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Statistical Inference on 2-Component Mixture of Topp-Leone Distribution, Bayesian and non-Bayesian Estimation

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Abstract. To study the heterogeneous nature of lifetimes of certain mechanical or engineering processes, a mixture model of some suitable lifetime distributions may be more appropriate and appealing as compared to simple models. This paper considers mixture of Topp-Leone distributions under classical and Bayesian perspective based on complete sample. The new distribution which exhibits decreasing and upside down bathtub shaped density while the distribution has the ability to model lifetime data with decreasing, increasing and upside down bathtub shaped failure rates. We derive several properties of the new distribution such as moments, moment generating function, conditional moment, mean deviation, Bonferroni and Lorenz curves and the order statistics of the proposed distribution. Moreover, we estimate the parameters of the model by using frequentist and Bayesian approaches. For Bayesian analysis, five loss functions, namely the squared error loss

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function (SELF), weighted squared error loss function (WSELF), modified squared error loss function (MSELF), precautionary loss function (PLF), and K-loss function (KLF) and uniform as well as gamma priors are considered to obtain the Bayes estimators and posterior risk of the unknown parameters of the model. Furthermore, credible intervals (CIs) and highest posterior density (HPD) intervals are also obtained. Monte Carlo simulation study is done to access the behavior of these estimators. For the illustrative purposes, a real-life application of the proposed distribution to a tensile strength data set is provided.

AMS Subject Classification: 60E05
Keywords and Phrases: Bayes estimators; Bayesian intervals; Loss functions; Mixture distribution; Posterior risks, Uniform prior.

1 Introduction

Of late, profound interest has been shown towards methodological development and practical applications of finite mixtures of lifetime distributions. Mixture models are quite versatile and thus have been frequently used in many real life applications. For example, Mendenhall and Hader (1958), while referring to practical situations encountered by engineers, pointed out that the failure of a system or a device may be divided into two or more different types of causes. Further, Acheson and McElwee (1952) categorized the failures of electronic tube into gaseous defects, mechanical defects, and normal deterioration of the cathode in order to know the proportion of failure due to a certain cause. Another example is that of an engineering system which consists of different subsystems. These subsystems may be homogeneous or heterogeneous. Heterogeneity nature of such systems can not be captured by a single probability models but it can be captured through mixture models.

A very important reason for which mixture models are receiving great attention is owing to the fact that majority of the commonly used distributions prove to be irrelevant with respect to population comprising of numerous subpopulations, that is, when we consider a population consisting of several subpopulations mixed in an unknown proportion. Take for example, a population of the lifetime of certain electrical elements or medicines which is divided into a number of subpopulations depending upon the possible causes of failures. Direct application of mixture models is feasible in situations where data are given only from overall mixture distributions. This type of direct application are mostly used whether data arising from the field of medicine, botany, zoology, paleontology, agriculture, economics, life testing, reliability, survival analysis etc. Various features of mixture models-type-I and type-II mixture models have been discussed in detail by Li (1983) and Li and Sedransk (1988). TypeI mixture models are those in which the probability distributions of the mixture model belongs to the same family while in type-II mixture models probability distributions do not belong to the same family. Researchers have successfully used mixture models in areas such as crime and justice, wind shear, engineering, physical, chemical and biological sciences [see Harris (1983); Kanji (1985); Jones and McLachlan (1990), Ateya (2014); Zhang and Huang(2015); Benaicha and Chaker(2014); Aslam et al.(2015)].

In the last few decades, researchers have focused their attention to Bayesian approaches to mixture models, especially mixture models with finite and infinite components. It was Newcomb (1886) who first developed the concept of the finite mixture distribution for modeling outliers. Many researchers considered classical analysis of a two-component mixture models in their studies. For example, Sankaran and Nair (2005) studied finite mixture of the Pareto distribution. Nadarajah and Kotz (2005) discussed the information matrix for a mixture of two Pareto distributions. Sultan et al. (2007) studied mixture of two inverse Weibull distributions. Kalantan and Alrewely (2019) studied 2-Component Laplace Mixture Model based on complete sample. With regard to censoring schemes based on Bayesian estimation of parameters of mixture models, readers may refer to the works of Saleem et al. (2010), Feroze and Aslam (2014), Ali (2014), Tahir et al. (2016, 2019), Sindhu et al.(2017, 2018), Attoui et al. (2018), Aslam et al.(2018, 2020), Cheema and Aslam (2020) and references cited there in. These contributions in mixture models are great motivations for the recent study.

Wide applicability of mixture modeling, motivates us to develop a two-component mixture of Topp-Leone (TCMTL) distribution for efficient modeling of tensile strength of polyester fibers data. The most frequently used hazard rate function in survival and reliability analysis is the bathtub-shaped one (Demiris et al. (2011)). Thus it imperative to have simple distributions capable of modelling bathtub-shaped hazard rates. The simplest one parameter distribution available in literature which exhibits bathtub-shaped hazard rates is the Topp-Leone distribution with support on (0,1) due to Topp and Leone (1955). Papke and Wooldridge (1996) observed that variables bounded between zero and one arise naturally in many economic setting such as the fraction of total weekly hours spent on working, the proportion of income spent on non-durable consumption, pension plan participation rates, industry market shares, television rating, fraction of land area allocate to agriculture, etc. Furthermore, when the reliability is measured as percentage or ratio, it is important to have models defined on the unit interval (see Genc, 2013) in order to have plausible results. Other motivations are that the proposed TCMTL distribution is capable of modeling increasing and bathtub shaped hazard rate and one real data application shows that it compares well with other three competing lifetime distributions in modeling tensile strength of polyester fibers data.

The purpose of this article is two fold. First we derive some basic properties of the 2-component mixture of Topp-Leone distributions such as moments, moment generating function, conditional moment, mean deviation, Bonferroni and Lorenz curves and the order statistics. Next, we estimate the parameters of the model by using frequentist and Bayesian approaches. For Bayesian analysis, five loss functions and uniform as well as gamma and beta priors are considered to obtain the Bayes estimators and posterior risk of the unknown parameters of the model. Besides, credible intervals (CIs) and highest posterior density (HPD) intervals are also obtained. To the best of our knowledge, 2-component mixture of Topp-Leone distributions is not discussed before using the aforementioned methods of estimation. Through this paper, we purport to provide some guidelines on selecting the best estimator that may be of significant interest to applied statisticians/practitioners/engineers.

The article is organized as follows. In the next section, we introduce the 2-component mixture of Topp-Leone distributions. Some statistical properties of the 2-component mixture of Topp-Leone distributions are presented in Section 3. In Section 4, classical and Bayesian methods of estimation are discussed. Monte Carlo simulation study is carried out to compare the different methods of estimation in Section 5. The potentiality of the new model is illustrated by means of an application to real data in Section 6. Finally, some concluding remarks are addressed in Section 7.

2 2-Component Mixture of Topp-Leone Distribution

Let X is a random variable which follows Topp-Leone distribution as

$$f_i(x) = \lambda_i(2-2x)(2x-x^2)^{\lambda_i - 1}, \quad 0 < x < 1, \lambda_i > 0, i = 1, 2$$

where λ_i $(i = 1, 2)$ are the parameter of Topp-Leone distribution. A finite mixture of 2-component densities with mixing weight p can be written as

$$f(x) = pf_1(x) + (1-p)f_2(x), \quad 0 < p < 1. \tag{1}$$

So the above equation can be written as

$$f(x) = 2p\lambda_1(1-x)[x(2-x)]^{\lambda_1 - 1} + 2\lambda_2(1-p)(1-x)[x(2-x)]^{\lambda_2 - 1}, \tag{2}$$

and the cumulative distribution function for 2-component mixture of Topp-Leone distribution is given by

$$\begin{aligned} F(x) &= pF_1(x) + (1-p)F_2(x), \quad 0 < p < 1 \\ &= p[x(2-x)]^{\lambda_1} + (1-p)[x(2-x)]^{\lambda_2}. \end{aligned} \tag{3}$$

The survival function and hazard rate function of the 2-component mixture of Topp-Leone distribution are, respectively, given by

$$S(x) = 1 - p[x(2-x)]^{\lambda_1} - (1-p)[x(2-x)]^{\lambda_2} \tag{4}$$

and

$$h(x) = \frac{2p\lambda_1(1-x)[x(2-x)]^{\lambda_1 - 1} + 2\lambda_2(1-p)(1-x)[x(2-x)]^{\lambda_2 - 1}}{1 - p[x(2-x)]^{\lambda_1} - (1-p)[x(2-x)]^{\lambda_2}}. \tag{5}$$

We denote a random variable X following the two component mixture Topp-Leone (TCMTL) distribution with parameters p, λ_1 and λ_2 by $X \sim TCMTL(p, \lambda_1, \lambda_2)$. The proposed distribution reduces to the Topp-Leone distribution if $p = 1$.

Figures 1, 2 and 3 display plots of the density and hrf of TCMTL distribution for different values of p, λ_1 and λ_2 . The plots reveal that the TCMTL density is bi-modal and left-skewed unimodal, while the hazard rate shape of TCMTL distribution is increasing and bathtub shaped (U- shaped).

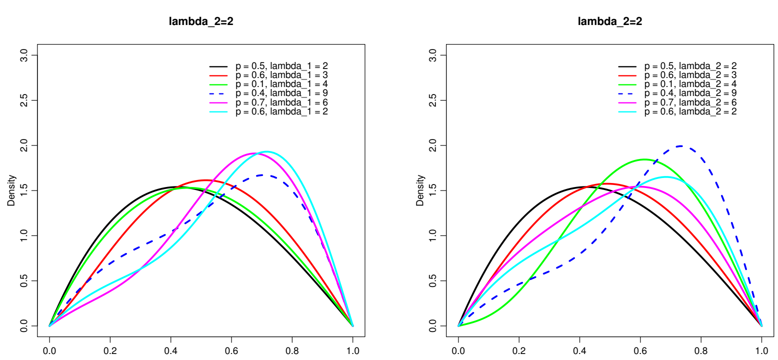

Figure 1: Plots of the density function for selected values of parameters.

