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Target Setting: A Generalized Concept of Centralized Resource Allocation

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Abstract. In this article, we first survey the relationship between Lozano and Villa's centralized resource allocation (CRA) and Mar-Molinero's simplified models. Then, we show how to obtain an optimal solution to the CRA model via the simplified model. Moreover, we propose a novel model by simplifying phase II of Lozano and Villa's model. This proposed model determines a peer efficient unit for each decision making unit (DMU) through some simple computations. The proposed simplified models help us to clarify some ambiguities in the CRA method and propose some solutions to solve them. Considering the importance of target setting in the management, we formulate another new model such that by preserving the aim of the CRA model, also projects DMUs onto efficient frontier according to desirable target as much as possible. The proposed model is illustrated by using a data set of earlier literature.

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1 Introduction

Evaluating units which work under the control of a unified organization is one of the most important issues in management science. Data envelopment analysis (DEA) as a non-parametric method to assessing decision making units (DMUs) was first proposed by Charnes et al.[5]. Following that, a great variety of researches was presented to generalize DEA method in various fields ([6])

Traditional DEA models evaluate each one of DMUs, separately. In the real world, there are many situations in which several units fall under the control of a centralized supervisor to support and handle them (e.g. bank branches, supermarket chains and university departments). In these cases, the centralized manager desires to improve the performance of a total system as well as the efficiency score of units. Heretofore, there have been many articles that handle the decision making units in a joint manner (e.g. [1, 2, 4, 7, 11, 13, 15, 17, 18]).

The concept of centralized resource allocation (CRA) was introduced by Lozano and Villa [13] for the first time. They claimed when units are under the control of a higher level supervising, the conventional DEA models do not guarantee minimizing of overall input consumption which is desirable for the central decision maker. In CRA method, only one model is solved to determine targets for all units. The idea behind CRA approach is reallocation input measures in a manner that minimizes the overall consumption of the inputs and maximizes the overall production of outputs.

Some researchers applied the approach of Lozano and Villa [13] with a few changes: [8, 9, 10, 12, 14, 16, 19] and [21] fall in this category. Some others have combined CRA models with other methods to achieve a suitable approach. Hosseinzadeh Lotfi et al.[17] proposed a CRA method for enhancing Russell models. Shamsi et al.[23] tried to achieve the goal of central organizations using CRA and multi objective models.

Asmild et al.[1] reconsidered the CRA method proposed by Lozano and Villa [13]. They presented some drawbacks for resource allocation in Lozano and Villa method; among them is the change of efficiency frontier after reallocation of input measures. To modify it, they suggested only inefficient units to be considered in the process of inputs reallocation. Mar-Molinero et al.[20] simplified one of CRA models proposed by Lozano and Villa [13]. They showed that the multiplier model introduced by Lozano and Villa [13] can be rewritten in a simpler form; consequently, based on duality theory, the radial model can be simplified too.

In this paper, the close relationship between Mar-Molinero et al. and Lozano and Villa's models is shown. We then simplify phase II of Lozano and Villa's model ([13]) and survey some prevailing relations between them. The aims of Lozano and Villa's models are achieved by solving two models with smaller dimensions. Lozano and Villa [13] did not mention any pattern to project DMUs onto the efficient frontier. They also did not present a perfect reason for projecting some efficient units on the other efficient units. Here, we interpret how to project units onto the frontier in CRA method and why some efficient units project onto the other efficient units. Moreover, the production possibility set (PPS) is determined after reallocation. Finally, a model to manage the projections of units is proposed.

In the rest of the present article, we illustrate Lozano and Villa's and Mar-Molinero et al.'s models and prove some relations among them in section 2. Section 3 presents a proper method to achieve a pattern to project units onto efficient frontier. Summaries and conclusions appear in section 4.

2 The CRA radial models and simplified ones with their relations

Lozano and Villa [13] proposed a variety of models to assess a system consisted of several units. Mar-Molinero et al. [20] presented a simplification on phase I of Lozano and Villa' model and stated some interpretations for their simplified model. Using Lozano and Villa' method, one can determine the efficient projections for units as well as assessing the main system; however, Mar-Molinero et al. [20] did not mentioned how the projections of units on efficient frontier can be obtained. In this section, we first review the mentioned models and show some relations between them. Afterwards, we determine the projection of units onto the efficient frontier by introducing another simplified model.

