Journal of Mathematical Extension
Vol. 16, No. 5, (2022) (4)1-17
URL: https://doi.org/10.30495/JME.2022.1752
ISSN: 1735-8299
Original Research Paper

On the Existence and Uniqueness of Fuzzy Differential Equations with Monotone Condition

S. S. Mansouri
University of Applied Science and Technology, Center of Poolad Peechkar

M. Gachpazan
Ferdowsi University of Mashhad

N. Ahmady *
Varamin-Pishva Branch, Islamic Azad University

E. Ahmady
Shahr-e-Qods Branch, Islamic Azad University

Abstract. This paper seeks to investigate the existence and uniqueness of solutions to fuzzy differential equations driven by Liu's process. For this purpose , we prove a novel existence and uniqueness theorem for fuzzy differential equations under Local Lipschitz and monotone conditions. This result allows us to consider and analyze solutions to a wide range of nonlinear fuzzy differential equations driven by Liu's process. To illustrate the main advantage of the approach some examples are finally given.

AMS Subject Classification:34A071; 34A12.
Keywords and Phrases: Fuzzy number, Fuzzy differential equation, Liu's process, Credibility space.

Received: July 2020; Accepted: December 2020
*Corresponding Author

1 Introduction

Fuzzy is a kind of uncertainty in the real world which has been first investigated by Zadeh [18], through proposing the concept of fuzzy sets via membership functions.

The first person who introduced credibility theory was Liu. He also presented the concept of credibility measure [12], credibility measure is a powerful tool for dealing with fuzzy phenomena.

Furthermore, Liu proposed the concept of fuzzy process [8]. This concept introduces a particular fuzzy process with stationary and independent increment named Liu's process which is just like a stochastic process described by Brownian motion.

Recently, literatures on the Liu process and its applications in other sciences, such as economics and optimal control has been published [19]. Liu inspired by stochastic notions and Ito process introduced fuzzy differential equations [9], which were driven by Liu process for better understand the fuzzy phenomena.

Solving European Pricing Problem in fuzzy environment is one of the most well known application of fuzzy differential equations driven by Liu's process [15]. Regarding to the importance of existence and uniqueness of a solution to fuzzy differential equations driven by Liu process, Liu investigated the existence and uniqueness of solution to the fuzzy differential equations by employing Lipschitz and Linear growth conditions [17]. Afterward, Fei studied the uniqueness of solution to the fuzzy differential equations driven by Liu process with non-Lipschitz coefficients [4].

Providing some weaker conditions to study the existence and uniqueness of solution to the fuzzy differential equations are the main goal of this paper. In this regard, we prove a novel existence and uniqueness theorem under the Local Lipschitz and monotone conditions.

The paper is arranged as follow. We will review some basic concepts about credibility theory, fuzzy variable, fuzzy process and Liu process in Section 2. A new existence and uniqueness theorem for fuzzy differential equations under Local Lipschitz and monotone conditions is proved in Section 3. Finally, in Section 4, Some examples are given to illustrate that the monotone condition is weaker than Linear growth condition.

2 Preliminaries

The emphasis in this section is mainly on introducing some concepts such as credibility measure, credibility space, fuzzy variables, independence of fuzzy variables, expected value, variance, fuzzy process, Liu process, and stopping time.
Suppose that Θ is a non-empty set and $\mathcal{P}$ is the power set of Θ. Each element of κ in $\mathcal{P}$ is called an event. For the purpose of presenting a basic definition of credibility, it is needed to consider a number $\mathbf{Cr}\{\kappa\}$ to each event κ. In order to make sure the number $\mathbf{Cr}\{\kappa\}$ has certain mathematical features that is intuitively expect these four axioms are accepted [8]:

1. Axiom (Normality) $\mathbf{Cr}\{\Theta\} = 1$.
2. Axiom (Monotonicity) $\mathbf{Cr}\{\kappa\} \leq \mathbf{Cr}\{\beta\}$ whenever $\kappa \subset \beta$.
3. Axiom (Self-Duality) $\mathbf{Cr}\{\kappa\} + \mathbf{Cr}\{\kappa^c\} = 1$ for any event κ.
4. Axiom (Maximality) $\mathbf{Cr}\{\mathbf{U}_i \kappa_i\} = \sup_i \mathbf{Cr}\{\kappa_i\}$ for any events $\{\kappa_i\}$ with

$$\sup_i \mathbf{Cr}\{\kappa_i\} < 0.5.$$

Definition 2.1. [17] The set function **Cr** is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.
A family $\mathcal{P}$ with these four properties is called a σ-algebra. The pair $(\Theta, \mathcal{P})$ is called a measurable space, and the elements of $\mathcal{P}$ is afterwards called $\mathcal{P}$-measurable sets instead of events.