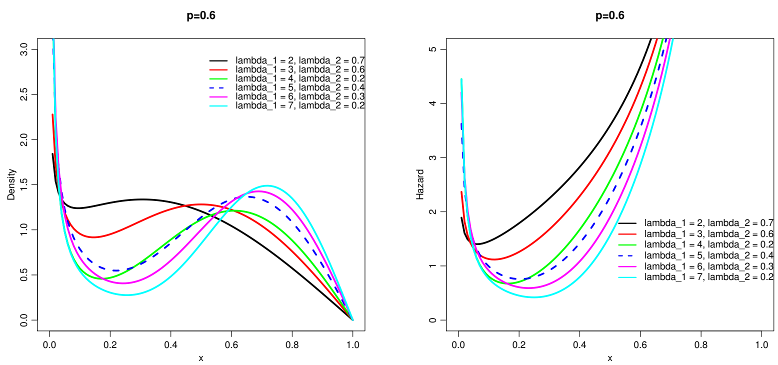

Figure 2: Plots of the density and hazard function for selected values of parameters.

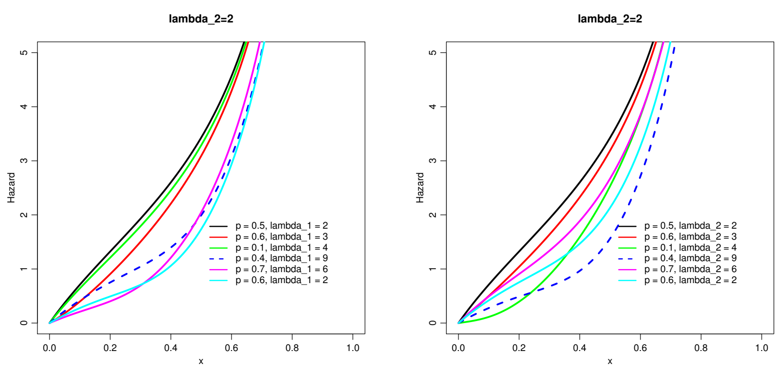

Figure 3: Plots of the hazard function for selected values of parameters.

3 Statistical and mathematical properties

In this section, we devoted to some statistical and mathematical properties of the $TCMTL$ distribution. Bonferroni and Lorenz curves are provided in Appendix B.

3.1 Moments and moment generating function

The moments, incomplete moments, moment generating function, skewness and kurtosis of a probability distribution are important tools to illustrate flexibility of the distribution. The nth moments the $TCMTL$ distribution is given by

$$\begin{aligned} E(X^n) &= \int_0^1 x^n f(x)dx = 2p\lambda_1 \int_0^1 x^{n+\lambda_1-1}(1-x)(2-x)^{\lambda_1-1}dx \\ & -2(1-p)\lambda_2 \int_0^1 x^{n+\lambda_2-1}(1-x)(2-x)^{\lambda_2-1}dx \\ &= 2p\lambda_1[J(n+\lambda_1,\ \lambda_1) - J(n+\lambda_1+1,\ \lambda_1)] \\ & +2(1-p)\lambda_2[J(n+\lambda_2,\ \lambda_2) - J(n+\lambda_2+1,\ \lambda_2)], \end{aligned}$$

where

$$J(a,\ b) = 2^{a+b-1}\int_0^{1/2} z^{a-1}(1-z)^{b-1}dz = 2^{a+b-1}Bet\left(a,\ b;\ \frac{1}{2}\right),$$

and $Bet(.,\ .;\ .)$ denotes the incomplete beta function defined by $Bet(a,\ b;\ u) = \int_0^u x^{a-1}(1-x)^{b-1}dx$. So we have

$$\begin{aligned} \mu'_n &= E(X^n) = p\lambda_1\ 2^{n+2\lambda_1}\left\{Bet\left(n+\lambda_1,\ \lambda_1;\ \frac{1}{2}\right) - 2Bet\left(n+\lambda_1+1,\ \lambda_1;\ \frac{1}{2}\right)\right\} \\ &+ (1-p)\lambda_2\ 2^{n+2\lambda_2}\left\{Bet\left(n+\lambda_2,\ \lambda_2;\ \frac{1}{2}\right) - 2Bet\left(n+\lambda_2+1,\ \lambda_2;\ \frac{1}{2}\right)\right\}. \quad (6) \end{aligned}$$

Here, we obtain the first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the $TCMTL$ distribution. The mean, variance, CV, CS, and CK are given by

$$\sigma^2 = \mu'_2 - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu'_2 - \mu^2}}{\mu} = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{E[(X-\mu)^3]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively. Table 1 lists the first six moments along with CS and CK of the $TCMTL$ distribution for some selected parameters values. These values can be obtained numerically using R. One can see from Table 1 that upto sixth order raw moments are increasing with respect to all considered parameter values, while standard deveiation (SD), coefficient of variation (CV), skewness (CS)and kurtosis (CK) are decreasing with respect to all considered parameter values.

Table 1: Moments of the $TCMTL$ distribution for some selected parameter values

μ_r'	$(p, \lambda_1, \lambda_2) = (0.5, 0.5, 0.5)$	$(p, \lambda_1, \lambda_2) = (0.6, 2, 2)$	$(p, \lambda_1, \lambda_2) = (0.8, 3, 3)$
μ_1'	0.2146	0.4667	0.5428
μ_2'	0.0959	0.2666	0.3357
μ_3'	0.0547	0.1714	0.2262
μ_4'	0.0355	0.1190	0.1619
μ_5'	0.0250	0.0873	0.1212
μ_6'	0.0185	0.0667	0.0939
SD	0.2231	0.2211	0.2025
CV	1.0400	0.4738	0.3730
CS	1.1513	0.1253	0.0711
CK	3.4939	2.1800	2.2658

The moment generating function of $TCMTL$ distribution can be computed as

$$\begin{aligned} M_X(t) &= p\lambda_1 \sum_{i=0}^{\infty} \frac{t^i \; 2^{i+2\lambda_1}}{i!} \left\{ Bet\left(i+\lambda_1,\; \lambda_1;\; \frac{1}{2}\right)\right\} \qquad (7) \\ &\quad - \left\{ Bet\left(i+\lambda_1+1,\; \lambda_1;\; \frac{1}{2}\right)\right\} \\ &+ (1-p)\lambda_2 \sum_{i=0}^{\infty} \frac{t^i \; 2^{i+2\lambda_2}}{i!} \left\{ Bet\left(i+\lambda_2,\; \lambda_2;\; \frac{1}{2}\right)\right\} \\ &\quad - \left\{ Bet\left(i+\lambda_2+1,\; \lambda_2;\; \frac{1}{2}\right)\right\}. \end{aligned}$$

3.2 Conditional moment and mean deviation

Here, we introduce an important lemma which will be used in the next section.

Lemma 1. Let X be a random variable with pdf given in (2) and let $J_n(t) = \int_0^t x^n f(x)dx$. Then we have

$$\begin{aligned} J_n(t) &= p\lambda_1 2^{n+2\lambda_1} \left\{ Bet\left(n+\lambda_1,\; \lambda_1;\; \frac{t}{2}\right)\right\} - \left\{ Bet\left(n+\lambda_1+1,\; \lambda_1;\; \frac{t}{2}\right)\right\} \\ &+ \lambda_2(1-p)2^{i+2\lambda_2} \left\{ Bet\left(n+\lambda_2,\; \lambda_2;\; \frac{t}{2}\right)\right\} \\ &\quad - \left\{ Bet\left(n+\lambda_2+1,\; \lambda_2;\; \frac{t}{2}\right)\right\}. \qquad (8) \end{aligned}$$

Proof. Using the equation (2), we have

$$\begin{aligned} J_n(t) &= \int_0^t x^n f(x)dx = 2p\lambda_1 \int_0^t x^{n+\lambda_1-1}(1-x)(2-x)^{\lambda_1-1}dx \\ &\quad -2(1-p)\lambda_2 \int_0^t x^{n+\lambda_2-1}(1-x)(2-x)^{\lambda_2-1}dx \\ &= 2p\lambda_1[I_t(n+\lambda_1,\; \lambda_1) - I_t(n+\lambda_1+1,\; \lambda_1)] \\ &\quad +2(1-p)\lambda_2[I_t(n+\lambda_2,\; \lambda_2) - I_t(n+\lambda_2+1,\; \lambda_2)] \end{aligned}$$

where

$$I_t(a,\; b) = \int_0^t y^{a-1}(2-y)^{b-1}dy.$$

The proof is complete.

The nth conditional moments of the $TCMTL$ distribution is given by

$$\eta_n(t) = E[X^n|x > t] = \frac{1}{1-F(t)}\int_t^\infty x^n f(x)dx = \frac{1}{S(t)}[E(X^n) - J_n(t)].$$

It can be expressed using (4), (6) and (7). The same remark hold for the nth reversed moments of the 2-component mixture of Topp-Leone distribution is given by

$$m_n(t) = E[X^n|x \leq t] = \frac{1}{F(t)}\int_0^t x^n f(x)dx = \frac{1}{F(t)}J_n(t).$$

4 Parameter Estimation

In this section, we obtain the estimation of the parameters of $TCMTL$ via three methods: maximum likelihood, bootstrap and Bayesian.

4.1 Maximum Likelihood estimators

Suppose $x_1, x_2, \ldots, x_n$ is a random sample from the TCMTL distribution with unknown parameters p, λ_1 and λ_2. The likelihood function of TCMTL distribution is given by :

$$L(p, \lambda_1, \lambda_2) = \prod_{1=1}^{n} f(x_i).$$

Assume that r is another parameter such that $r \in \{0, 1, 2, \ldots, n\}$, we have

$$L(p, \lambda_1, \lambda_2, r) \quad \propto \quad (p\lambda_1)^r \left((1-p)\lambda_2\right)^{n-r} \mathrm{e}^{-\lambda_1\Delta_1 - \lambda_2\Delta_2}, \qquad (9)$$

where

$$\Delta_1 = -\sum_{i=1}^{r} \log(2x_i - x_i^2)$$

and

$$\Delta_2 = -\sum_{i=r+1}^{n} \log(2x_i - x_i^2).$$

So the log-likelihood function is given as

$$\begin{aligned} l(p, \lambda_1, \lambda_2, r) &= r(\log p + \log \lambda_1) + (n-r)(\log(1-p) + \log \lambda_2) \\ &\quad -\lambda_1 \Delta_1 - \lambda_2 \Delta_2. \end{aligned}$$

Without loss of generality, we assume that $\hat{r} \in \{0, 1, 2, \ldots, n\}$. Taking the partial derivatives of the log-likelihood function with respect to parameter vector $(p, \lambda_1, \lambda_2)$, the MLEs of the parameters p, λ_1 and λ_2 are obtained by equating them to zero. The MLEs are obtained as

$$\hat{p} = \frac{\hat{r}}{n}, \hat{\lambda_1} = \frac{\hat{r}}{\Delta_1}, \hat{\lambda_2} = \frac{n - \hat{r}}{\Delta_2}.$$

4.2 Bootstrap estimation

The uncertainty in the parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or non-parametric (resampling with replacement from the original data set) bootstraps resampling method (Efron and Tibshirani (1994)). These two parametric and nonparametric bootstrap procedures are described below.