2.1 CRA radial model and simplified one

Consider a data set consisting of n DMUs. Let j, k be indices for DMUs (i.e. $j, k = 1, ..., n$). Each DMU$_j$ consumes a m-vector $x_j = (x_{1j}, ..., x_{mj})$ of inputs to produce a s-vector $y_j = (y_{1j}, ..., y_{sj})$ of outputs. Lozano and Villa introduced the following model (1) to evaluate the system

$$\begin{array}{lll} \theta^* = & \min & \theta \\ & s.t. & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{jk} x_{ij} \leq \theta \sum_{k=1}^{n} x_{ik}, \quad i = 1, ..., m, \\ & & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{jk} y_{rj} \geq \sum_{k=1}^{n} y_{rk}, \quad r = 1, ..., s, \\ & & \sum_{j=1}^{n} \lambda_{jk} = 1, \quad k = 1, ..., n, \\ & & \lambda_{jk} \geq 0, \quad \theta free. \end{array} \tag{1}$$

The value of λ_{jk} is the share of DMU$_j$ in constructing the projection of DMU$_k$ onto the frontier and θ is the radial contraction of aggregated input vector. Mar-Molinero et al. [20] stated assuming a common value for θ , in all of individual models, model (1) may be obtained only adding up all similar constraints. They subsequently introduced the simplified model (2). In model (2), ρ has the same concept of θ and μ_j substitutes to $\sum_{k=1}^{n} \lambda_{jk}$,

$$\begin{array}{lll} \rho^* = & \min & \rho \\ & s.t. & \sum_{j=1}^{n} \mu_j x_{ij} \leq \rho \sum_{k=1}^{n} x_{ik}, \quad i = 1, ..., m, \\ & & \sum_{j=1}^{n} \mu_j y_{rj} \geq \sum_{k=1}^{n} y_{rk}, \quad r = 1, ..., s, \\ & & \sum_{j=1}^{n} \mu_j = n, \\ & & \mu_j \geq 0, \quad \rho free. \end{array} \tag{2}$$

Just as it can be seen, model (2) may be obtained from model (1) adding up on the third set of constrains and introducing $\sum_{k=1}^{n} \lambda_{jk}$ as a new variable. It is obvious that model (1) involves $n^2 + 1$ variables and $m + s + n$ constraints while model (2) involves only $n + 1$ variables and $m + s + 1$ constraints; therefore, the latter is computationally more economic. Theorem 2.1 states a close relation between models (1) and (2).

Theorem 2.1. *If* $(\{\lambda^*_{jk}, j, k = 1, ..., n\}, \theta^*)$ *is an optimal solution to model (1), then* $(\{\sum_{k=1}^{n} \lambda^*_{jk}, j = 1, ..., n\}, \theta^*)$ *is an optimal solution to model (2) and if* $(\{\mu^*_j, j = 1, ..., n\}, \rho^*)$ *is an optimal solution to model (2), then* $(\{\frac{1}{n}\mu^*_j, j, k = 1, ..., n\}, \rho^*)$ *is an optimal solution to model (1).*

Proof. Let $(\{\lambda^*_{jk}, j, k = 1, ..., n\}, \theta^*)$ be an optimal solution to model (1). It is obvious that setting $\mu_j = \sum_{k=1}^{n} \lambda^*_{jk}$ for $j = 1, ..., n$ and $\rho = \theta^*$ implies a feasible solution to model (2). It is also an optimal one; otherwise, there must be another feasible solution to model (2) such as $(\{\hat{\mu}_j, j = 1, ..., n\}, \hat{\rho})$ that $\hat{\rho} < \theta^*$. Setting $\theta = \hat{\rho}$ and $\lambda_{jk} = \frac{1}{n}\hat{\mu}_j$ for each $j, k \in \{1, ..., n\}$, we obtain a feasible solution to model (1) with an objective value less than θ^* which is a contradiction.

On the other hand, suppose $(\{\mu^*_j, j = 1, ..., n\}, \rho^*)$ is an optimal solution to model (2). Introduce ρ^* as θ and $\frac{1}{n}\mu^*_j$ as λ_{jk} for each $j, k \in \{1, ..., n\}$. This is of course a feasible solution to model (1) with the same objective value. This solution is also optimal to model (1); otherwise, referring to the previous part, one can obtain another feasible solution to model (2) with a better objective value than ρ^* which is a contradiction. □

Corollary 2.2. *Model (1) can also be introduced as a model to evaluate the virtual unit* $(\frac{1}{n}\sum_{j=1}^{n} x_j, \frac{1}{n}\sum_{j=1}^{n} y_j)$*.*

Proof. Theorem 2.1 shows not only the optimal values of model (1) and simplified model (2) are equal but also it is possible to obtain an optimal solution for one model using the other one; therefore, they are equivalent. On the other hand, model (2) is the same as BCC model to assess a virtual unit with average input and output vectors (i.e. $(\frac{1}{n}\sum_{j=1}^{n} x_j, \frac{1}{n}\sum_{j=1}^{n} y_j)$) which is hereafter named mean point; It is sufficient to divide both sides of all constraints in model (2) by n.[3] So, it implies that model (1) naturally evaluates the mean point. □

[3] Authors have surveyed the properties of aggregate point in T_c in another research which is similar to mean point in T_v.