Definition 2.2. [17] The triple $(\Theta, \mathcal{P}, \mathbf{Cr})$ is a credibility space if Θ be a nonempty set, $\mathcal{P}$ the power set of Θ, and **Cr** a credibility measure.

Let $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A filtration is a family $\{\mathcal{P}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{P}$ (i.e. $\mathcal{P}_t \subset \mathcal{P}_s \subset \mathcal{P}$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $\mathcal{P}_t = \bigcap_{s>t} \mathcal{P}_s$ for all $t \leq 0$. When the credibility space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{P}_0$ contains all **Cr**-null

sets.
We also define $\mathcal{P}_\infty = \sigma(U_{t\geq 0}\mathcal{P}_t)$ (i.e. σ-algebra generated by $U_{t\geq 0}\mathcal{P}_t$.)
$\mathcal{P}$-measurable fuzzy variable are denoted by $\mathbf{L}^p(\Theta, \mathbf{R^d})$ that will be defined later.
A process is called $\mathcal{P}_t$-adapted, if for all $t \in [0, t]$ the fuzzy variable $x(t)$ is $\mathcal{P}_t$-measurable.

Definition 2.3. [17] A fuzzy variable is defined as a (measurable) function
$\xi : (\Theta, \mathcal{P}, \mathbf{Cr}) \longrightarrow \mathbf{R}$.

Definition 2.4. [17] If we suppose that ς is a fuzzy variable. Then the expected value of ς is as follows:

$$\mathbf{E}[\varsigma] = \int_0^{+\infty} \mathbf{Cr}\{\varsigma \geq s\}\mathrm{ds} - \int_{-\infty}^{0} \mathbf{Cr}\{\varsigma \leq s\}\mathrm{ds}$$

these two integrals are finite.

Definition 2.5. [9] The credibility distribution $\eta(x)$ of a fuzzy variable ς is defined by

$$\eta(w) = \max\{1, 2\mathbf{Cr}(\varsigma = w)\}, \quad w \in \mathbf{R}.$$

Definition 2.6. [9] A credibility distribution $\eta(w)$ is regular on condition that it is a continuous and strictly increasing function with respect to w at which $0 < \eta(w) < 1$, and

$$\lim_{w\to-\infty} \eta(w) = 0, \quad \lim_{w\to+\infty} \eta(w) = 1.$$

In addition, the inverse function $\eta^{-1}(\alpha)$ can be called the inverse credibility distribution of ς.

Definition 2.7. [10] Considering $\mathbf{T}$ be an index set and $(\Theta, \mathcal{P}, \mathbf{Cr})$ be a credibility space. A fuzzy process can be described as a function from $\mathbf{T} \times (\Theta, \mathcal{P}, \mathbf{Cr})$ to the set of real numbers.

A fuzzy process is basically a sequence of fuzzy variables indexed by time or space.

Definition 2.8. [9] A fuzzy process $\mathbf{C}_t$ is a Liu process if the following are met

1. $\mathbf{C}_0 = 0$,
2. $\mathbf{C}_t$ has stationary and independent increments,
3. every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed fuzzy variable with expected value **et** and variance $\sigma^2 t^2$ whose membership function is

$$\eta(w) = 2(1 + \exp(\tfrac{\pi|w-et|}{\sqrt{6}\sigma t}))^{-1}, \qquad -\infty < w < +\infty.$$

Based on Liu process, Liu integral is defined as a fuzzy counterpart of Ito integral as follows.

Definition 2.9. [9] Suppose that $g(t)$ is a fuzzy process and $\mathbf{C}_t$ is a standard Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1\leq i\leq k} |t_{i+1} - t_i|.$$

Then the Liu integral of $g(t)$ with respect to $\mathbf{C}_t$ is

$$\int_a^b g(t)d\mathbf{C}_t = \lim_{\Delta\to 0} \sum_{i=1}^{k} g(t_i).(\mathbf{C}_{t_{i+1}} - \mathbf{C}_{t_i})$$

provided that the limit exists almost surely and is a fuzzy variable.

Theorem 2.10. *[3] Let $\mathbf{C}_t$ be a standard Liu process, and $f(t, c)$ a continuously differentiable function. Define $w(t) = f(t, \mathbf{C}_t)$. Then we have the following chain rule*

$$dw(t) = \tfrac{\partial f}{\partial t}(t, \mathbf{C}_t)dt + \tfrac{\partial f}{\partial c}(t, \mathbf{C}_t)d\mathbf{C}_t,$$

which is called Liu formula.
Let us define a sequence of credibilistic stopping times.