Parametric bootstrap procedure:

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, employing the MLE procedure based on a random sample.

2. Generate a bootstrap sample $\{x_1^*, \ldots, x_m^*\}$ using θ and obtain its bootstrap estimate, say $\widehat{\theta^*}$, from the bootstrap sample based on the MLE procedure.

3. Repeat Step 2 $NBOOT$ times.

4. Order $\widehat{\theta^*}_1, \ldots, \widehat{\theta^*}_{NBOOT}$ as $\widehat{\theta^*}_{(1)}, \ldots, \widehat{\theta^*}_{(NBOOT)}$. Then, obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals for the parameters.

In case of the $TCMTL$ distribution, the parametric bootstrap estimators (PBs) of p, λ_1 and λ_2 are $\hat{p}_{PB}, \hat{\lambda}_{1PB}$ and $\hat{\lambda}_{2PB}$, respectively.

Nonparametric bootstrap procedure

1. Generate a bootstrap sample $\{x_1^*, \ldots, x_m^*\}$ with replacement from the original data set. Obtain the bootstrap estimate of θ with the MLE procedure, say $\widehat{\theta}^*$, using the bootstrap sample.

2. Repeat Step 2 $NBOOT$ times.

3. Order $\widehat{\theta}^*{}_1, \ldots, \widehat{\theta}^*{}_{NBOOT}$ as $\widehat{\theta}^*{}_{(1)}, \ldots, \widehat{\theta}^*{}_{(NBOOT)}$. Then, obtain γ-quantiles and $100(1-\alpha)\%$ confidence intervals for the parameters.

In case of $TCMTL$ distribution, the nonparametric bootstrap estimators (NPBs) of p, λ_1 and λ_2 are $\hat{p}_{NPB}, \hat{\lambda}_{1NPB}$ and $\hat{\lambda}_{2NPB}$, respectively.

4.3 Bayesian Estimation:

Bayesian inference procedure have been taken into consideration by many statistical researchers, especially, researchers in the field of survival analysis and reliability engineering. This section discusses the Bayes procedure to derive the point and interval estimates of the parameters p, λ_1 and λ_2 based on complete sample. In our Bayesian analysis, we have assumed five loss functions namely, squared error loss function (SELF), weighted squared error loss function (WSELF), modified squared error loss function (MSELF), precautionary loss function (PLF) and K-loss function (KLF). The expression of the loss functions and the corresponding Bayes estimators and posterior risk functions are provided in Table 2 [see, Ahmad et al. (2020) and Dey et al. (2015, 2017)]. It may be noted that if all the parameters p, λ_1 and λ_2 are unknown, conjugate prior may not exists. In such cases, there are several ways to choose the priors. However, if one has adequate information about the parameter(s), it is better to choose informative prior(s), otherwise, it is preferable to use noninformative prior(s). Thus it is not unreasonable to propose uniform priors and independent gamma and beta priors for the parameters to estimate the Bayes estimators and posterior risks, because gamma and beta distributions are very flexible.

Table 2: Bayes estimator and posterior risk under different loss functions

loss function	Bayes estimator	Posterior risk
$L_1 = SELF = (\theta - d)^2$	$E(\theta\|x)$	$Var(\theta\|x)$
$L_2 = WSELF = \frac{(\theta-d)^2}{\theta}$	$(E(\theta^{-1}\|x))^{-1}$	$E(\theta\|x) - (E(\theta^{-1}\|x))^{-1}$
$L_3 = MSELF = \left(1 - \frac{d}{\theta}\right)^2$	$\frac{E(\theta^{-1}\|x)}{E(\theta^{-2}\|x)}$	$1 - \frac{E(\theta^{-1}\|x)^2}{E(\theta^{-2}\|x)}$
$L_4 = PLF = \frac{(\theta-d)^2}{d}$	$\sqrt{E(\theta^2\|x)}$	$2\left(\sqrt{E(\theta^2\|x)} - E(\theta\|x)\right)$
$L_5 = KLF = \left(\sqrt{\frac{d}{\theta}} - \sqrt{\frac{\theta}{d}}\right)$	$\sqrt{\frac{E(\theta\|x)}{E(\theta^{-1}\|x)}}$	$2\left(\sqrt{E(\theta\|x)E(\theta^{-1}\|x)} - 1\right)$

Suppose that the likelihood function associated with main model $f(x, \Theta)$, where $\Theta = (\Theta_1, ..., \Theta_k)$ is a parameter vector, is given as $L(\Theta, data)$. Under the assumption of independent prior distributions $\pi(\Theta_i)$ $(i = 1, ..., k)$, the joint posterior distribution is given as

$$\pi^*(\Theta|data) = K \prod_{i=1}^{k} \pi(\Theta_i) L(\Theta, data),$$

where

$$K = \frac{1}{\int \cdots \int \prod_{i=1}^{k} \pi(\Theta_i) L(\Theta, data) d\Theta_1 ... d\Theta_k}.$$

Moreover, the marginal posterior *pdf* of parameter θ_i $(i = 1, ..., k)$, can be given

$$\pi(\Theta_i|data) = \int ... \int \pi^*(\Theta|data) d\Theta_{j_1} ... d\Theta_{j_k}, \tag{10}$$

where $i, j_1, ..., j_k = 1, ..., k$, $i \neq j_1 ... \neq j_k$ and also Θ_i is ith member of a vector Θ.

In the next, we consider two prior distributions for the parameters of $TCMTL$ distribution. In each case, we provide exact Bayesian estimator and associated posterior risk under the assumption of loss functions in Table 2.

4.3.1 Bayes estimators under uniform priors based on different loss functions

We assume that the parameters p, λ_1 and λ_2 of $TCMTL$ distribution have independent uniform prior distributions as given by

$$p \sim U(0,1),\ \lambda_1 \sim U(0,\infty),\ \lambda_2 \sim U(0,\infty),$$

Hence, the joint prior density function is formulated as follows:

$$\pi(p,\lambda_1,\lambda_2) \quad \propto \quad 1;\ 0<p<1,\ \lambda_1>0,\ \lambda_2>0. \tag{11}$$

The joint posterior distribution in terms of a given likelihood function $L(data)$ and joint prior distribution $\pi(p,\lambda_1,\lambda_2)$ is defined as

$$\pi^*(p,\lambda_1,\lambda_2|data) \propto \pi(p,\lambda_1,\lambda_2)L(data). \tag{12}$$

Hence, we get joint posterior density of parameters p, λ_1 and λ_2 for complete sample data by combining the likelihood function (9) and joint prior density (11). Therefore, the joint posterior density function is given by

$$\pi^*(p,\lambda_1,\lambda_2|\underline{x},\hat{r}) \quad = \quad K\,(p\lambda_1)^{\hat{r}}\,((1-p)\lambda_2)^{n-\hat{r}}\,\mathrm{e}^{-\lambda_1\Delta_1-\lambda_2\Delta_2}$$

where K is given as

$$\begin{aligned} K^{-1} &= \int_0^1\int_0^\infty\int_0^\infty (p\lambda_1)^{\hat{r}}\,((1-p)\lambda_2)^{n-\hat{r}}\,\mathrm{e}^{-\lambda_1\Delta_1-\lambda_2\Delta_2}d\lambda_1 d\lambda_2 dp. \\ &= \frac{Be(\hat{r}+1,n-\hat{r}+1)\Gamma(\hat{r}+1)\Gamma(n-\hat{r}+1)}{\Delta_1^{\hat{r}+1}\Delta_2^{n-\hat{r}+1}}, \end{aligned}$$

and

$$\Delta_1 = -\sum_{i=1}^{\hat{r}}\log(2x_i-x_i^2), \Delta_2 = -\sum_{i=\hat{r}+1}^{n}\log(2x_i-x_i^2).$$

Hence, the marginal posterior density of parameters p, λ_1 and λ_2 are in explicit form as

$$p|(\underline{x},\hat{r}) \sim Beta(\hat{r}+1,n-\hat{r}+1),$$

$$\lambda_1|(\underline{x},\hat{r}) \sim Gamma(\hat{r}+1,\Delta_1),$$

and

$$\lambda_2|(\underline{x},\hat{r}) \sim Gamma(n-\hat{r}+1,\Delta_2).$$

4.3.2 Bayes estimators and posterior risk based on $SELF$

The most used loss function is $SELF$ which is symmetrical loss function and assigns equal losses to over estimation and underestimation. The Bayes Estimators and the corresponding posterior risk under $SELF$ can be obtain as

$$\hat{p} = E[p|\underline{x}] = \frac{Beta(\hat{r}+2,n-\hat{r}+1)}{Beta(\hat{r}+1,n-\hat{r}+1)}$$

and

$$\begin{aligned}\rho(\hat{p}) &= E[p^2|\underline{x}] - E[p|\underline{x}]^2 = \frac{Beta(\hat{r}+3,n-\hat{r}+1)}{Beta(\hat{r}+1,n-\hat{r}+1)} \\ &\quad -[\frac{Beta(\hat{r}+2,n-\hat{r}+1)}{Beta(\hat{r}+1,n-\hat{r}+1)}]^2\end{aligned}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i=1,2)$ and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = E[\lambda_1|\underline{x}] = \frac{\hat{r}+1}{\Delta_1},\ \hat{\lambda}_2 = E[\lambda_2|\underline{x}] = \frac{n-\hat{r}+1}{\Delta_2},$$

and

$$\rho(\hat{\lambda}_1) = E[\lambda_1^2|\underline{x}] - E[\lambda_i|\underline{x}]^2 = \frac{\Gamma(\hat{r}+3)}{\Gamma(\hat{r}+1)\Delta_1^2} - \left(\frac{\hat{r}+1}{\Delta_1}\right)^2,$$

$$\rho(\hat{\lambda}_2) = E[\lambda_2^2|\underline{x}] - E[\lambda_2|\underline{x}]^2 = \frac{\Gamma(n-\hat{r}+3)}{\Gamma(n-\hat{r}+1)\Delta_2^2} - \left(\frac{n-\hat{r}+1}{\Delta_2}\right)^2.$$

