Journal of Mathematical Extension
Vol. 16, No. 1, (2022) (10)1-12
URL: https://doi.org/10.30495/JME.2022.1738
ISSN: 1735-8299
Original Research Paper

Some Results of Fixed Points on C^*-Algebra-Valued b-Metric Spaces

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Abstract. In this paper we present some fixed point theorems for mappings satisfying contractive conditions on C^*-algebra-valued b-metric spaces. Specifically we extend some fixed point results on metric spaces to C^*-algebra valued case and prove the related fixed point theorems. Also several theorems in existence of n-periodic points are given.

AMS Subject Classification: 37C25; 47H10; 46L07
Keywords and Phrases: fixed point, C^*-algebra-valued b-metric spaces, orbitally continuous mapping, n-periodic point

1 Introduction

Forty years ago, Bogdan Rzepecki [12], presented a generalized metric d_E on a set X in a way that $d_E : X \times X \to S$ where E is a Banach space

Received: July 2020; Accepted: December 2020
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and S is a normal cone with partial order $\preceq$. In that article, the author generalized the fixed point theorems of Maia type [10]. Seven years later, Shy-Der Lin [5] presented the notion metric spaces by substituting real numbers with cone in the metric function. In 2007, the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space was declared by Huang and Zhang [3]. In 2013 Liu and Xu [7] came to some new conclusions by substituting the usual real contraction constant with a vector constant and scalar multiplication with vector multiplication and confirmed that their results were different from those in cone metric spaces. Lately, many results on fixed point theorems have been developed in cone metric spaces [6],[1]. Ma et al. introduced the concept of C^*-algebra-valued metric spaces and give a fixed point theorem for C^*-algebra-valued contraction mappings [9]. In [8], C^*-algebra-valued b-metric spaces were presented and some applications related to operator and integral equations were given. To make it clear, now we remind some fundamental definitions, notations, and conclusions of C^*-algebras. The descriptions of C^*-algebras can be attained in [11].

A complete normed algebra is called a *Banach algebra*. An *involution* on an algebra $\mathbb{E}$ is a conjugate-linear map $e \to e^*$ on $\mathbb{E}$, such that $e^{**} = e$ and $(eg)^* = g^*e^*$ for all $e, g \in \mathbb{E}$. The pair $(\mathbb{E}, *)$ is called an *involutive* algebra, or a $*$-*algebra*. A *Banach* $*$-*algebra* is a $*$-*algebra* $\mathbb{E}$ together with a complete submultiplicative norm such that $\|e^*\| = \|e\|$ $(e \in \mathbb{E})$. If, in addition, $\mathbb{E}$ has a unit such that $\|1\| = 1$, we call $\mathbb{E}$ a *unital Banach* $*$-*algebra*. A C^*-*algebra* is a unital Banach $*$-algebra such that, $\|e^*e\| = \|e\|^2 (e \in \mathbb{E})$.

An element a of a C^*-algebra $\mathbb{E}$ is positive if $e = e^*$ and its spectrum $\sigma(e) \subset I\!R^+$. We write $e \succeq 0$ to mean that e is positive, and denote by $\mathbb{E}^+$ the set of positive elements of $\mathbb{E}$.

Theorem 1.1. *[11] Let $\mathbb{E}$ be a C^*-algebra and $e \in \mathbb{E}^+$. Then there exists a unique element $g \in \mathbb{E}^+$ such that $g^2 = e$.*

If $\mathbb{E}$ is a C^*-algebra, we make $\mathbb{E}_h = \{t \in \mathbb{E} : t = t^*\}$ a poset by defining $e \preceq g$ to mean $g - e \in \mathbb{E}^+$. The relation $\preceq$ is translation-invariant; that is, $e \preceq g \Rightarrow e + k \preceq g + k$ for all $e, g, k \in \mathbb{E}_h$. Also, $e \preceq g \Rightarrow re \preceq rg$ for all $r \in I\!R^+$, and $e \preceq g \Leftrightarrow -e \succeq -g$. Let

$\mathbb{E}' = \{e \in \mathbb{E} : eg = ge, \quad \forall g \in \mathbb{E}\}$, and $\mathbb{E}'_+ = \mathbb{E}^+ \cap \mathbb{E}'$.

Theorem 1.2. *[9] If $\mathbb{E}$ is a C^*-algebra, then*

1. *The set $\mathbb{E}^+$ is equal to $\{e^*e \mid e \in \mathbb{E}\}$.*
2. *If $e, g \in \mathbb{E}_h$ and $k \in \mathbb{E}$, then $e \preceq g \Rightarrow k^*ek \preceq k^*gk$.*
3. *If $0_{\mathbb{E}} \preceq e \preceq g$, then $\|e\| \leq \|g\|$.*
4. *If $\mathbb{E}$, is unital and e, g are positive invertible elements, then $e \preceq g \Rightarrow 0_{\mathbb{E}} \preceq g^{-1} \preceq e^{-1}$.*

Notice that in a C^*-algebra, one cannot conclude that $eg \succeq 0_{\mathbb{E}}$ wherever $e, g \succeq 0_{\mathbb{E}}$.