Definition 2.11. [10] A fuzzy variable $\tau : \Theta \to [0, \infty]$ (it may take the value ∞) is called an $\{\mathcal{P}_t\}$-stopping time (or simply, stopping time) if $\{\theta : \tau(\theta) \leq t\} \in \mathcal{P}_t$ for any $t \geq 0$

$$\begin{cases} \tau_h = \inf\{t \geq 0 : |w(t)| \geq k\}, \\ \sigma_1 = \inf\{t \geq 0 : x(w(t)) \geq 2\varepsilon\}, \\ \sigma_{2i} = \inf\{t \geq \sigma_{2i-1} : x(w(t)) \leq \varepsilon\} \quad i = 1, 2, \ldots, \\ \sigma_{2i+1} = \inf\{t \geq \sigma_{2i} : x(w(t)) \geq 2\varepsilon\} \quad i = 1, 2, \ldots, \end{cases}$$

where throughout this paper we set $\inf \phi = \infty$.

Definition 2.12. [10] If $W = \{W_t\}_{t\geq 0}$ is a measurable process and τ is a stopping time, then $\{W_{\tau\wedge t}\}_{t\geq 0}$ is called a stopped process of W.

There are some useful inequalities for fuzzy variables such as Hölder inequality and Chebyshev inequality.

Theorem 2.13. *[4](Hölder's Inequality) Let* $\mathbf{n}$ *and* $\mathbf{m}$ *be two positive real numbers with* $\frac{1}{\mathbf{n}} + \frac{1}{\mathbf{m}} = 1$, ξ *and* η *be independent fuzzy variables with*

$$\mathbf{E}\ [\ |\varsigma|^{\mathbf{n}}\ ] \leq +\infty \quad \textit{and} \quad \mathbf{E}\ [\ |\rho|^{\mathbf{m}}\ ] \leq +\infty.$$

We have

$$\mathbf{E}\ [\ |\varsigma\rho|\ ] \leq \sqrt[\mathbf{n}]{\mathbf{E}\ [\ |\varsigma|^{\mathbf{n}}\ ]}\ \sqrt[\mathbf{m}]{\mathbf{E}\ [\ |\rho|^{\mathbf{m}}\ ]}.$$

Theorem 2.14. *[4](Chebychev's Inequality) Let* $\varsigma : \theta \to \mathbf{R}^k$ *be a fuzzy variable such that* $\mathbf{E}\ [\ |\varsigma|^{\mathbf{n}}\ ] \leq +\infty$ *for some* $\mathbf{n}$, $0 \leq \mathbf{n} \leq \infty$.
Then Chebychev's inequality:

$$\mathbf{Cr}[\ |\varsigma| \geq \lambda\ ] \leq \frac{1}{\lambda^{\mathbf{n}}}\mathbf{E}\ [\ |\varsigma|^{\mathbf{n}}\ ] \quad \textit{for all} \quad \lambda \geq 0.$$

Before, ending this section it is essential to introduce some symbols that are used in next sections.
Notation 1: $\ell^{\mathbf{n}}(\theta, \mathbf{R}^{\mathbf{d}})$ the family of $\mathbf{R}^{\mathbf{d}}$-valued fuzzy variables ς with $\mathbf{E}|\xi|^{\mathbf{p}} < \infty$.
Notation 2: $\ell^{\mathbf{p}}([a, b], \mathbf{R}^{\mathbf{d}})$ the family of $\mathbf{R}^{\mathbf{d}}$-valued $\mathcal{P}_t$-adapted processes $\{h(t)\}_{a\leq t\leq b}$ such that $\int_a^b |h(t)|^{\mathbf{n}}\mathrm{d}t < \infty$ almost surely.
Notation 3: $M^{\mathbf{n}}([a, b], \mathbf{R}^{\mathbf{d}})$ the family of processes $\{h(t)\}_{a\leq t\leq b}$ in $\ell^{\mathbf{n}}([a, b], \mathbf{R}^{\mathbf{d}})$ such that $\int_a^b |h(t)|^{\mathbf{n}} dt < \infty$.
Notation 4: $\ell^{\mathbf{n}}(\mathbf{R}_+, \mathbf{R}^{\mathbf{d}})$ the family of processes $\{h(t)\}_{t>0}$ such that for every $T > 0$, $\{h(t)\}_{a\leq t\leq T} \in \ell^{\mathbf{n}}([0, T], \mathbf{R}^{\mathbf{d}})$.