4.3.3 Bayes estimators and posterior risk based on $WSELF$

The Bayes estimators and the corresponding posterior risk under $WSELF$ can be obtain as

$$\hat{p} = E[p^{-1}|\underline{x}]^{-1} = \frac{Beta(\hat{r}+1,n-\hat{r}+1)}{Beta(\hat{r},n-\hat{r}+1)}$$

and

$$\rho(\hat{p}) = E[p|\underline{x}] - E[p^{-1}|\underline{x}]^{-1} = \frac{Beta(\hat{r}+2, n-\hat{r}+1)}{Beta(\hat{r}+1, n-\hat{r}+1)} - \frac{Beta(\hat{r}+1, n-\hat{r}+1)}{Beta(\hat{r}, n-\hat{r}+1)}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = E[\lambda_1^{-1}|\underline{x}]^{-1} = \frac{\hat{r}}{\Delta_1}, \ \hat{\lambda}_2 = E[\lambda_2^{-1}|\underline{x}]^{-1} = \frac{n-\hat{r}}{\Delta_2},$$

and

$$\rho(\hat{\lambda}_1) = E[\lambda_1|\underline{x}] - E[\lambda_1^{-1}|\underline{x}]^{-1} = \frac{\hat{r}+1}{\Delta_i} - \frac{\hat{r}}{\Delta_1},$$

$$\rho(\hat{\lambda}_2) = E[\lambda_2|\underline{x}] - E[\lambda_2^{-1}|\underline{x}]^{-1} = \frac{n-\hat{r}+1}{\Delta_2} - \frac{n-\hat{r}}{\Delta_2}.$$

4.3.4 Bayes estimators and posterior risk based on $MSELF$

The Bayes estimators and the corresponding posterior risk under $MSELF$ can be obtain as

$$\hat{p} = \frac{E[p^{-1}|\underline{x}]}{E[p^{-2}|\underline{x}]} = \frac{Beta(\hat{r}, n-\hat{r}+1)}{Beta(\hat{r}-1, n-\hat{r}+1)}$$

and

$$\rho(\hat{p}) = 1 - \frac{E[p^{-1}|\underline{x}]^2}{E[p^{-2}|\underline{x}]} = 1 - \frac{\left(Beta(\hat{r}, n-\hat{r}+1)\right)^2}{Beta(\hat{r}+1, n-\hat{r}+1)Beta(\hat{r}-1, n-\hat{r}+1)}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = \frac{E[\lambda_1^{-1}|\underline{x}]}{E[\lambda_1^{-2}|\underline{x}]} = \frac{\hat{r}-1}{\Delta_1}, \ \hat{\lambda}_2 = \frac{E[\lambda_2^{-1}|\underline{x}]}{E[\lambda_2^{-2}|\underline{x}]} = \frac{n-\hat{r}-1}{\Delta_2},$$

and

$$\rho(\hat{\lambda}_1) = 1 - \frac{E[\lambda_1^{-1}|\underline{x}]^2}{E[\lambda_1^{-2}|\underline{x}]} = 1 - \frac{\hat{r}-1}{\hat{r}},$$

$$\rho(\hat{\lambda}_2) = 1 - \frac{E[\lambda_2^{-1}|\underline{x}]^2}{E[\lambda_2^{-2}|\underline{x}]} = 1 - \frac{n-\hat{r}-1}{n-\hat{r}}.$$

4.3.5 Bayes estimators and posterior risk based on PLF

The Bayes estimators and the corresponding posterior risk under $WSELF$ can be obtained as

$$\hat{p} = \sqrt{E[p^2|\underline{x}]} = \sqrt{\frac{Beta(\hat{r}+3, n-\hat{r}+1)}{Beta(\hat{r}+1, n-\hat{r}+1)}}$$

and

$$\begin{aligned}\rho(\hat{p}) &= 2\left(\sqrt{E[p^2|\underline{x}]} - E[p|\underline{x}]\right) \\ &= 2\left(\sqrt{\frac{Beta(\hat{r}+3, n-\hat{r}+1)}{Beta(\hat{r}+1, n-\hat{r}+1)}} - \frac{Beta(\hat{r}+2, n-\hat{r}+1)}{Beta(\hat{r}+1, n-\hat{r}+1)}\right),\end{aligned}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$ and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = \sqrt{E[\lambda_1^2|\underline{x}]} = \sqrt{\frac{\Gamma(\hat{r}+3)}{\Gamma(\hat{r}+1)\Delta_1^2}}, \ \hat{\lambda}_2 = \sqrt{E[\lambda_2^2|\underline{x}]} = \sqrt{\frac{\Gamma(n-\hat{r}+3)}{\Gamma(n-\hat{r}+1)\Delta_2^2}},$$

and

$$\rho(\hat{\lambda}_1) = 2\left(\sqrt{E[\lambda_1^2|\underline{x}]} - E[\lambda_1|\underline{x}]\right) = 2\left(\sqrt{\frac{\Gamma(\hat{r}+3)}{\Gamma(\hat{r}+1)\Delta_1^2}} - \frac{\Gamma(\hat{r}+2)}{\Gamma(\hat{r}+1)\Delta_1}\right),$$

$$\rho(\hat{\lambda}_2) = 2\left(\sqrt{E[\lambda_2^2|\underline{x}]} - E[\lambda_2|\underline{x}]\right) = 2\left(\sqrt{\frac{\Gamma(n-\hat{r}+3)}{\Gamma(n-\hat{r}+1)\Delta_2^2}} - \frac{\Gamma(\hat{r}+2)}{\Gamma(\hat{r}+1)\Delta_2}\right),$$

4.3.6 Bayes estimators and posterior risk based on KLF

The Bayes Estimators and the corresponding posterior risk under KLF can be obtain as

$$\hat{p} = \sqrt{\frac{E[p|\underline{x}]}{E[p^{-1}|\underline{x}]}} = \sqrt{\frac{Beta(\hat{r}+2, n-\hat{r}+1)}{Beta(\hat{r}, n-\hat{r}+1)}}$$

and

$$\begin{aligned}\rho(\hat{p}) &= 2\left(\sqrt{E[p|\underline{x}]E[p^{-1}|\underline{x}]}-1\right)\\ &= 2\left(\sqrt{\frac{Beta(\hat{r}+2,n-\hat{r}+1)Beta(\hat{r},n-\hat{r}+1)}{\big(Beta(\hat{r}+1,n-\hat{r}+1)\big)^2}}-1\right)\end{aligned}$$

respectively. Similarly the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = \sqrt{\frac{E[\lambda_1|\underline{x}]}{E[\lambda_1^{-1}|\underline{x}]}} = \sqrt{\frac{\hat{r}(\hat{r}+1)}{\Delta_1^2}},\ \hat{\lambda}_2 = \sqrt{\frac{E[\lambda_2|\underline{x}]}{E[\lambda_2^{-1}|\underline{x}]}} = \sqrt{\frac{(n-\hat{r})(n-\hat{r}+1)}{\Delta_2^2}},$$

and

$$\rho(\hat{\lambda}_1) = 2\left(\sqrt{E[\lambda_1|\underline{x}]E[\lambda_1^{-1}|\underline{x}]}-1\right) = 2\left(\sqrt{\frac{\hat{r}+1}{\hat{r}}}-1\right),$$

$$\rho(\hat{\lambda}_2) = 2\left(\sqrt{E[\lambda_2|\underline{x}]E[\lambda_2^{-1}|\underline{x}]}-1\right) = 2\left(\sqrt{\frac{n-\hat{r}+1}{n-\hat{r}}}-1\right).$$

4.4 Bayes estimators and posterior risk based under the beta and gamma priors

We assume that the parameters p, λ_1 and λ_2 of $TCMTL$ distribution have independent prior distributions as

$$p \sim Beta(a,b), \lambda_1 \sim Gamma(c,d), \lambda_2 \sim Gamma(e,f),$$

where a,b,c,d,e and f are positive constants. Hence, the joint prior density function is formulated as follows:

$$\begin{aligned}\pi(p,\lambda_1,\lambda_2) \;\propto\;& p^{a-1}(1-p)^{b-1}\lambda_1^{c-1}\lambda_2^{e-1}e^{-(d\lambda_1+f\lambda_2)};\\ & 0<p<1,\ \lambda_1>0,\ \lambda_2>0.\end{aligned} \tag{13}$$

We call this joint prior distribution as informative prior(IP). We can now get the joint posterior density of parameters p, λ_1 and λ_2 for complete sample data by combining the likelihood function (9) and joint prior density (13). Therefore, the joint posterior density function is given by

$$\begin{aligned}\pi^*(p, \lambda_1, \lambda_2 | \underline{x}, \hat{r}) &= K p^{\hat{r}+a-1}(1-p)^{n-\hat{r}+b-1}(\lambda_1)^{\hat{r}+c-1}(\lambda_2)^{n-\hat{r}+e-1} \\ &\times \ \mathrm{e}^{-\lambda_1(\Delta_1+d)-\lambda_2(\Delta_2+f)},\end{aligned}$$

where $K = \frac{Be(\hat{r}+a, n-\hat{r}+b)\Gamma(\hat{r}+c)\Gamma(n-\hat{r}+e)}{\Delta_1^{\hat{r}+a}\Delta_2^{n-\hat{r}+e}}$. Hence, the marginal posterior density of parameters p, λ_1 and λ_2 have known densities as

$$p|(\underline{x}, \hat{r}) \sim Beta(\hat{r}+a, n-\hat{r}+b), \ \lambda_1|(\underline{x}, \hat{r}) \sim Gamma(\hat{r}+c, \Delta_1+d),$$

and

$$\lambda_2|(\underline{x}, \hat{r}) \sim Gamma(n-\hat{r}+e, \Delta_2+f).$$

4.4.1 Bayes estimators and posterior risk based on $SELF$

The Bayes estimators and the corresponding posterior risk under $SELF$ can be obtain as

$$\hat{p} = E[p|\underline{x}] = \frac{Beta(\hat{r}+a+1, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+b)}$$

and

$$\begin{aligned}\rho(\hat{p}) &= E[p^2|\underline{x}] - E[p|\underline{x}]^2 = \frac{Beta(\hat{r}+a+2, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+b)} \\ &- [\frac{Beta(\hat{r}+a+1, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+b)}]^2\end{aligned}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = E[\lambda_1|\underline{x}] = \frac{\hat{r}+c}{\Delta_1+d}, \ \hat{\lambda}_2 = E[\lambda_2|\underline{x}] = \frac{n-\hat{r}+b}{\Delta_2+f},$$

and

$$\rho(\hat{\lambda}_1) = E[\lambda_1^2|\underline{x}] - E[\lambda_1|\underline{x}]^2 = \frac{\Gamma(\hat{r}+c+2)}{\Gamma(\hat{r}+c)(\Delta_1+d)^2} - \left(\frac{\hat{r}+c}{\Delta_1+d}\right)^2,$$

$$\rho(\hat{\lambda}_2) = E[\lambda_2^2|\underline{x}] - E[\lambda_2|\underline{x}]^2 = \frac{\Gamma(n-\hat{r}+e+2)}{\Gamma(n-\hat{r}+e)(\Delta_2+f)^2} - \left(\frac{n-\hat{r}}{\Delta_2+f}\right)^2.$$