Definition 1.3. Let $\mathbb{E}$ be a C^*-algebra, and P be a nonempty set. Let $b \in \mathbb{E}'_+$ such that $\|b\| \geq 1$. A mapping $D =: P \times P \to \mathbb{E}^+$ is said to be a C^*-algebra-valued b-metric on P if the following conditions hold for all $e, g, k \in P$:

1. $D(e, g) = 0_{\mathbb{E}}$ if and only if $e = g$.
2. $D(e, g) = D(g, e)$.
3. $D(e, k) \preceq b[D(e, g) + D(g, k)]$.

The triple $(P, \mathbb{E}, D)$ is called a C^*-*algebra-valued* b-*metric space* with coefficient b. By definition a C^*-*algebra-valued metric space* is a C^*-algebra-valued I-metric space for identity element I. In fact, an ordinary C^*-algebra valued metric space is a C^*-algebra valued b-metric space but, the following example illustrates that, in general, a C^*-algebra-valued b-metric space is not necessary a C^*-algebra-valued metric space.

Example 1.4. $P = l_p$ is the set of all sequences $\{e_n\}$ in $I\!R$ such that $\sum_{n=1}^{\infty} |e_n|^p < \infty$ where $0 < p < 1$. If $\mathbb{E} = M_2(I\!R)$ and $e = \{e_n\}, g = \{g_n\}$ in l_p, define $D : P \times P \to \mathbb{E}$ as follows:

$$D(e, g) = \begin{pmatrix} (\sum_{n=1}^{\infty} |e_n - g_n|^p)^{\frac{1}{p}} & 0 \\ 0 & (\sum_{n=1}^{\infty} |e_n - g_n|^p)^{\frac{1}{p}}. \end{pmatrix}$$

We have D is a C^*-algebra-valued b-metric with coefficient $b = \begin{pmatrix} 2^{\frac{1}{p}} & 0 \\ 0 & 2^{\frac{1}{p}} \end{pmatrix}$ such that $\|b\| = 2^{\frac{1}{p}}$. But D is not a usual C^*-algebra-valued metric on P.

Definition 1.5. Let $(P, \mathbb{E}, D)$ be a C^*-algebra-valued b-metric space, $t \in P$, and $\{t_n\}$ a sequence in P. Then:

1. $\{t_n\}$ converges to t with respect to D whenever for any $\varepsilon > 0$, there is an $N \in I\!N$ such that $\|(D(t_n, t)\| < \varepsilon$ for all $n > N$. We denote this by $Lim_{n\to\infty} t_n = t$ or $t_n \to t$.

2. $\{t_n\}$ is a Cauchy sequence with respect to D if for each $\varepsilon > 0$, there is an $N \in I\!N$ such that $\|(D(t_n, t_m)\| < \varepsilon$ for all $n, m > N$.

3. $(P, \mathbb{E}, D)$ is complete if every Cauchy sequence in P is convergent with respect to D.

Definition 1.6. Suppose that $(P, \mathbb{E}, D)$ is a C^*-algebra-valued b-metric space. We call a mapping $F : P \to P$ is a contractive mapping on P, if there exists an $\lambda \in \mathbb{E}$ with $\|\lambda\| < 1$ such that $D(Ft, Fs) \preceq \lambda^* D(t, s)\lambda$ for all $t, s \in P$.

Theorem 1.7. *[9] If $(P, \mathbb{E}, D)$ is a complete C^*-algebra-valued b-metric space and F is a contractive mapping, then F has a unique fixed point in P for F.*

Theorem 1.8. *[9] Let $(P, \mathbb{E}, D)$ be a complete C^*-algebra-valued metric space. Suppose that a mapping $F : P \to P$ satisfies*
$D(Ft, Fs) \preceq \lambda[D(Ft, s) + D(Fs, t)]$, for all $t, s \in P$ where $\lambda \in \mathbb{E}'_+$ and $\|\lambda\| < \dfrac{1}{2}$. Then there exists a unique fixed point in P.

Theorem 1.9. *[8] Let $(P, \mathbb{E}, D)$ be a complete C^*-algebra-valued b-metric space. Suppose that a mapping $F : P \to P$ satisfies*
$D(Ft, Fs) \preceq \lambda[D(Ft, t) + D(Fs, s)]$, for all $t, s \in P$ where $\lambda \in \mathbb{E}'_+$ and $\|\lambda\| < \dfrac{1}{2}$. Then there exists a unique fixed point in P.

2 Main Results

As we mentioned before, many researchers investigated fixed point theorems in C^*-algebra-valued metric spaces. In this section, we prove some fixed point theorems for mapping with different contractive conditions in the setting of these spaces.