2.2 Target Setting

Here, we deal with simplification of phase II model introduced by Lozano and Villa [13] (model (3)). This model, apart from finding the overall reduction in input measures and expansion in output measures, is used to determine the efficient projection for units.

$$
\begin{array}{lll}
\alpha^* = & \max & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
 & s.t. & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{jk} x_{ij} + s_i^- = \theta^* \sum_{k=1}^{n} x_{ik}, \quad i = 1, ..., m, \\
 & & \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{jk} y_{rj} - s_r^+ = \sum_{k=1}^{n} y_{rk}, \quad r = 1, ..., s, \\
 & & \sum_{j=1}^{n} \lambda_{jk} = 1, \quad k = 1, ..., n, \\
 & & \lambda_{jk} \geq 0, \quad j, k = 1, ..., n, \\
 & & s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, ..., m, r = 1, ..., s.
\end{array} \tag{3}
$$

In model (3), θ^* is the optimal objective value in model (1). The unit $(\sum_{j=1}^{n} \lambda_{jk}^* x_j \ , \sum_{j=1}^{n} \lambda_{jk}^* y_j)$ is introduced by Lozano and Villa as the projection of DMU$_k$ onto the efficient frontier. We claim that an optimal solution to model (3) (and therefore the efficient projection for DMUs) can be obtained by solving the simplified model (4).

$$
\begin{array}{lll}
\beta^* = & \max & \sum_{i=1}^{m} d_i^- + \sum_{r=1}^{s} d_r^+ \\
 & s.t. & \sum_{j=1}^{n} \mu_j x_{ij} + d_i^- = \rho^* \sum_{k=1}^{n} x_{ik}, \quad i = 1, ..., m, \\
 & & \sum_{j=1}^{n} \mu_j y_{rj} - d_r^+ = \sum_{k=1}^{n} y_{rk}, \quad r = 1, ..., s, \\
 & & \sum_{j=1}^{n} \mu_j = n, \\
 & & \mu_j \geq 0, \quad j = 1, ..m, \\
 & & d_i^- \geq 0, d_r^+ \geq 0, \quad i = 1, ..., m, r = 1, ..., s.
\end{array} \tag{4}
$$

The two next theorems follow our claim about model (3) and proposed model (4).

Theorem 2.3. *If* $(\lambda^*_{jk}, s^{-*}_i, s^{+*}_r)$ *for* $j,k = 1,...,n$, $i = 1,...,m$ *and* $r = 1,...,s$ *is an optimal solution to model (3), then* $(\sum\limits_{k=1}^{n} \lambda^*_{jk}, s^{-*}_i, s^{+*}_r)$ *that* $j = 1,...,n$, $i = 1,...,m$, $r = 1,...,s$ *is an optimal solution to model (4). Moreover, if* $(\mu^*_j, d^{-*}_i, d^{+*}_r)$, $j = 1,...,n$, $i = 1,...,m$, $r = 1,...,s$ *is an optimal solution to model (4), then setting* $\lambda^*_{jk} = \frac{1}{n}\mu^*_j, s^{-*}_i = d^{-*}_i, s^{+*}_r = d^{+*}_r$ *obtains an optimal solution to model (3) where* $j,k = 1,...,n$, $i = 1,...,m$, $r = 1,...,s$.

Proof. Suppose $(\lambda^*_{jk}, s^{-*}_i, s^{+*}_r)$ for $j,k = 1,...,n$, $i = 1,...,m$, $r = 1,...,s$ is an optimal solution to model (3). It is easily shown that $(\mu_j = \sum_{k=1}^{n} \lambda^*_{jk}, d^-_i = s^{-*}_i, d^+_r = s^{+*}_r)$, $j = 1,...,n$, $i = 1,...,m$, $r = 1,...,s$ is a feasible solution to model (4). Assume it is not optimal; thus, there must be another feasible solution to model (4) such as $(\hat{\mu}_j, , \hat{d}^-_i, \hat{d}^+_r)$ with a better objective value i.e. $\sum_{i=1}^{m} \hat{d}^-_i + \sum_{r=1}^{s} \hat{d}^+_r > \alpha^*$. Setting $s^-_i = \hat{d}^-_i$, $i = 1,...,m$ and $s^+_r = \hat{d}^+_r$, $r = 1,...,s$ and also $\lambda_{jk} = \frac{1}{n}\hat{\mu}_j$, $j,k = 1,...,n$, we obtain a feasible solution to model (3) with an objective value better than α^* which is a contradiction.

On the other hand, let $(\mu^*_j, d^{-*}_i, d^{+*}_r)$, $j = 1,...,n$, $i = 1,...,m$, $r = 1,...,s$ be an optimal solution to model (4), we set $\lambda^*_{jk} = \frac{1}{n}\mu^*_j, s^{-*}_i = d^{-*}_i, s^{+*}_r = d^{+*}_r$. Now, we claim that $\lambda^*_{jk} = \frac{1}{n}\mu^*_j, s^{-*}_i = d^{-*}_i, s^{+*}_r = d^{+*}_r$ is the optimal solution to model (3); otherwise, another feasible solution to model (4) can be obtained with an objective value better than $\sum_{i=1}^{m} \hat{d}^{-*}_i + \sum_{r=1}^{s} \hat{d}^{+*}_r$ which is a contradiction. □

Corollary 2.4. *The virtual units* $(\sum_{j=1}^{n} \frac{1}{n}\mu^*_j x_j, \sum_{j=1}^{n} \frac{1}{n}\mu^*_j y_j)$ *can be introduced as the projection of all units. This corollary rises from the second part of proof of theorem 2.3.*

Corollary 2.5. *If* $\mu^*_j = 0$ *in all optimal solutions to model (4), then for each* $k \in \{1,...,n\}$, $\lambda^*_{jk} = 0$ *in each optimal solution to model (3). This point is concluded from the first part of proof of theorem 2.3.*

It is worth to note that according to corollary 2.5, if DMU_j does not belong to the set of reference units of mean point, then it has no share in constructing the efficient projection of DMUs.