4.4.2 Bayes estimators and posterior risk based on $WSELF$

The Bayes estimators and the corresponding posterior risk under $WSELF$ can be obtain as

$$\hat{p} = E[p^{-1}|\underline{x}]^{-1} = \frac{Beta(\hat{r}+a, n-\hat{r}+b)}{Beta(\hat{r}+a-1, n-\hat{r}+b)}$$

and

$$\begin{aligned}\rho(\hat{p}) &= E[p|\underline{x}] - E[p^{-1}|\underline{x}]^{-1} = \frac{Beta(\hat{r}+a+1, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+1b)} \\ &- \frac{Beta(\hat{r}+a, n-\hat{r}+b)}{Beta(\hat{r}+a-1, n-\hat{r}+b)}\end{aligned}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = E[\lambda_1^{-1}|\underline{x}]^{-1} = \frac{\hat{r}+c-1}{\Delta_1+d}, \ \hat{\lambda}_2 = E[\lambda_2^{-1}|\underline{x}]^{-1} = \frac{n-\hat{r}+e-1}{\Delta_2+f},$$

and

$$\rho(\hat{\lambda}) = E[\lambda_1|\underline{x}] - E[\lambda_1^{-1}|\underline{x}]^{-1} = \frac{\hat{r}+c}{\Delta_1+d} - \frac{\hat{r}+c-1}{\Delta_1+d},$$

$$\rho(\hat{\lambda}_i) = E[\lambda_i|\underline{x}] - E[\lambda_i^{-1}|\underline{x}]^{-1} = \frac{n-\hat{r}+e}{\Delta_2+f} - \frac{n-\hat{r}+e-1}{\Delta_2+f}.$$

4.4.3 Bayes estimators and posterior risk based on $MSELF$

The Bayes estimators and the corresponding posterior risk under $MSELF$ can be obtain as

$$\hat{p} = \frac{E[p^{-1}|\underline{x}]}{E[p^{-2}|\underline{x}]} = \frac{Beta(\hat{r}+a-1, n-\hat{r}+b)}{Beta(\hat{r}+a-2, n-\hat{r}+b)}$$

and

$$\rho(\hat{p}) = 1 - \frac{E[p^{-1}|\underline{x}]^2}{E[p^{-2}|\underline{x}]} = 1 - \frac{\left(Beta(\hat{r}+a-1, n-\hat{r}+b)\right)^2}{Beta(\hat{r}+a, n-\hat{r}+b)Beta(\hat{r}+a-2, n-\hat{r}+b)}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = \frac{E[\lambda_1^{-1}|\underline{x}]}{E[\lambda_1^{-2}|\underline{x}]} = \frac{\hat{r}+c-2}{\Delta_1+d},\ \hat{\lambda}_2 = \frac{E[\lambda_2^{-1}|\underline{x}]}{E[\lambda_2^{-2}|\underline{x}]} = \frac{n-\hat{r}+e-2}{\Delta_2+f},$$

and

$$\rho(\hat{\lambda}_1) = 1 - \frac{E[\lambda_1^{-1}|\underline{x}]^2}{E[\lambda_1^{-2}|\underline{x}]} = 1 - \frac{\hat{r}+c-2}{\hat{r}+c-1},$$
$$\rho(\hat{\lambda}_2) = 1 - \frac{E[\lambda_2^{-1}|\underline{x}]^2}{E[\lambda_2^{-2}|\underline{x}]} = 1 - \frac{n-\hat{r}+e-2}{n-\hat{r}+e-1}.$$

4.4.4 Bayes estimators and posterior risk based on PLF

The Bayes estimators and the corresponding posterior risk under PLF can be obtain as

$$\hat{p} = \sqrt{E[p^2|\underline{x}]} = \sqrt{\frac{Beta(\hat{r}+a+2, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+b)}}$$

and

$$\begin{aligned}\rho(\hat{p}) &= 2\left(\sqrt{E[p^2|\underline{x}]} - E[p|\underline{x}]\right)\\ &= 2\left(\sqrt{\frac{Beta(\hat{r}+a+2, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+b)}} - \frac{Beta(\hat{r}+a+1, n-\hat{r}+b)}{Beta(\hat{r}+a, n-\hat{r}+b)}\right)\end{aligned}$$

respectively. Similarly, the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = \sqrt{E[\lambda_1^2|\underline{x}]} = \sqrt{\frac{\Gamma(\hat{r}+c+2)}{\Gamma(\hat{r}+c)(\Delta_1+d)^2}},\ \hat{\lambda}_2 = \sqrt{E[\lambda_2^2|\underline{x}]} = \sqrt{\frac{\Gamma(n-\hat{r}+e+2)}{\Gamma(n-\hat{r}+e)(\Delta_2+f)^2}},$$

and

$$\begin{aligned}\rho(\hat{\lambda}_1) &= 2\left(\sqrt{E[\lambda_1^2|\underline{x}]} - E[\lambda_1|\underline{x}]\right)\\ &= 2\left(\sqrt{\frac{\Gamma(\hat{r}+c+2)}{\Gamma(\hat{r}+c)(\Delta_1+d)^2}} - \frac{\Gamma(\hat{r}+c+1)}{\Gamma(\hat{r}+c)(\Delta_1+d)}\right),\end{aligned}$$

$$\begin{aligned}\rho(\hat{\lambda}_2) &= 2\left(\sqrt{E[\lambda_2^2|\underline{x}]} - E[\lambda_2|\underline{x}]\right)\\ &= 2\left(\sqrt{\frac{\Gamma(n-\hat{r}+e+2)}{\Gamma(n-\hat{r}+e)(\Delta_2+f)^2}} - \frac{\Gamma(n-\hat{r}+e+1)}{\Gamma(n-\hat{r}+e)(\Delta_2+f)}\right).\end{aligned}$$

4.4.5 Bayes estimators and posterior risk based on KLF

The Bayes estimators and the corresponding posterior risk under KLF can be obtain as

$$\hat{p} = \sqrt{\frac{E[p|\underline{x}]}{E[p^{-1}|\underline{x}]}} = \sqrt{\frac{Beta(\hat{r}+a+1, n-\hat{r}+b)}{Beta(\hat{r}+a-1, n-\hat{r}+b)}}$$

and

$$\begin{aligned}\rho(\hat{p}) &= 2\left(\sqrt{E[p|\underline{x}]E[p^{-1}|\underline{x}]} - 1\right)\\ &= 2\left(\sqrt{\frac{Beta(\hat{r}+a+1, n-\hat{r}+b)Beta(\hat{r}+a-1, n-\hat{r}+b)}{\big(Beta(\hat{r}+a, n-\hat{r}+b)\big)^2}} - 1\right)\end{aligned}$$

respectively. Similarly the Bayes estimator of $\lambda_i (i = 1, 2)$, and associated posterior risk are given respectively, as follows:

$$\hat{\lambda}_1 = \sqrt{\frac{E[\lambda_1|\underline{x}]}{E[\lambda_1^{-1}|\underline{x}]}} = \sqrt{\frac{(\hat{r}+c)(\hat{r}+c-1)}{(\Delta_1+d)^2}},$$

$$\hat{\lambda}_2 = \sqrt{\frac{E[\lambda_2|\underline{x}]}{E[\lambda_2^{-1}|\underline{x}]}} = \sqrt{\frac{(n-\hat{r}+e)(n-\hat{r}+e-1)}{(\Delta_2+f)^2}},$$

and

$$\rho(\hat{\lambda}_i) = 2\left(\sqrt{E[\lambda_i|\underline{x}]E[\lambda_i^{-1}|\underline{x}]} - 1\right) = 2\left(\sqrt{\frac{\hat{r}+c}{\hat{r}+c-1}} - 1\right)$$

$$\rho(\hat{\lambda}_i) = 2\left(\sqrt{E[\lambda_i|\underline{x}]E[\lambda_i^{-1}|\underline{x}]} - 1\right) = 2\left(\sqrt{\frac{n-\hat{r}+e}{n-\hat{r}+e-1}} - 1\right).$$

5 Simulation

In this section, we provide some simulation results based on maximum likelihood and Bayesian methods.

5.1 Simulation study for Maximum likelihood method

We consider the performance of the MLEs of the parameters with respect to sample size n and for different parameter values for the $TCMTL$ distribution. Let $\hat{p}$, $\widehat{\lambda_1}$ and $\widehat{\lambda_2}$ be the MLEs of the parameters p, λ_1 and λ_2, respectively. We calculate the mean squared errors ($MSEs$) and bias of the MLEs of the parameters p, λ_1 and λ_2 based on simulation results of 2000 independence replications. Results of the simulation study are summarized in Table 3 for different values of n, p, λ_1 and λ_2. From Table 3, we observe that with the increase in the values of p and λ_1, MSEs decrease. Also, with the decrease in the values of λ_2, MSEs decrease. While, when p decreases and λ_1 and λ_2 increases, MSEs of p and λ_1 increases whereas MSEs of λ_2 decreases. Further, as the sample size increases, the average biases and the MSEs decrease. It verifies the consistency properties of MLEs.