3 Main result

In this section, the required tools to prove our proposed theorem for existence and uniqueness, such as Lemma's and inequalities our mentioned.

Lemma 3.1. *(**Burkholder -Davis-Gundy** inequality for* Liu process*):*
Let $g \in \ell^p(\mathbf{R}_+, \mathbf{R}^{d\times m})$. Define for $t > 0$,

$$w(t) = \int_0^t g(s)d\mathbf{C}_s \quad and \quad \kappa(t) = \int_0^t |g(s)|^2 ds.$$

Then,

$$\mathbf{E}|\kappa(t)| \leq \mathbf{E}(\sup_{0\leq s\leq t} |w(s)|^2) \leq 4\mathbf{E}|\kappa(t)| \tag{1}$$

Proof $w(t)$ and $\kappa(t)$ are bounded, then for integer number n, where $n > 1$, we define the stopping time as

$$\tau_n = \inf\{t \geq 0 : |w(t)| \vee \kappa(t) \geq n\}.$$

If (1) can be state by the stopped processes $w(t\wedge\tau_n)$ and $\kappa(t\wedge\tau_n)$, then the general case follows by putting $n \to \infty$. Moreover, for simplicity we set

$$w^*(t) = \sup_{0\leq s\leq t} |w(s)|.$$

Consider, the following inequality

$$E|w(t)|^2 \leq E\int_0^t |g(s)|^2 ds = E(\kappa(t)) \tag{2}$$

then by the use of Doob martingale inequality[3],

$$E|w^*(t)| \leq 4E|w(t)|^2$$

by substituting this into (2) yields,

$$E|w^*(t)| \leq 4E(\kappa(t))$$

which is the right-hand-side inequality of (1). In order to demonstrate the left-hand-side one,

$$y(t) = \int_0^t g(s)ds.$$

Then,

$$\mathbf{E}|y(t)|^2 = \mathbf{E}\int_0^t |g(s)|^2 ds = \mathbf{E}|\kappa(t)|. \tag{3}$$

On the other hand, the integration by parts formula yields,

$$w(t) = \int_0^t dw(s) + \int_0^t w(s)ds$$

$$= y(t) + \int_0^t w(s)ds.$$

Therefore,

$$|y(t)| \leq |w(t)| + \int_0^t |w(s)|ds \leq 2w^*(t).$$

Here, by substituting this into (3) one sees that,

$$\mathbf{E}|\kappa(t)| \leq 4\mathbf{E}[|w^*(t)|^2].$$

This implies,

$$\tfrac{1}{4}\mathbf{E}|\kappa(t)| \leq \mathbf{E}[|w^*(t)|^2].$$

Finally, the proof is complete .□

In the theory of ordinary differential equations, stochastic differential equations, and fuzzy differential equations the integral inequalities of Gronwall type have been used in a wide scope. In order to prove the results on stability, existence, and uniqueness.

Lemma 3.2. *(**Gronwall**'s inequality for* Liu process*): Let* $\mathrm{T} > 0$, $c > 0.$*, and* $\vartheta(.)$ *be a credibility measurable bounded nonnegative function on* $[0, \mathrm{T}]$*, and* $\beta(.)$ *be a nonnegative integrable function on* $[0, \mathrm{T}]$*. If*

$$\vartheta(t) \leq c + \int_0^t \beta(s)\vartheta(s)ds \qquad for \quad all \quad 0 \leq t \leq T, \tag{4}$$

then

$$\vartheta(t) \leq c\exp(\int_0^t \beta(s)ds) \qquad for \quad all \quad 0 \leq t \leq T. \tag{5}$$

Proof. Let

$$e(t) = c + \int_0^t \beta(s)\vartheta(s)ds \qquad for \qquad 0 \le t \le T.$$

Then $\vartheta(t) \le e(t)$. Moreover, by the chain rule of classical calculus, we have

$$\log(e(t)) = \log(c) + \int_0^t \frac{\beta(s)\vartheta(s)}{e(s)}ds \le \log(c) + \int_0^t \beta(s)ds.$$