Theorem 2.6. *If $(\mu_j^*, d_i^{-*}, d_r^{+*})$ where $j = 1, ..., n$, $i = 1, ..., m$, $r = 1, ..., s$ is an optimal solution to model (4), then any set of non-negative variables λ_{jk} that satisfy system of equations (5) gives an optimal solution to model (3).*

$$\begin{aligned} &\sum_{k=1}^{n} \lambda_{jk} = \mu_j^* \quad j \in \{1, ..., n\} \\ &\sum_{j=1}^{n} \lambda_{jk} = 1, \quad k \in \{1, ..., n\}. \end{aligned} \tag{5}$$

Proof. System (5) is feasible, since $\lambda_{jk} = \frac{1}{n}\mu_j^*$ satisfies all equalities. Let $\{\bar{\lambda}_{jk} \geq 0 : j, k = 1, ..., n\}$ satisfy equations (5). Since $\sum_{j=1}^{n} \bar{\lambda}_{jk} = 1$, the third set of constraints in model (3) are tight. At the same time, $\sum_{k=1}^{n} \bar{\lambda}_{jk} = \mu_j^*$ implies that the following equalities are true.

$$\begin{aligned} &\sum_{k=1}^{n}\sum_{j=1}^{n} \bar{\lambda}_{jk} x_{ij} + d_i^{-*} = \sum_{j=1}^{n} \mu_j^* x_{ij} + d_i^{-*} = \rho^* \sum_{k=1}^{n} x_{ik} \quad i = 1, ..., m \\ &\sum_{k=1}^{n}\sum_{j=1}^{n} \bar{\lambda}_{jk} y_{rj} - d_r^{+*} = \sum_{j=1}^{n} \mu_j^* y_{rj} - d_r^{+*} = \sum_{k=1}^{n} y_{rk} \qquad r = 1, ..., s. \end{aligned}$$

Since $\rho^* = \theta^*$, we conclude $(\bar{\lambda}_{jk}, d_i^{-*}, d_r^{+*})$ as a feasible solution to model (3) and according to theorem 2.3, this is also an optimal solution. □

Based on theorem 2.6, with each optimal solution to model (4) it can suffice to solve a system of equations, given in (5), to obtain an optimal solution to model (3). The following theorem states one important note about efficient projections for units.

Theorem 2.7. *All projections for units obtained by model (3) lie on the same hyperplane.*

Proof. Let $\{(\sum_{j=1}^{n} \lambda_{jk}^* x_j, \sum_{j=1}^{n} \lambda_{jk}^* y_j) : k = 1, ..., n\}$ be a set of efficient projections for DMUs obtained by model (3). Based on theorem 2.3, $\mu_j^* = \sum_{k=1}^{n} \lambda_{jk}^*$ is the optimal solution to model (4). Assume $H : u^* y - v^* x + \xi^* = 0$ to be a hyperplane involving $(\frac{1}{\mathrm{n}} \sum_{j=1}^{n} \mu_j^* x_j, \frac{1}{\mathrm{n}} \sum_{j=1}^{n} \mu_j^* y_j)$ as a projection of mean point onto the efficient frontier. It is claimed that all projected units $(\sum_{j=1}^{n} \lambda_{jk}^* x_j, \sum_{j=1}^{n} \lambda_{jk}^* y_j)$ lie on H. Suppose in contradict there is a unit, say DMU_o, whose projection dose not position on H i.e. $u^*(\sum_{j=1}^{n} \lambda_{jo}^* y_j) - v^*(\sum_{j=1}^{n} \lambda_{jo}^* x_j) + \xi^* < 0$. This implies that $u^*(\sum_{k=1}^{n} \sum_{j=1}^{n} \lambda_{jk}^* y_j) - v^*(\sum_{k=1}^{n} \sum_{j=1}^{n} \lambda_{jk}^* x_j) + \sum_{k=1}^{n} \xi^* < 0$. Therfore $u^*(\frac{1}{\mathrm{n}} \sum_{j=1}^{n} \mu_j^* y_j) - v^*(\frac{1}{\mathrm{n}} \sum_{j=1}^{n} \mu_j^* x_j) + \xi^* < 0$ which is a contradiction. □

One drawback of Lozano and Villas models is the existence of many variables and constraints while simplified models (2) and (4) have much less variables and constraints. Asmild et al. [1] reckoned unpredictability of the new frontier (after reallocation) as a weakness to the method proposed by Lozano and Villa [13]. Considering theorem 2.7, we may see that the new frontier is not unpredictable but new PPS after reallocation is the set of all production possibilities which can produce by reference units positioned on hyperplane H. Lozano and Villa [13] did not present a clarified reason for projecting an efficient unit onto another efficient unit. Asmild et al. [1] also added it is not acceptable that a unit which is projected onto another efficient unit can be selected as a peer for other units.