5.2 Simulation study for Bayesian method

This section is devoted to calculate the bias and posterior risk values of Bayes estimators under different loss functions based on Monte Carlo simulation. We generated samples of different sizes $n = \{40, 50, 60, 70, 100\}$ from the $TCMTL$ distribution for real value of parameters (i) $(p, \lambda_1, \lambda_2) = (0.5, 2, 3)$ and (ii) $(p, \lambda_1, \lambda_2) = (0.45, 4, 5)$. Tables 4 and 5 report the bias and posterior risk values of Bayes estimators under prior distributions defined in (11) and aforementioned five loss functions as shown in Table 2. These results provided by considering hyper parameters values as $(a, b) = (2, 1), (c, d) = (4, 2), (e, f) = (6, 2)$ for case (i) and $(a, b) = (10, 1), (c, d) = (8, 2), (e, f) = (12, 1)$ for case (ii) based on 10000 replicates of $MCMC$ procedure in OpenBUGS software. The posterior risk is reported for different loss functions which is different from mean squared error because posterior risk is more comprehensive measure for comparison of different loss functions in Bayesian setup. It is evident

Table 3: MSE and bias (values in parentheses) of the MLEs of the parameters p, λ_1 and λ_2.

		$p = 0.5$	$\lambda_1 = 2$	$\lambda_2 = 3$
n	40	0.2660 (-0.0563)	0.6715 (-0.3981)	1.4307 (1.0646)
	50	0.2638 (-0.0480)	0.6324 (-0.3822)	1.4198 (1.0450)
	60	0.2624 (-0.0441)	0.6164 (-0.3498)	1.3922 (1.0011)
	70	0.2550 (-0.0362)	0.6261 (-0.3614)	1.4087 (1.0095)
	100	0.2315 (-0.0359)	0.6181 (-0.3438)	1.3404 (0.9047)
		$p = 0.45$	$\lambda_1 = 4$	$\lambda_2 = 5$
n	40	0.3202 (-0.2880)	1.9729 (-1.8759)	0.2660 (-0.0867)
	50	0.3205 (-0.2738)	1.9159 (-1.8179)	0.2671 (-0.0829)
	60	0.3273 (-0.2653)	1.8785 (-1.7791)	0.2651 (-0.0830)
	70	0.3153 (-0.2143)	1.7129 (-1.8118)	0.2557 (-0.0851)
	100	0.3021 (-0.2271)	1.6489 (-1.7481)	0.2117 (-0.0821)
		$p = 0.65$	$\lambda_1 = 3$	$\lambda_2 = 1$
	40	0.0879 (-0.2965)	0.5461 (0.7390)	0.2534 (0.5034)
	50	0.0620 (-0.2491)	0.3922 (0.6262)	0.1812 (0.4256)
	60	0.0467 (-0.2162)	0.2992 (0.5470)	0.1376 (0.3709)
	70	0.0368 (-0.1919)	0.2389 (0.4888)	0.1091 (0.3303)
	100	0.0213 (-0.1459)	0.1415 (0.3761)	0.0636 (0.2523)
		$p = 0.65$	$\lambda_1 = 5$	$\lambda_2 = 1$
	40	0.0071 (-0.0843)	0.14226 (0.3772)	0.0986 (0.3140)
	50	0.0050 (-0.0706)	0.1096 (0.3311)	0.0678 (0.2603)
	60	0.0037 (-0.0612)	0.0882 (0.2969)	0.0501 (0.2238)
	70	0.0029 (-0.0543)	0.0730 (0.2702)	0.0389 (0.1972)
	100	0.0017 (-0.0414)	0.0468 (0.2162)	0.0218 (0.1476)
		$p = 0.5$	$\lambda_1 = 5$	$\lambda_2 = 2$
	40	0.1204 (-0.3470)	4.1462 (2.0362)	0.5812 (0.7624)
	50	0.0943 (-0.3072)	3.1551 (1.7763)	0.4389 (0.6625)
	60	0.0770 (-0.2776)	2.5127 (1.5852)	0.3488 (0.5906)
	70	0.0644 (-0.2538)	2.0411 (1.4287)	0.2861 (0.5349)
	100	0.0422 (-0.2055)	1.2682 (1.1262)	0.1799 (0.4242)

from Tables 4 and 5 that MSELF and KLF have smaller posterior risk as compare to other loss functions. As the sample size increases, the posterior risk of all Bayes estimates decreases which verifies the consistency properties of all the estimators. We also observe from Tables 4 and 5 that in the case of parameter p, SELF performs better than their counterparts in terms of posterior risk, while in case of parameters (λ_1 and λ_2), both the loss functions MSELF and KLF perform equally and they have the least posterior risks as compared to other loss functions.

Table 4: Bias and posterior risk values of Bayesian estimators under different loss functions based on simulation data set for different sample sizes.

$n=40$	$p=0.5$		$\lambda_1=2$		$\lambda_2=3$	
Loss function	Estimate	Risk	Estimate	Risk	Estimate	Risk
SELF	-0.0081	0.0027	0.4720	0.1375	-0.4281	0.1424
WSELF	-0.0137	0.0055	0.4162	0.0558	-0.4830	0.0549
MSELF	-0.0194	0.0117	0.3603	0.0231	-0.5372	0.0215
PLF	-0.0054	0.0054	0.4996	0.0553	-0.5004	0.0551
KLF	-0.0109	0.0114	0.4439	0.0230	-0.4557	0.0217
$n=50$	$p=0.5$		$\lambda_1=2$		$\lambda_2=3$	
SELF	-0.0044	0.0023	0.4553	0.1121	-0.4478	0.1121
WSELF	-0.0092	0.0048	0.4098	0.0438	-0.4916	0.0438
MSELF	-0.0141	0.0099	0.3645	0.0188	-0.5355	0.0175
PLF	-0.0021	0.0046	0.4780	0.0454	-0.5220	0.0437
KLF	-0.0068	0.0097	0.4324	0.0188	-0.4698	0.0174
$n=60$	$p=0.5$		$\lambda_1=2$		$\lambda_2=3$	
SELF	-0.0020	0.0020	0.4550	0.0925	-0.4760	0.0916
WSELF	-0.0060	0.0039	0.4173	0.0377	-0.5123	0.0362
MSELF	-0.0010	0.0080	0.3795	0.0156	-0.5484	0.0145
PLF	-0.0019	0.0039	0.4737	0.0375	-0.5263	0.0361
KLF	-0.0040	0.0080	0.4360	0.0155	-0.4942	0.0145
$n=70$	$p=0.5$		$\lambda_1=2$		$\lambda_2=3$	
SELF	-0.0027	0.0016	0.4470	0.0740	-0.4798	0.0828
WSELF	-0.0060	0.0032	0.4171	0.0299	-0.5122	0.0324
MSELF	-0.0093	0.0066	0.3874	0.0123	-0.5441	0.0128
PLF	-0.0011	0.0032	0.4621	0.0302	-0.5379	0.0327
KLF	-0.0044	0.0066	0.4320	0.0123	-0.4960	0.0130
$n=100$	$p=0.5$		$\lambda_1=2$		$\lambda_2=3$	
SELF	-0.0039	0.0011	0.4365	0.05531	-0.5072	0.0585
WSELF	-0.0062	0.0023	0.4138	0.0226	-0.5307	0.0234
MSELF	-0.0086	0.0048	0.3912	0.0093	-0.5540	0.0095
PLF	-0.0027	0.0023	0.4478	0.0226	-0.5522	0.0234
KLF	-0.0050	0.0047	0.4251	0.0093	-0.5190	0.0095

Table 5: Bias and posterior risk values of Bayesian estimators under different loss functions based on simulation data set for different sample sizes.

$n = 40$	$p = 0.45$		$\lambda_1 = 4$		$\lambda_2 = 5$	
Loss function	Estimate	Risk	Estimate	Risk	Estimate	Risk
SELF	0.0984	0.0025	0.5610	0.4343	0.4522	0.5656
WSELF	0.0937	0.0046	0.4656	0.0954	0.3477	0.1045
MSELF	0.0890	0.0087	0.3701	0.0214	0.2425	0.0197
PLF	0.1007	0.0046	0.6084	0.0947	-0.3916	0.1032
KLF	0.0960	0.0085	0.5131	0.0213	0.3997	0.0194
$n = 50$	$p = 0.45$		$\lambda_1 = 4$		$\lambda_2 = 5$	
SELF	0.0906	0.0021	0.5748	0.3681	0.2914	0.4457
WSELF	0.0866	0.0040	0.4953	0.0795	0.2064	0.0850
MSELF	0.0825	0.0076	0.4170	0.0174	0.1207	0.0165
PLF	0.0930	0.0039	0.6149	0.0801	-0.3851	0.0839
KLF	0.0886	0.0074	0.5349	0.0176	0.2487	0.0163
$n = 60$	$p = 0.45$		$\lambda_1 = 4$		$\lambda_2 = 5$	
SELF	0.0824	0.0019	0.5431	0.2860	0.1947	0.3989
WSELF	0.0789	0.0035	0.4800	0.0631	0.1184	0.0764
MSELF	0.0753	0.0066	0.4168	0.0141	0.0423	0.0149
PLF	0.0841	0.0035	0.5744	0.0627	-0.4256	0.0765
KLF	0.0806	0.0066	0.5114	0.0140	0.1564	0.0149
$n = 70$	$p = 0.45$		$\lambda_1 = 4$		$\lambda_2 = 5$	
SELF	0.0781	0.0016	0.5518	0.2480	0.1130	0.2967
WSELF	0.0750	0.0031	0.4972	0.0546	0.0547	0.0583
MSELF	0.0718	0.0060	0.4426	0.0121	-0.0038	0.0116
PLF	0.0796	0.0030	0.5789	0.0543	-0.4211	0.0579
KLF	0.0765	0.0059	0.5244	0.0121	0.0838	0.0115
$n = 100$	$p = 0.45$		$\lambda_1 = 4$		$\lambda_2 = 5$	
SELF	0.0708	0.0011	0.5550	0.1961	-0.0662	0.2088
WSELF	0.0686	0.0022	0.5122	0.0428	-0.1086	0.0424
MSELF	0.0664	0.0043	0.4697	0.0094	-0.1511	0.0087
PLF	0.0719	0.0022	0.5765	0.0430	-0.4235	0.0422
KLF	0.0697	0.0042	0.5336	0.0094	-0.0875	0.0086

6 Application of $TCMTL$

The goal here is to show the application of $TCMTL$ model under the methods (maximum likelihood, bootstrap and Bayesian) discussed in the Section 4 via a real data set. In order to achieve this goal, we consider a real data set related to 30 measurements of tensile strength of polyester fibers taken from Quesenberry and Hales (1980). The data are:

0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926.