This implies

$$e(t) \le c\exp(\int_0^t \beta(s)ds) \qquad for \qquad 0 \le t \le T.$$

According to $\vartheta(t) \le e(t)$, the proof is complete. □

Throughout this paper, we consider the fuzzy differential equations

$$dw(t) = p(w(t), t)dt + q(w(t), t)d\mathbf{C}_t \tag{6}$$

where $\mathbf{C}_t$ is a standard Liu process and p, q are some given functions. $w(t)$ is the solution of (6) which is a fuzzy process in the sense of Liu. By the definition of fuzzy differential, this equation is equivalent to the following fuzzy integral equation:

$$w(t) = w_0 + \int_{t_0}^t p(w(s), s)ds + \int_{t_0}^t q(w(s), s)d\mathbf{C}_s. \tag{7}$$

As this point we have the following conditions .
(**I**) **Local Lipschitz condition**: For each integer $n \ge 1$, there exists a positive constant number $\mathbf{L}_n$ such that

$$|p(w(t), t) - p(y(t), t)|^2 \vee |q(w(t), t) - q(y(t), t)|^2 \le \mathbf{L}_n |w(t) - y(t)|^2,$$

for those $w(t), y(t) \in \mathbf{R}^n$ with $|w(t)| \vee |y(t)| \le n$.
(**II**) **Linear growth condition**: There exists a positive number $\mathbf{L}$ such that

$$|p(w(t), t)|^2 \vee |q(w(t), t)|^2 \le \mathbf{L}(1 + |w(t)|^2),$$

(**III**) **Monotone condition**: there exists a positive constant $\mathbf{K}$ such that

$$w(t)^T f(w(t),t) + \tfrac{1}{2}|q(w(t),t)|^2 \leq \mathbf{K}(1+|w(t)|^2)$$

for all $w(t) \in \mathbf{R}^n$.

Remark 3.3. The monotone condition (**III**) is considered, there exists a positive constant $\mathbf{G}$ such that the solution of (6) satisfies

$$\mathbf{E}(\sup_{t_0\leq t\leq T} |w(t)|^2) \vee \mathbf{E}(\sup_{t_0\leq t\leq T} |q(t)|^2) \leq \mathbf{G}, \tag{8}$$

where $\mathbf{G} = \mathbf{G}(T, K, w_0)$ is a constant independent of h, ($h = \frac{1}{m}$ be a given step size with integer $m \geq 1$ and $T = Nh$).

Proof: By using Liu'formula and condition (**III**) for $t \in [t_0, T]$ we have

$$|w(t)|^2 =$$
$$|w_0|^2 + \textstyle\int_{t_0}^t [2(w^T p(w(s),s)) + \frac{1}{2}|q(w(s),s)|^2]ds + \int_{t_0}^t 2(w^T q(w(s),s))d\mathbf{C}_s,$$

then

$$1+|w(t)|^2 \leq 1+|w_0|^2 + 2k\textstyle\int_{t_0}^t [1+|w(s)|^2]ds + 2\int_{t_0}^t 2(w^T q(w(s),s))d\mathbf{C}_s,$$

we obtain

$$\begin{aligned} \mathbf{E}\sup(1+|w(t)|^2) &\leq (1+|w_0|^2) + \mathbf{E}\sup\int_{t_0}^t 2k[1+|w(s)|^2]ds \\ &+ \mathbf{E}\sup|\int_{t_0}^t 2(w^T q(w(s),s))d\mathbf{C}_s|. \end{aligned} \tag{9}$$

By the Lemma 3.1, we have

$$\mathbf{E}\sup|\textstyle\int_{t_0}^t 2(w(s)^T q(w(s),s))d\mathbf{C}_s| \leq 2\mathbf{E}(\int_{t_0}^t 4(|w(s)|^2|q(w(s),s)|^2)ds)^{\frac{1}{2}}$$

$$\leq 2\mathbf{E}(\sup(1+|w(t)|^2)\textstyle\int_{t_0}^t 4|q(w(s),s)|^2 ds)^{\frac{1}{2}}$$

$$\leq 3\mathbf{E}(\sup(1+|w(t)|^2)\textstyle\int_{t_0}^t 4k(1+|w(t)|^2)ds)^{\frac{1}{2}}$$

$$\leq 0.5\mathbf{E}\sup(1+|w(t)|^2)+\frac{9}{2}\mathbf{E}\int_{t_0}^{t}4k(1+|w(t)|^2)ds. \qquad (10)$$

Substituting (10) into (9) and using the Hölder inequality

$$\mathbf{E}\sup(1+|w(t)|^2)\leq[2(1+|w(0)|^2)+40k(\mathbf{E}\textstyle\int_{t_0}^{t}(1+|w(t)|^2)ds)],$$

so we obtain

$$\mathbf{E}\sup(1+|w(t)|^2)\leq[2(1+|w(0)|^2)+80k(\textstyle\int_{t_0}^{t}\mathbf{E}\sup(1+|w(t)|^2)ds)].$$

By the Gronwall inequality, we must get

$$\mathbf{E}\sup(|w(t)|)\leq\mathbf{G}$$

where $\mathbf{G}=(T,k,w_0)$ is a constant independent of h. Similarly, we can show that $\mathbf{E}\sup(|y(t)|)\leq\mathbf{G}$. □
It is known that some functions, such as $\sin^2 w$ and $-|w|^2w$, do not satisfy in Lipschitz and Linear growth conditions. Therefore, we prove the following theorem under weaker conditions that ensure the existence and uniqueness of the solution to equation (6).