In a short statement, hitherto, we proposed the simplified model (4) which can be substitute to Lozano and Villa's one ([13]). We also clarified drawbacks of their method and tried to resolvable some of them. Now, we will propose model (6) to obtain a more desirable solution for CRA model (3).This model find targets for units such that unnecessary transfers are avoided as much as possible.

Assume that we have an optimal solution to model (4). Based on theorem 2.6, each non-negative solution to equations (5) gives an optimal solution to model (3). Let's consider a utility function $f(\lambda)$ which is desirable to optimize. We propose solving model (6):

$$\begin{array}{lll} \min(\max) & f(\lambda) & \\ s.t. & \sum\limits_{k=1}^{n} \lambda_{jk} = \mu_j^* & j = 1, ..., n, \\ & \sum\limits_{j=1}^{n} \lambda_{jk} = 1 & k = 1, ..., n, \\ & \lambda_{jk} \geq 0. & \end{array} \tag{6}$$

The proposed method by model (6) gives the decision maker a possibility to exert the opinion of individual units in selecting a more desirable projections among optimal solutions to model (3). The constraints of model (6) force λ_{jk} s to be optimal in model (3); as a result, using model (6), we can obtain optimal values to λ_{jk} s such that specific targets for units could be achieved. This yet has a weakness; since model (4) may not have a unique optimal solution and variables λ_{jk} in model (6) are depend to μ_j^* s, it may not obtain the most desirable solution. Complying our goal, we propose another model in section 3.

3 Following specific goals in projecting units

In the previous section, it was explained that model (6) is not a comprehensive model to achieve the most desirable projections for units. This section presents another model which is independent of optimal values μ_j^* . To this end, it is needed to first prepare some preliminary. Next, our method is presented and illustrated through an example.

3.1 Finding a perfect reference set for mean point

It is worth to remember, models (2) and (4) are two phases of BCC envelopment form to evaluate the mean point (as under assessment unit). Banker [3] proposed an original model consisting of slake variables in constraints and objective function. Using his method for mean point, model (7) is obtained as follows:

$$
\begin{array}{lll}
\min & \rho - \varepsilon(\sum\limits_{i=1}^{m} d_i^- + \sum\limits_{r=1}^{s} d_r^+) & \\
s.t. & \sum\limits_{j=1}^{n} \mu_j x_{ij} + d_i^- = \rho \sum\limits_{k=1}^{n} x_{ik} & i = 1, ..., m, \\
& \sum\limits_{j=1}^{n} \mu_j y_{rj} - d_r^+ = \sum\limits_{k=1}^{n} y_{rk} & r = 1, ..., s, \\
& \sum\limits_{j=1}^{n} \mu_j = n, & \\
& \mu_j \geq 0 & j = 1, ..m, \\
& d_i^- \geq 0, d_r^+ \geq 0, & i = 1, ..., m, r = 1, ..., s, \\
& \rho & free.
\end{array}
\tag{7}
$$

In model (7), ε is non-Archimedean small number. Solving model (7) is equivalent to solving both models (2) and (4); therefore, it does not only find a radial contraction in aggregated input vector but also seeks the further reduction of any input measures and/or expansion of any output productions. Now, consider the dual form of model (7) as follows:

$$
\begin{array}{lll}
\max & \sum_{r=1}^{s} u_r \sum_{k=1}^{n} y_{rk} + n\xi & \\
s.t. & \sum_{i=1}^{m} v_i \sum_{k=1}^{n} x_{ik} = 1, & \\
 & \sum_{r=1}^{s} u_r \sum_{j=1}^{n} y_{rj} - \sum_{i=1}^{m} v_i \sum_{j=1}^{n} x_{ij} + \xi \le 0, & j,k = 1,...,n, \\
 & v_i \ge \varepsilon, u_r \ge \varepsilon, & , i = 1,...,m, r = 1,...,s, \\
 & \xi free. &
\end{array} \tag{8}
$$

Using model (8), we may determine a hyperplane in which the mean point is projected on. It is worth to remember that the projection of all observed units also lies on this hyperplane. However, each one of these models may have multiple optimal solutions. Sueyoshi and Sekitani [24] studied the existence of multiple projections for DMU under assessment. They emphasized that, despite the existence of multiple reference DMUs, there exists a unique set concluding all reference units and named it the perfect reference set. They proposed a model using envelopment and multiplier forms together and considering strong complimentary slack conditions (SCSC). Their model finds a perfect reference set for DMU under assessment. We use their method for the mean point. In this way, we can find any DMU_j that may have a $\mu_j^* \neq 0$ in at least one of the optimal solutions to model (7) (in other words, each DMU_j whose $\mu_j^* = 0$ in all optimal solutions to model (7) can be recognized). We may also determine a face of production possibility set (PPS) with the least dimension involving all reference units (See [24]). Referring to Sueyoshi and Sekitani [24], we present model (9).