Graphical measure: The total time test (TTT) plot due to Aarset (1987) is an important graphical approach to verify whether the data can be applied to a specific distribution or not. The TTT plot for this data set presented in Figure 5 indicates that the empirical hazard rate functions of tensile strength of polyester fibers data is upside-down bathtub shaped. Therefore, the $TCMTL$ distribution is appropriate to fit these data.

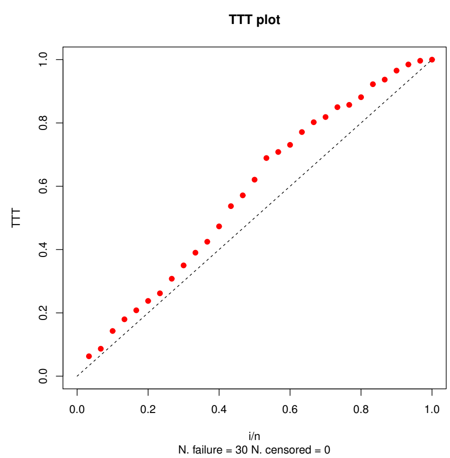

Figure 4: Scaled-TTT plot of the tensile strength of polyester fibers data set.

We compare the fits of the $TCMTL$ distribution with some competative models (Topp-Leone (TL), Beta and Kumaraswamy (Ku)) and their densities are given by:

$$f_{TL}(x) = 2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1}, \quad 0 < x < 1,$$

$$f_{Beta}(x) = \frac{x^{\alpha-1}(1-x)^{\beta}}{Beta(a,b)}, \quad 0 < x < 1,$$

$$f_{Ku}(x) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \quad 0 < x < 1.$$

In Table 6, we report the $MLEs$ of the parameters and the goodness of fit measures, namely Kolmogorov-Smirnov (KS), Anderson-Darling (A^*) and Cramér-von Mises (W^*) statistics and its p-value (parenthesis) for the given data set. Based on the results in Table 6, we conclude that the $TCMTL$ distribution provides the best fit with the lowest values of these statistics. For a visual comparison, the histogram of the data set, fitted *pdf* of the $TCMTL$ distribution are plotted in Figure 5. Also, the plots of empirical and fitted *cdf* function, $P-P$ plot and $Q-Q$ plot for the $TCMTL$ are displayed in Figure 5.

Table 6: Parameter estimates and goodness of fit measures with corresponding p-values

Model	$MLEs$ of parameters	A^*	W^*	$K.S$
$TCMTL$	$\hat{p}=0.0591$ $\hat{\lambda}_1=64.328$ $\hat{\lambda}_2=1.0446$	0.1236 (0.999)	0.0172 (0.999)	0.061(0.999)
$Topp-Leone$	$\hat{\alpha}=1.0392$	0.3277 (0.9153)	0.0332 (0.9666)	0.0665 (0.9981)
$Beta$	$\hat{\alpha}=0.9666,\ \hat{\beta}=1.6204$	0.1703 (0.9966)	0.0221 (0.9953)	0.0669 (0.9979)
$Kumaraswamy$	$\hat{\alpha}=0.9627,\ \hat{\beta}=1.6084$	0.1633 (0.9975)	0.0207 (0.9969)	0.0650 (0.9987)

6.1 Bootstrap inference for $TCMTL$ parameters

In this section, we obtain point and 95% confidence interval (CI) for the parameters of the $TCMTL$ distribution by parametric and non-parametric bootstrap methods. We provide results of bootstrap methods in Table 7 for tensile strength data set.

Table 7: Bootstrap point and interval estimation of the parameters p, λ_1 and λ_2 for the tensile strength data.

	parametric bootstrap		non-parametric bootstrap	
	point estimation	CI	point estimation	CI
p	0.064	(0, 0.517)	0.078	(0, 0.447)
λ_1	92.790	(7.50, 362)	96.785	(27.52, 168.27)
λ_2	1.048	(0.505, 2.090)	1.041	(0, 1.742)

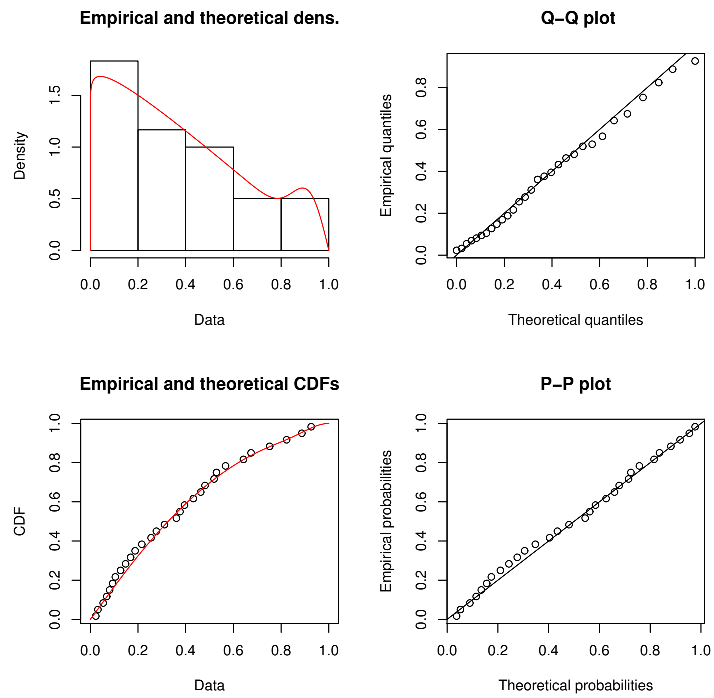

Figure 5: Histogram and fitted density plot, plot of empirical and fitted *cdf*, $P-P$ plot and $Q-Q$ plot for the tensile strength of polyester fibers data set.

Next, we report Bayes estimates and associated posteriors risks based on five loss functions and different priors in Tables 8, 9, 10 and 11. Tables 9 and 11 provides 95% credible and HPD intervals for each parameter of the $TCMTL$ distribution. The posterior samples extracted by using Gibbs sampling technique. Moreover, we provide the posterior summary plots in Figures 8, 9 and 10. These plots confirm that the convergence of Gibbs sampling process is occurred. By comparing the results (Tables 8-11) of uniform priors and IP in terms of their respective posterior risks and interval estimates for the parameters under the assumed loss functions, we may conclude that Bayes estimates and interval estimates based on uniform priors are more efficient than the IP. For the case of parameter p, it is observed that SELF performs better than their counterparts in terms of posterior risk, while in case of parameters (λ_1 and λ_2), both the loss functions MSELF and KLF perform equally and they have the least posterior risks as compared to other loss functions.

Table 8: Bayesian estimates and their posterior risks of the parameters under different loss functions based on tensile strength of polyester fibers data set (under the beta and gamma priors)

Bayes	$\widehat{P}$		$\widehat{\lambda_1}$		$\widehat{\lambda_2}$	
Loss functions	Estimate	Risk	Estimate	Risk	Estimate	Risk
SELF	0.0550	8.485e-05	64.3744	2.3216	1.1393	0.0422
WSELF	0.0535	0.0015	64.3383	0.0362	1.1020	0.0373
MSELF	0.0520	0.0287	64.3020	0.0006	1.0645	0.0340
PLF	0.0558	0.0015	64.3925	0.0361	1.1581	0.0367
KLF	0.0543	0.0286	64.3564	0.0006	1.1205	0.0336

Table 9: Credible and HPD intervals of the parameters p, λ_1 and λ_2 for tensile strength of polyester fibers data set (under beta and gamma priors).

	Credible interval	HPD interval
p	(0.0484, 0.0611)	(0.0373, 0.0729)
λ_1	(63.330, 65.430)	(61.560,67.340)
λ_2	(0.9905, 1.2760)	(0.7588, 1.5350)

Table 10: Bayesian estimates and their posterior risks of the parameters under different loss functions based on tensile strength of polyester fibers data set (under the uniform priors).

Bayes	$\widehat{P}$		$\widehat{\lambda_1}$		$\widehat{\lambda_2}$	
Loss functions	Estimate	Risk	Estimate	Risk	Estimate	Risk
SELF	0.0676	5.073e-06	64.0371	0.0014	1.1924	0.0201
WSELF	0.0675	7.938e-05	64.3363	2.199e-05	1.1770	0.0154
MSELF	0.0674	0.0012	64.0378	3.431e-07	1.1630	0.0119
PLF	0.0676	7.505e-05	64.0373	2.201e-05	1.2008	0.0168
KLF	0.0675	0.0012	64.0560	3.434e-07	1.1847	0.0130

Table 11: Credible and HPD intervals of the parameters p, λ_1 and λ_2 for tensile strength of polyester fibers data set (under the uniform priors).

	Credible interval	HPD interval
p	(0.0665 0.0693)	(0.06284, 0.07000)
λ_1	(64.0100 64.0500)	(64.0010,64.1100)
λ_2	(1.0850 1.2680)	(1.0010, 1.4710)

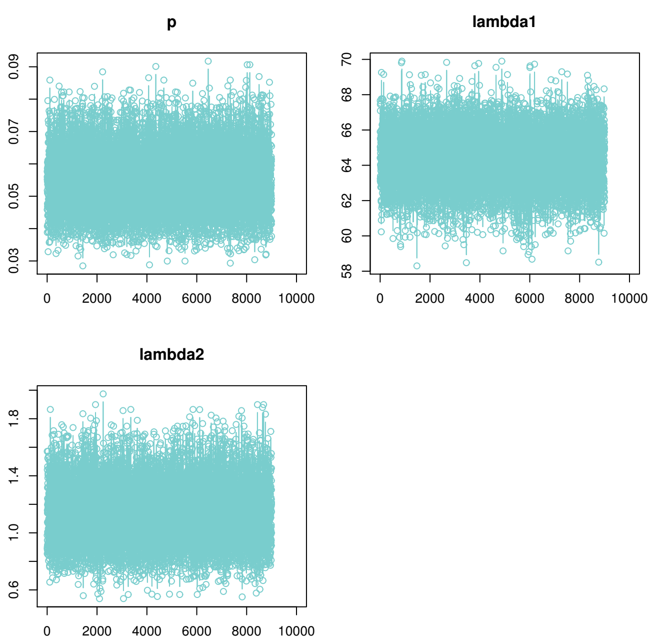

Figure 6: Plots of Bayesian analysis and performance of Gibbs sampling for tensile strength of polyester fibers data set. Trace plots of each parameter of $TCMTL$ distribution.