Theorem 3.4. *(Existence and Uniqueness of solution): Assume the locally Lipschitz condition (**I**) holds, but the monotone condition (**III**) can be used instead of the linear growth condition (**II**). Then there is a unique solution $w(t)$ to equation (6) in $M^2((t_0,T],\mathbf{R}^d)$.*

Proof: For each $n\geq 1$, define the truncation function

$$p_n(w(t),t)=\{\begin{array}{ll} p(w(t),t) & |w(t)|\leq n\\ p(\frac{nw(t)}{|w(t)|},t) & |w(t)|>n\end{array}$$

$$q_n(w(t),t)=\{\begin{array}{ll} q(w(t),t) & |w(t)|\leq n\\ q(\frac{nw(t)}{|w(t)|},t) & |w(t)|>n,\end{array}$$

then p_n and q_n satisfy Lipschitz condition. So that, equation

$$w_n(t) = w(0) + \int_{t_0}^{t} p_n(w_n(s), s)ds + \int_{t_0}^{t} q_n(w_n(s), s)d\mathbf{C}_s, \ t_0 \leq t \leq T \tag{11}$$

Due to remark (3.3), $w(t)$ is a unique solution to equation (6) in $\mathbf{M}^2((t_0, T], \mathbf{R}^d)$. In addition, $w_n(t) \in \mathbf{M}^2((t_0, T], \mathbf{R}^d)$, also $w_{n+1}(t)$ is the unique solution of equation (6), and for $t_0 \leq t \leq T$ we have:

$$\begin{aligned} w_{n+1}(t) &= w(0) \\ &+ \int_{t_0}^{t} p_{n+1}(w_{n+1}(s), s)ds + \int_{t_0}^{t} q_{n+1}(w_{n+1}(s), s)d\mathbf{C}_s, \end{aligned} \tag{12}$$

and $w_{n+1}(t) \in \mathbf{M}^2((t_0, T], \mathbf{R}^d)$.
Define the stopping time

$$\tau_n = T \wedge \inf\{t \in [t_0, T] : |(w_n)_t| \geq n\}.$$

By using the Hölder inequality, we have

$$\begin{aligned} \mathbf{E}|w_{n+1}(t) &- w_n(t)|^2 \\ &\leq 2\mathbf{E}|\int_{t_0}^{t}[p_{n+1}(w_{n+1}(s), s)]ds - \int_{t_0}^{t}[p_n(w_n(s), s)]ds|^2 \\ &+ 2\mathbf{E}|\int_{t_0}^{t}[q_{n+1}(w_{n+1}(s), s)]d\mathbf{C}s - \int_{t_0}^{t}[q_n(w_n(s), s)]d\mathbf{C}_s|^2 \\ &\leq 2(t - t_0)\mathbf{E}\int_{t_0}^{t}|p_{n+1}(w_{n+1}(s), s) - p_n(w_n(s), s)|^2 ds \\ &+ \mathbf{E}\int_{t_0}^{t}|q_{n+1}(w_{n+1}(s), s) - q_n(w_n(s), s)|^2 ds \\ &\leq 4(t - t_0)\mathbf{E}\int_{t_0}^{t}[|p_{n+1}(w_{n+1}(s), s) - p_{n+1}(w_n(s), s)|^2 \\ &+ |p_{n+1}(w_n(s), s) - p_n(w_n(s), s)|^2]ds \\ &+ 4\mathbf{E}\int_{t_0}^{t}[|q_{n+1}(w_{n+1}(s), s) - q_{n+1}(w_n(s), s)|^2 \\ &+ |q_{n+1}(w_n(s), s) - q_n(w_n(s), s)|^2]ds. \end{aligned}$$