$$
\begin{array}{lll}
\max & \eta & \\
s.t. & \text{all constraints in (7) and (8)}, & \\
& \rho-\varepsilon(\sum_{i=1}^{m} d_i^- + \sum_{r=1}^{s} d_r^+) = \sum_{r=1}^{s} u_r \sum_{k=1}^{n} y_{rk} + n\xi, & \\
& v_i + d_i^- \geq \eta+\varepsilon, & i=1,...,m, \\
& u_r + d_r^+ \geq \eta+\varepsilon, & r=1,...,s, \\
& \mu_j - (\sum_{r=1}^{s} u_r \sum_{j=1}^{n} y_{rj} - \sum_{i=1}^{m} v_i \sum_{j=1}^{n} x_{ij} + \xi) \geq \eta, & j=1,...,n, \\
& \eta \geq 0. &
\end{array}
\tag{9}
$$

Let $J = \{j : \mu_j^* \neq 0, \text{in at least one optimal solution to model (7)}\}$. Solving model (9), the set J for mean point can be obtained; it also determines a hyperplane H : $u^*y - v^*x+\xi^* = 0$ including the efficient projection of mean point. It is worth to note that the hyperplane H involves also all units belonging to J.

Solving model (9) is time consuming, because of the great number of variables and constraints. It may also have multiple optimal solutions but what persuades us to use this model is determining a unique and perfect reference set (J). Note that, if for a given DMU$_j$, $\mu_j^* > 0$ in one optimal solution to model (7) but $\mu_j^* = 0$ in another optimal solution then j belongs to set J. On this foundation, the set J is perfect and also unique; even if model (9) has been multiple optimal solutions.

3.2 Proposed approach

Now, according to uniqueness of J, we propose model (10) to optimize utility function $f(\lambda)$ (stated in Section 2.2).

$$
\begin{array}{lll}
\max(\min) & f(\lambda) & \\
s.t. & \sum_{k=1}^{n} \sum_{j\in J} \lambda_{jk} x_{ij} \leq \rho^* \sum_{k=1}^{n} x_{ik}, & i=1,...,m, \\
& \sum_{k=1}^{n} \sum_{j\in J} \lambda_{jk} y_{rj} \geq \sum_{k=1}^{n} y_{rk}, & r=1,...,s, \\
& \sum_{j\in J} \lambda_{jk} = 1, & k=1,...,n, \\
& \lambda_{jk} \geq 0, & k=1,...,n, j\in J.
\end{array}
\tag{10}
$$

Model (10) is similar to model (1) but with a few differences. The unknown multiplier in model (1) in right hand side of the first set of constraint is not a variable in model (10) but is the optimal value. In addition, index j in constraints of model (10) is limited to members of J. This means only those units can construct the efficient projection for units that belong to J (see corollary 2.5). The following theorems state further illustrations on model (10).

Theorem 3.1. *Model (10) is feasible.*

Proof. Suppose that $(\lambda^*_{jk}\,,\, j,k \in \{1,...,n\}, \theta^*)$ is an optimal solution to model (1). Based on corollary 2.5, it must be $\lambda^*_{jk} = 0$ for any $j \in \{1,...,n\}/J$ so that it can be a feasible solution to model (10) too. □

Considering the proof of theorem 3.1, we conclude that the set of optimal values $\lambda^*_{jk} > 0$, $j,k \in \{1,...,n\}$ obtained from model (3) is a subset of feasible solutions of model (10) (note that each optimal solution to model (3) is an optimal solution to model (1)). We also claim that any feasible solution to model (10) gives an optimal solution to model (3). To this end, we state the following theorem.

Theorem 3.2. *Any feasible solution $(\lambda_{jk}\,,\, k \in \{1,...,n\}, j \in J)$ to model (10) satisfies the equation of hyperplane H obtained from model (8) i.e. $u^*(\sum_{j\in J}\lambda_{jk}y_j) - v^*(\sum_{j\in J}\lambda_{jk}x_j) + \xi^* = 0$ for each $k = 1,...,n$.*

Proof. We know $\mu^*_j \neq 0$ for each $j \in J$; then, considering 4^{th} set of constraints of model (7), $\mu^*_j > 0$ for each $j \in J$. So, based on the complimentary slack conditions it must be $u^*y_j - v^*x_j + \xi^* = 0$ for each $j \in J$. Trough multiplying both sides of these equalities by λ_{jk} and adding up on j, we obtain $u^*(\sum_{j\in J}\lambda_{jk}y_j) - v^*(\sum_{j\in J}\lambda_{jk}x_j) + \xi^* = 0$ for each $k \in \{1,...,n\}$ (notice $\sum_{j\in J}\lambda_{jk} = 1$ for $k = 1,...,n$). Proof is complete. □

For each $j \in J$, (x_j, y_j) belongs to production possibility set. By convexity axiom of PPS ([3]), we get $(\sum_{j\in J}\lambda_{jk}x_j, \sum_{j\in J}\lambda_{jk}y_j) \in PPS$ for each $k = 1,...,n$. On the other hand, based on theorem 3.2, all projections of DMUs lie on H. This means all obtained projections for DMUs through using model (10), belong to H∩PPS . Thus, the following corollary can be stated.