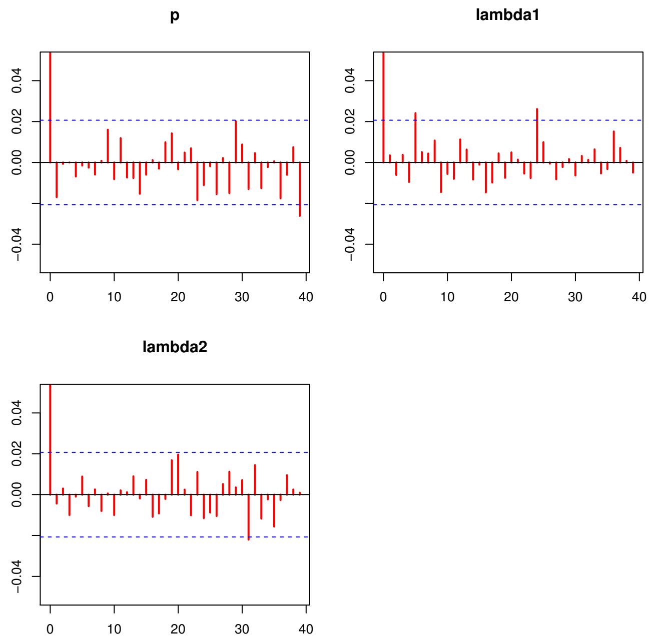

Figure 7: Plots of Bayesian analysis and performance of Gibbs sampling for tensile strength of polyester fibers data set. Autocorrelation plots of each parameter of $TCMTL$ distribution.

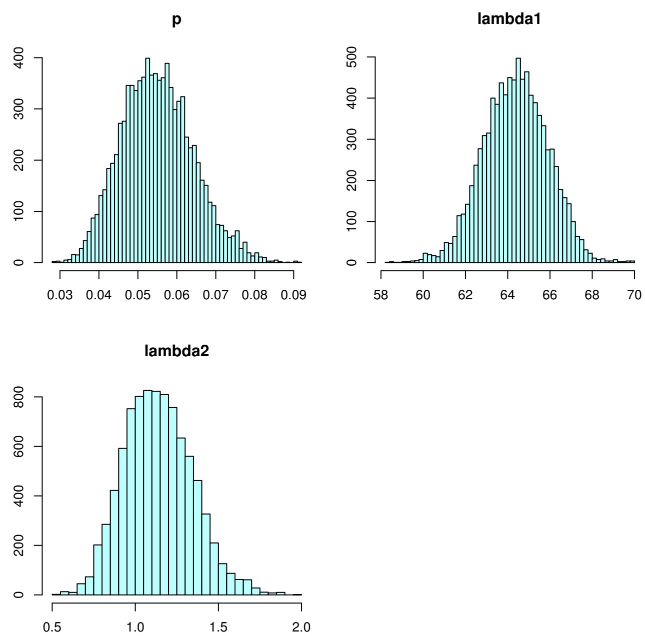

Figure 8: Plots of Bayesian analysis and performance of Gibbs sampling for tensile strength of polyester fibers data set. Histogram plots of each parameter of $TCMTL$ distribution.

7 Conclusion

In this study, we have presented the classical and Bayesian methods of estimation of $TCMTL$ distribution. In addition, we have derived some statistical properties of $TCMTL$ distribution. From simulation study and real-life data analysis, we may conclude that the Bayesian estimation has an advantage because of its small posterior risks as compared to the MLE method. If we compare the estimates with respect to loss functions, the MSELF and KLF performs better as compared to their counterparts. Thus, for the $TCMTL$ distribution, either MSELF or KLF is a suitable choice for the estimation of parameters. It is also observed that, under different loss functions, uniform prior performs better as compared to IP. In future, this work can be extended using 3-component mixture of TL distribution or some more flexible probability distribution using informative and non-informative priors based on censored data.

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Appendix: A

R and OpenBUGS codes

```
x=c(0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188,
0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529,
0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926)
output < − fitdist( x,"TL",start=list(t=u1,l1=u2,l2=u3))
summary(output); plot(output); gofstat(output); ks.test(x,"pTL", 0.0591,64,328,
1,0446).
n=length(x)
model{
for (i in 1:n) { x[i]∼dTL(x[i],t,l1,l2)
    }
t ∼ dunif(0,1)
l1 ∼ dgamma(0.0001,0.0001)
```

```
l2 ~ dgamma(0.0001,0.0001)
}
```

Appendix: B

Bonferroni and Lorenz curves

Bonferroni and Lorenz curves are used to measure the inequality of the distribution of a random variable X. They are applied in many fields such as economics, reliability, demography, insurance, etc. These index are defined as:

$$B(P)=\frac{1}{P\mu}\int_0^Q xf(x)dx$$

and

$$L(P)=\frac{1}{\mu}\int_0^Q xf(x)dx,$$

respectively, where $Q=F^{-1}(P)$. If X has the *pdf* in (2), then, Bonferroni curve of the 2-component mixture of Topp-Leone distribution can be computed as

$$\begin{aligned} B(P) &= \frac{p\lambda_1 2^{2\lambda_1+1}}{P\mu}\left\{Bet\left(\lambda_1+1,\ \lambda_1;\ \frac{Q}{2}\right)-2Bet\left(\lambda_1+2,\ \lambda_1;\ \frac{Q}{2}\right)\right\} \\ &\quad +\frac{(1-p)\lambda_1 2^{2\lambda_2+1}}{P\mu}\left\{Bet\left(\lambda_2+1,\ \lambda_1;\ \frac{Q}{2}\right)-2Bet\left(\lambda_2+2,\ \lambda_2;\ \frac{Q}{2}\right)\right\}. \end{aligned}$$

The Lorenz curve of the $TCMTL$ distribution is

$$\begin{aligned} L(P) &= \frac{p\lambda_1 2^{2\lambda_1+1}}{\mu}\left\{Bet\left(\lambda_1+1,\ \lambda_1;\ \frac{Q}{2}\right)-2Bet\left(\lambda_1+2,\ \lambda_1;\ \frac{Q}{2}\right)\right\} \\ &\quad +\frac{(1-p)\lambda_1 2^{2\lambda_2+1}}{\mu}\left\{Bet\left(\lambda_2+1,\ \lambda_1;\ \frac{Q}{2}\right)-2Bet\left(\lambda_2+2,\ \lambda_2;\ \frac{Q}{2}\right)\right\}. \end{aligned}$$

The area between the line $L(F(x))=F(x)$ and the Lorenz curve, known as the area of concentration, may be regarded as a measure of inequality of income, so it is important in insurance, economics and other fields like reliability and medicine. Figures 4 and 5 show some shapes for $L(P)$ and $B(P)$ functions.

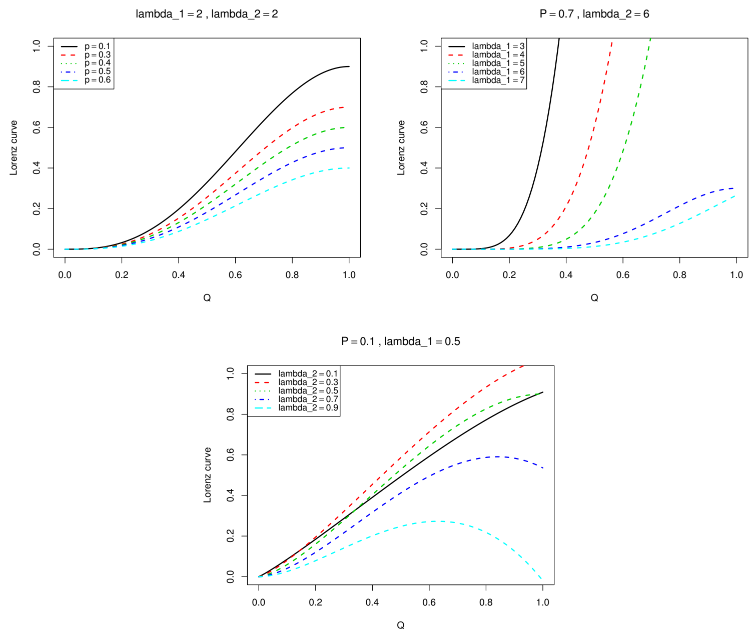

Figure 9: Some plots of Lorenz curve for selected parameter values.

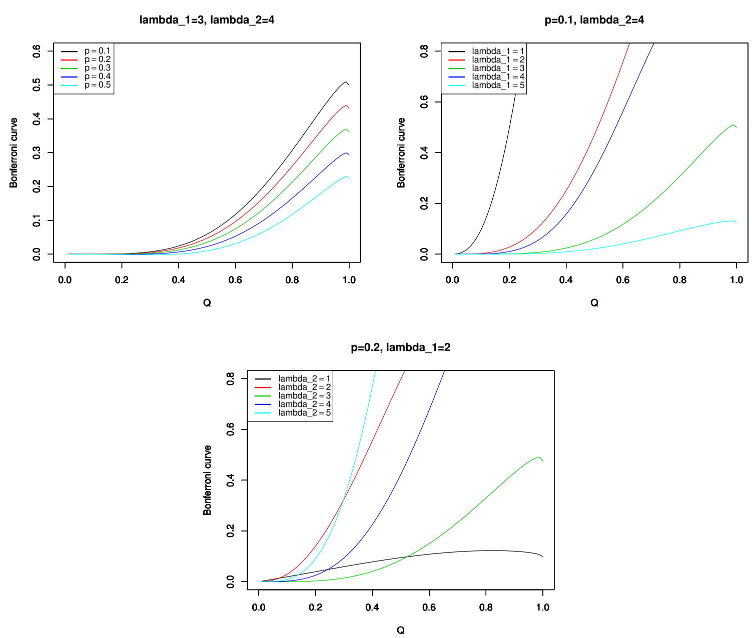

Figure 10: Some plots of Bonferroni curve for selected parameter values.