For $t_0 \leq t \leq \tau_n$, getting

$$p_{n+1}(w_n(s),s) = p_n(w_n(s),s) = p(w_n(s),s),$$

$$q_{n+1}(w_n(s),s) = q_n(w_n(s),s) = q(w_n(s),s),$$

again by substituting $w_{n+1}(t_0+s) = w_n(t_0+s) = w(s), s \in (t_0, 0]$, we obtain:

$$\begin{aligned}
\mathbf{E}(\quad \sup & {}_{t_0 \le r \le t} |w_{n+1}(t) - w_n(t)|^2) \\
\le & \quad 4(t-t_0)\mathbf{E}\int_{t_0}^{t} |p_{n+1}(w_{n+1}(s),s) - p_{n+1}(w_n(s),s)|^2 ds \\
+ & \quad 4\mathbf{E}\int_{t_0}^{t} |q_{n+1}(w_{n+1}(s),s) - q_{n+1}(w_n(s),s)|^2 ds \\
\le & \quad 4(t-t_0+1)\mathbf{E}\int_{t_0}^{t} K_n |(w_{n+1}(s),s) - (w_n(s),s)|^2 ds \\
\le & \quad 4(t-t_0+1)K_n \int_{t_0}^{t} \mathbf{E}(\sup_{t_0 \le r \le s} |(w_{n+1}(s),s) - (w_n(s),s)|^2) ds.
\end{aligned}$$

From the Gronwall inequality, we see that

$$\mathbf{E}(\sup_{t_0 \le s \le t} |w_{n+1}(t) - w_n(t)|^2) = 0 \qquad t_0 \le t \le \tau,$$

so this means that for $t_0 \le t \le \tau_n$, we always have

$$w_n(t) = w_{n+1}(t). \tag{13}$$

We give that τ_n is increasing, that is as $n \to \infty,_n \uparrow T$ a.s. By the monotone (**III**) condition, for almost all $\omega \in \Omega$, there exists an integer $n_0 = n_0(\omega)$ such that $\tau_n = T$ as $n \ge n_0$. Now define $w(t)$ by $w(t) = w_{n_0}(t), t \in [t_0, T]$. In order to verify that $w(t)$ is the solution of (6). By (13), $w(t \wedge \tau_n) = w_n(t \wedge \tau_n)$, and by (11), it follows that

$$w(t \wedge \tau_n) = \varsigma(0) + \int_{t_0}^{t \wedge \tau_n} p_n(w(s),s)ds + \int_{t_0}^{t \wedge \tau_n} q_n(w(s),s)d\mathbf{C}_s$$

$$= \varsigma(0) + \int_{t_0}^{t \wedge \tau_n} p(w(s),s)ds + \int_{t_0}^{t \wedge \tau_n} q(w(s),s)d\mathbf{C}_s.$$

Upon letting $n \to \infty$ then yields

$$w(t \wedge \tau_n) = \varsigma(0) + \int_{t_0}^{t \wedge \tau_n} p(w(s),s)ds + \int_{t_0}^{t \wedge \tau_n} q(w(s),s)d\mathbf{C}_s$$

that is

$$w(t) = \varsigma(0) + \int_{t_0}^{t} p(w(s), s)ds + \int_{t_0}^{t} q(w(s), s)d\mathbf{C}_s.$$

We can see $w(t)$ is the solution of (6), and $w(t) \in \mathbf{M}^2((t_0, T], \mathbf{R}^d)$. Therefor, the existence is complete. □
Here, some examples are given to illustrate the results of theorem (3.4).

Example 3.5. By considering the one-dimensional fuzzy differential equation, we have

$$dw(t) = [w(t) - w^3(t)]dt + w^2(t)d\mathbf{C}_t \qquad on \quad t \in [t_0, T] \tag{14}$$

Here $\mathbf{C}_t$ is a one-dimensional standard fuzzy process.

Clearly, the Local Lipschitz condition (**I**) is satisfied but the Linear growth condition (**II**) is not. On the other hand, note that

$$w(t)[w(t) - w^3(t)] + \tfrac{1}{2}w^4(t) \leq w^2(t) < 1 + w^2(t).$$

That is, the monotone condition is fulfilled. Hence theorem (3.4), guarantees that equation (14) has a unique solution.