Corollary 3.3. *There are a one-to-one correspondence between optimal set of λ^*_{jk} s in model (3) and the set of feasible solutions to model (10).*

It is worth to note that introducing the target for units means presenting some change in construction of units for finding a better performance. Since each change needs cost, so using model (10) we may reduce the costs by avoiding undesirable transportations of units as much as possible. Therefore, using model (10) (and naturally model (9)) will be worthwhile for the problems with costly transportations.

In what follows, we illustrate the aforementioned method through an example.

3.3 Example

To illustrate the proposed approach, the data set of Lozano and Villa [13] that is appeared in Table 1 is used. The "Existing" columns give input and output measures of DMUs. The "BCC-I" and "Lozano and Villa" columns show the projection of DMUs on the efficient frontier obtained from BCC model (input orientation) and Lozano and Villa's radial model, respectively.

Table 1: Data set of Lozano and Villa [13]

	Existing			BCC-I			Lozano and Villa		
DMUs	x_1	x_2	y	$\widetilde{x}_1$	$\widetilde{x}_2$	$\widetilde{y}$	x_1^*	x_2^*	y^*
A	6	2	1	6	2	1	3.5	3.5	1
B	4	3	1	4	3	1	4	3	1
C	3	4	1	3	4	1	3	4	1
D	2	5	1	2	5	1	4	3	1
E	4	7	1	2.55	4.45	1	2	5	1
F	5	5	1	3.5	3.5	1	4	3	1
G	5	3	1	4.54	2.73	1	4	3	1
Total	29	29	7	25.59	24.68	7	24.5	24.5	7

Suppose that the manager desires to project DMUs onto efficient frontier which in addition to preserving the aim of model (1), are near projections obtained by BCC-I model. To this means, norm 1 is considered. We should minimize the summation $\sum_{k=1}^{n} \|(x_k, y_k)_{LV} - (\widetilde{x}_k, \widetilde{y}_k)\|_1$

which $(x_k, y_k)_{LV}$ is the projection of DMU_k based on definition by Lozano and Villa [13] and $(\widetilde{x}_k, \widetilde{y}_k)$ is the projection obtained by BCC-I model. To solve the proposed problem, we introduce the utility function $f(\lambda) = \sum_{k=1}^{n}\sum_{i=1}^{m}\left|\sum_{j\in J}\lambda_{jk}x_{ij} - \theta_k^* x_{ik}\right|$ as an objective function in model (10) in which θ_k^* is the BCC efficiency score of DMU_k. The obtained model is nonlinear because of the objective function but it is convertible to a linear model (11) (see [22]). The concluded results by running model (9) are summarized in Table 2. Table 3 shows the obtained projections for units using model (11). PPS_{BCC} (projected in plan $x_1 - x_2$) for given units is graphed in Fig. 1. All obtained projections via BCC-I beside model (11) are shown in Fig. 1.

$$
\begin{array}{lll}
\min & f(\lambda) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}(\varphi_{1ij} + \varphi_{2ij}) & \\
s.t. & \sum\limits_{k\in J}\sum\limits_{j=1}^{n}\lambda_{jk}x_{ik} \le \rho^* \sum\limits_{j=1}^{n} x_{ij}, & i = 1, ..., m, \\
& \sum\limits_{k\in J}\sum\limits_{j=1}^{n}\lambda_{jk}y_{rk} \ge \sum\limits_{j=1}^{n} y_{rj}, & r = 1, ..., s, \\
& \sum\limits_{k\in J}\lambda_{jk}x_{ik} - \theta_j^* x_{ij} = \varphi_{1ij} - \varphi_{2ij}, & i = 1, ..., m, j = 1, ..., n, \\
& \sum\limits_{k\in J}\lambda_{jk} = 1, & j = 1, ..., n, \\
& \lambda_{jk} \ge 0, & j = 1, ..., n, k \in J, \\
& \varphi_{1ij} \ge 0, \varphi_{2ij} \ge 0, & i = 1, ..., m, j = 1, ..., n.
\end{array}
\tag{11}
$$

Table 2: Results of model (9)

$\rho^* = 0.8448$
$J = \{B, C, D\}$
$\lambda_B^* = 5.2414 \quad \lambda_C^* = 0.0172 \quad \lambda_D^* = 1.7414$
$H : 0.0172y - 0.0172x_1 - 0.0172x_2 = 0$

The dashed line in Fig. 1 shows the hyperplane H. It can be seen that the mean point is projected onto H and all observed units are projected on H as well. Also, the new PPS after reallocation input measures is the shadowed area depicted in Fig. 1. It is noticeable that the efficient

Table 3: The resulted projection for units by model (11)

DMU	x_1	x_2	y
A	4	3	1
B	4	3	1
C	3.5	3.5	1
D	2	5	1
E	3	4	1
F	4	3	1
G	4	3	1
Total	24.5	24.5	7

unit A is projected on the other efficient unit B, since A does not lie on the hyperplane H.

