2)}{H_2}, \quad \omega_{j1}^2 = 0, \quad j \geq 3. \tag{27}$$

By applying (20) and (27) we obtain

$$\omega_{1j}^2 = 0, \quad j \geq 3. \tag{28}$$

Moreover, it follows from (23), (28) and (18) that

$$\omega_{12}^j = 0, \quad \omega_{21}^j = 0, \quad j \geq 3.$$

In the same way, we derive that

$$\omega_{11}^j = 0, \qquad \omega_{22}^j = 0, \qquad \omega_{jj}^1 = (\frac{a_1 + a_2}{2a_2})\frac{e_1(H_2)}{H_2}, \qquad j \geq 3. \tag{29}$$

Now, it follows from the Codazzi's equation that

$$\sum_k (a_j - a_k)\omega_{ij}^k e_k = \sum_k (a_i - a_k)\omega_{ji}^k e_k, \quad i, j \geq 3.$$

Therefore, we get

$$\omega_{ij}^1 = \omega_{ji}^1, \qquad \omega_{ij}^2 = \omega_{ji}^2, \qquad i, j \geq 3.$$

Then, (15), (16) and (25) imply that $[e_1, e_2](H_2) = 0$. Hence, we have

$$e_2 e_1(H_2) = 0.$$

From (15), (19) and (26) we also have

$$e_j e_1(H_2) = 0, \qquad j \geq 3.$$

Applying Gauss's equation to $\langle R(e_1, e_2)e_1, e_2\rangle$, $\langle R(e_1, e_j)e_1, e_j\rangle$ and $\langle R(e_2, e_j)e_2, e_j\rangle$, we respectively obtain that

$$e_1(\frac{e_1(H_2)}{H_2}) + \frac{a_2}{2(a_1 - a_2)}(\frac{e_1(H_2)}{H_2})^2 + 2a_1(a_1 - a_2)H_2 = 0, \tag{30}$$

$$e_1(\frac{e_1(H_2)}{H_2}) - \frac{a_1 + a_2}{2a_2}(\frac{e_1(H_2)}{H_2})^2 - 2a_1a_2H_2 = 0,$$

$$(\frac{e_1(H_2)}{H_2})^2 - 4a_2(a_1 - a_2)H_2 = 0. \tag{31}$$

On the other hand, from (17), (24), (29) and the definition of L_1, we find

$$L_1H_2 = b\left[e_1(e_1(H_2)) + \frac{c(e_1(H_2))^2}{H_2}\right]\sqrt{H_2}, \tag{32}$$

for some real numbers b and c.

Now, using (8), (11) and (32), we obtain that

$$b\left[e_1(e_1(H_2)) + \frac{c(e_1(H_2))^2}{H_2}\right]\sqrt{H_2} = dH_2^2\sqrt{H_2} + \lambda H_2, \tag{33}$$

for some real numbers b, c and d.
By substituting (31) into (30), we get

$$e_1(e_1(H_2)) = \left[\frac{2a_1 - 3a_2}{2}(a_1 + a_2)\right]H_2^2. \tag{34}$$

Also, substituting (31) into equation (33) gives

$$e_1(e_1(H_2)) = \left[\frac{bca_2(a_1 - a_2) - d}{b} - \frac{\lambda}{H_2\sqrt{H_2}}\right]H_2^2. \tag{35}$$

By comparing (34) and (35), we conclude that H_2 is constant, which is a contradiction.

This completes the proof of the theorem.

References

[1] L.J.Alías, N.Gürbüz, An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures, *Geom. Dedicata,* 121, (2006), 113-127.

[2] B.Y.Chen, Some pinching and classification theorems for minimal submanifolds, *Arch. Math.* 60 (1993), 568-578.

[3] B.Y.Chen, Some new obstructions to minimal and Lagrangian isometric immersions, *Japan. J. Math,* 26 (2000), 105-127.

[4] B.Y.Chen, *Pseudo-Riemannian Geometry, $\delta(r)$-Invariants And Applications,* World Scientific, Hackensack, New Jersey, 2011.

[5] B.Y.Chen, O.J.Garay, $\delta(2)$-ideal null 2-type hypersurfaces of Euclidean space are spherical cylinders, *Kodai Math. J,* 35 (2012), 382-391.

[6] S.M.B.Kashani, On some L_1-finite type (hyper)surfaces in $\mathbb{R}^{n+1}$, *Bull. Korean Math. Soc,* 46, 1 (2009), 35-43.

[7] A.Mohammadpouri, S.M.B.Kashani, On some L_k-finite-type Euclidean hypersurfaces, *ISRN Geom,* (2012), 23 p.

[8] A.Mohammadpouri, S.M.B.Kashani, Quadric hypersurfaces of L_r-finite type, *Beitrge zur Algebra und Geometrie,* (2012), 625-641.

[9] A.Mohammadpouri, S.M.B.Kashani, F. Pashaie, On some L_1-finite type Euclidean surfaces, *Acta Math. Vietnam,* 38 (2013) 303-316.

[10] A.Mohammadpouri, F. Pashaie, On the classification of hypersurfaces in Euclidean spaces satisfying $L_r\overrightarrow{H}_{r+1} = \lambda\overrightarrow{H}_{r+1}$,*Proyecciones,* (2016), 1-10.

[11] R.C.Reilly, Variational properties of functions of the mean curvatures for hypersurfaces in space forms, *J.Differential Geom,* 8, 3 (1973), 465-477.

[12] H.Rosenberg, Hypersurfaces of constant curvature in space forms, *Bull. Sci. Math,* 117, (1993), 211-239.

[13] B.Segre, Famiglie di ipersuperficie isoparametrische negli spazi euclidei ad un qualunque numero di dimensioni, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur,* 27, (1938), 203-207.

[14] B.G.Yang, X.M.Liu, r-minimal hypersurfaces in space forms, *J. Geom. Phys,* 59, (2009), 685-692.

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