Example 3.6. Consider the following fuzzy differential equation

$$dw(t) = [-w^3(t)]dt + [\sin w^2(t)]d\mathbf{C}_t \qquad \forall\ t \geq 0. \tag{15}$$

Clearly, the equation do not satisfy the linear growth condition (**II**). On the other hand, we have

$$w(-w^3) + \frac{1}{2}(\sin w^2)^2 \leq -w^4 + (\sin w^2)^2 \leq 2(1 + w^2). \tag{16}$$

In other words, the equation satisfies condition (**III**). Moreover, we also have

$$\begin{aligned} 2w^T p(w(t), t) + |q(w(t), t)|^2 &\leq |w(t)|^2 + K(1 + |w(t)|^2) \\ &\leq (1 + 2K)(1 + |w(t)|^2). \end{aligned} \tag{17}$$

It is clearly that (**III**) follows (**II**). It is worth nothing to say that, the of this example can be used for many nonlinear FDEs.

4 Conclusion

The existence and uniqueness theorem is one of the most useful and basic theorems in the theory of fuzzy differential equations. However, there are few people who have considered weaker conditions. In the present paper, we have aimed to prove a novel existence and uniqueness theorem under the Local Lipschitz and monotone conditions.

References

[1] X. Chen, X. Qin, A new existence and uniqueness theorem for fuzzy differential equations, *International Journal of Fuzzy Systems.*, 13(2)(2013), 148-151.

[2] Z. Ding, M. Ma, A. Kandel, Existence of the solutions of fuzzy differential equations with parameters, *Information Sciences.*, 99(1) (1999), 1205-1217.

[3] W. Dai, Lipschitz continuity of Liu process, *Proceedings of the Eighth International Conference on Information and Management Science. China.*, (2009) 756-760.

[4] W. Fei, Uniqueness of solutions to fuzzy differential equations driven by Liu process with non-Lipschitz coefficients, *Internatinal Confrance On Fuzzy and Knowledge Discovery.*, (2009) 565-569.

[5] M. Friedman, M. Ming, A. Kandel, Numerical procedures for solving fuzzy differential and integral equations,*Internat Confrance Fuzzy Logic and Applications.*, (1997) 18-21.

[6] J. Gao, Credibilistic Option Pricing, *Journal of Uncertain Systems.*, 2(4) (2008) 243-247.

[7] O. Kaleva, Fuzzy differential equation, *Fuzzy Sets and Systems.*, 24(2) (1987) 301-317.

[8] X. Li, B. Liu, A suffcient and necessary condition for credibility measures, *International Journal of Uncertainty, Fuzziness and Knowledge Based Systems.*, 14(5) (2004) 527-535.

[9] B. Liu, A survey of credibility theory, *Fuzzy Optimization and Decision Making.*, 5 (4) (2006) 387-408.

[10] B. Liu, *Uncertainty Theory*, Springer-Verlag Press,(2007).

[11] B. Liu, Fuzzy process, hybrid process and uncertain process, *Journal of Uncertain Systems.*, 2(1) (2008) 3-16.

[12] B. Liu, Y. Liu, Expected value of fuzzy variable and fuzzy expected value models,*IEEE Transactions on Fuzzy Systems.*, 10 (4) (2002) 445-450.

[13] J. Peng, A General Stock Model for Fuzzy Markets, *Journal of Uncertain Systems.*, 2 (4) (2008) 248-254.

[14] Z. Qin, X. Li, Option pricing formula for fuzzy financial market, *Journal of Uncertain Systems.*, 2(1) (2008) 17-21.

[15] Z. Qin, X. Gao, Fractional Liu process with application to finance, *Mathematical and Computer Modeling.*, 50 (9) (2009), 1538-1543.

[16] C. You, Multi-dimensional Liu process, differential and integral. *Proceedings of the First Intelligent Computing Conference.*, (2007) 153-158.

[17] C. You, W. Wang, H. Huo, Existence and Uniqueness Theorems for Fuzzy Differential Equations, *Journal of Uncertain Systems.*, 7 (4) (2013) 303-315.

[18] L. A. Zadeh, Fuzzy sets, *Information and Control.*, 8(1) (1965) 338-353.

[19] Y. Zhu, A Fuzzy option control with application to portfolio selection, *Journal of Uncertain Systems.*, 3(4) (2009) 270-279.

Samira Siah Mansouri
Assistant Professor of Mathematics
University of Applied Science and Technology, Center of Poolad Peechkar.
Varamin, Iran.
E-mail: samira.mansouri91@yahoo.com

Mortaza Gachpazan
Department of Mathematics
Associate Professor of Mathematics
Department of Applied Mathematics, School of Mathematical Sciences, Ferdowsi University Of Mashhad.
, Mashhad, Iran.
E-mail: gachpazan@um.ac.ir

Nazanin Ahmady
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin, Iran.
E-mail: n.ahmadi@iauvaramin.ac.ir

Elham Ahmady
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.
E-mail: e.ahmadi@qodsiau.ac.ir