As Table 2 shows, we are able to determine the hyperplane H using model (9), while this has been ignored in [13] and other articles in CRA field.

Comparing the results in Table 1 and 3, concludes that although obtained targets for DMUs are exactly the same as those computed by Lozano and Villa, but assignment is in a way that proposed peer efficient unit for each one is close to its BCC-I projection as much as possible.

To clarify the issue, we have analyzed the obtained results by solving model (11) with two other methods prepared in the CRA literature based on model (6). Again, consider $Di = \sum_{k=1}^{n} \|(\hat{x}_k, \hat{y}_k) - (\widetilde{x}_k, \widetilde{y}_k)\|_1$ as distance between the targets of DMU$_k$ (i.e. $(\hat{x}_k, \hat{y}_k)$) and the projection $(\widetilde{x}_k, \widetilde{y}_k)$ obtained by BCC-I model, in norm-1.

Fang [9] added some constraints to the radial model proposed by Lozano and Villa to preserve efficiency scores of DMUs. The resulted targets of Fang's method are given in the second column of Table 4. Computing Di for them concludes that $Di = 15.27$ Whiles corresponding value obtained from data of Table 3 is "6.71"; furthermore, Fang's model does not present notable changes in total inputs and outputs. In the third column of Table 4, we report the results of Hosseinzadeh Lotfi et al. [17]. They used the CRA idea for the enhanced Russell model. Whereas the resulted projections for DMUs locate on efficient frontier (especially on hyperplane H), but the value of Di in their method (17.91)

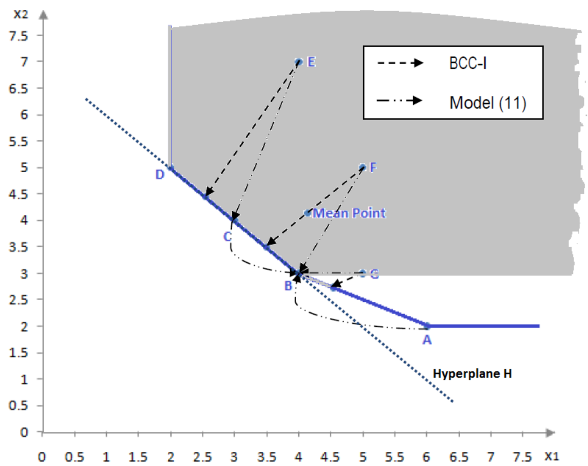

Figure 1: PPS_{BCC} for data set of Table 1

is far from the obtained value of model (11) (i.e. "6.71"). Also, in their method contrary to the notable decrease in the total amount of input 1, there is not any decrease in the total amount of input 2; which is a defect in the real problems, if the cost of some inputs is notably greater than other ones. The last column of Table 4 is earmarked to results of model (6), considering $f(\lambda) = \sum_{k=1}^{n} \|(x_k, y_k)_{LV} - (\widetilde{x}_k, \widetilde{y}_k)\|_1$ as its objective function. We observe that the optimal objective value of this model is "6.81" which is closed to "6.71" obtained from model (11); this is due to the same procedure of these two models. The only defect of model (6) is its dependence on the optimal solution of model (4) that is possibly caused to lesser desirable targets if data set is changed.

As a last note, we declare the specific utility function used in model (11) is an example and eager researchers can examine different functions.

4 Summaries and conclusions

In this paper, we have presented two models with a number of variables and constraints noticeably less than Lozano and Villa's radial models. It was shown that how one can determine the optimal solution to Lozano

Table 4: The resulted projection for units by some methods

	Fang			Lotfi			Model(6)		
DMUs	$\hat{x}_1$	$\hat{x}_2$	$\hat{y}$	$\hat{x}_1$	$\hat{x}_2$	$\hat{y}$	$\hat{x}_1$	$\hat{x}_2$	$\hat{y}$
A	4	3	1	2	5	1	4	3	1
B	4	3	1	2	5	1	4	3	1
C	4	3	1	4	3	1	3.5	3.5	1
D	3.28	3.72	1	3	4	1	2	5	1
E	4	7	1	2	5	1	3	4	1
F	4.67	5.33	1	3	4	1	4	3	1
G	4.4	3.3	1	4	3	1	4	3	1
Total	28.35	28.35	7	20	29	7	24.5	24.5	7

and Villa' s models through using the simplified models. Another benefit of our simplified model is its clear interpretation based on traditional DEA models. While [13] did not state any pattern to project DMUs onto efficient frontier or why they did project an efficient unit on another efficient unit, this article finds the pattern of projecting process. It was shown that all projections of all DMUs settle on one hyperplane; moreover, we can determine this hyperplane. Although, the units under supervision of a central decision maker have no authority in choosing a peer efficient unit, using the proposed approach, one can exert their interest as much as possible.

We presented our method in this article based on variable returns to scale assumption to accord with [13] and [20]. Our method can be used on constant, non-decreasing or non-increasing returns to scale assumption as well. An avid reader may research that whether the proposed approach is adaptable to the non-radial models.

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