Definition 2.1. A mapping F on a C^*-algebra-valued b-metric space $(P, \mathbb{E}, D)$ is said to be orbitally continuous if $Lim_{i\to\infty} F^{n_i}(t) = z$ implies that $Lim_{i\to\infty} F(F^{n_i})(t) = Fz$ for any t in P, where $\{n_i\}_{i\geq 1} \subset I\!N$. The C^*-algebra-valued b-metric space $(P, \mathbb{E}, D)$ is named F-orbitally complete if every cauchy sequence of the form $\{F^{n_i}(t)\}_{i\in I\!N}$, $t \in \mathbb{E}$ converges on $(P, \mathbb{E}, D)$.

In the following, under some appropriate conditions, we generalize [11,theorem 2.3] from the ordinary metric space to the C^*-algebra case.

Theorem 2.2. *Let $F : P \to P$ be an orbitally continuous mapping over C^*-algebra-valued metric space $(P, \mathbb{E}, D)$ and $(P, \mathbb{E}, D)$ is F-orbitally complete such that*

$$U(t, s) - D(Ft, s) \preceq \lambda^* D(t, s)\lambda \tag{1}$$

where $\lambda \in \mathbb{E}^+$ with $\|\lambda\| < 1$ and every $t, s \in P$, when

$$U(t, s) \in \{D(t, Ft), D(Ft, Fs), D(Fs, s)\}.$$

Then, for all $t \in P$, we have $\{F^n(t)\} \to z$ when z is a fixed point of F.

Proof. Fix $t_0 \in P$. For $n \geq 1$ set $t_1 = Ft_0$ and recursively $t_{n+1} = F(t_n) = F^{n+1}(t_0)$. It is clear that the sequence $\{t_n\}_{n\in I\!N}$, is Cauchy when the equation $t_{n+1} = t_n$ holds for some $n \in I\!N$. Consider the case $t_{n+1} \neq t_n$ for all $n \in I\!N$. By replacing t, s with t_{n-1} , t_n in (1), respectively, one can get

$$U(t_{n-1}, t_n) - D(Ft_{n-1}, t_n) \preceq \lambda^* D(t_{n-1}, t_n)\lambda$$

but $\|\lambda\| < 1$, so this case yields contradiction. Thus

$$D(t_n, t_{n+1}) \preceq \lambda^* D(t_{n-1}, t_n)\lambda$$

By B, we denote the element $D(t_1, t_0)$ in $\mathbb{E}$. So we have

$$\begin{aligned} D(t_n, t_{n+1}) &\preceq \lambda^* D(t_n, t_{n-1})\lambda \preceq (\lambda^*)^2 D(t_{n-1}, t_{n-2})\lambda^2 \\ &\preceq \ldots \preceq (\lambda^*)^n D(t_1, t_0)\lambda^n = (\lambda^*)^n B\lambda^n. \end{aligned}$$

So for $n+1>m$, we get

$$\begin{aligned} D(t_{n+1},t_m) &\preceq D(t_{n+1},t_n)+D(t_n,t_{n-1})+\ldots+D(t_{m+1},t_m) \\ &= \sum_{k=m}^{n}(\lambda^*)^k B\lambda^k = \sum_{k=m}^{n}(\lambda^*)^k B^{\frac{1}{2}}B^{\frac{1}{2}}\lambda^k \\ &= \sum_{k=m}^{n}(B^{\frac{1}{2}}\lambda^k)^*(B^{\frac{1}{2}}\lambda^k) \preceq \sum_{k=m}^{n}\|B^{\frac{1}{2}}\|^2\|\lambda^k\|^2 I \\ &\preceq \|B^{\frac{1}{2}}\|^2\sum_{k=m}^{n}\|\lambda\|^{2k}I \preceq \|B^{\frac{1}{2}}\|^2\frac{\|\lambda\|^{2k}}{1-\|\lambda\|} \to 0 \end{aligned}$$

as $m\to\infty$. Therefore $\{t_n\}$ is a Cauchy sequence in $(P,\mathbb{E},D)$. By the completeness of $(P,\mathbb{E},D)$ there exists $z\in P$ such that $lim_{n\to\infty}t_n = z$. Since $(P,\mathbb{E},D)$ is F-orbitally complete,

$$Lim_{n\to\infty}t_n = Lim_{n\to\infty}F^n(t_0) = z.$$

Since F is orbitally continuous, we have

$$F(z) = Lim_{n\to\infty}F(F^n(t_0)) = Lim_{n\to\infty}F^{n+1}(t_0) = z.$$

Then, z is a fixed point for F. □

Motivated by the ideas and results presented in [2], for b-metric spaces, we state and prove the following theorem in a C^*-algebra-valued b-metric space.

Theorem 2.3. *Suppose that F be an orbitally continuous self-map in the C^*-algebra-valued b-metric space $(P,\mathbb{E},D)$ and exist e_1,e_2,e_3,e_4 in $\mathbb{E}$ such that self mapping F satisfies the conditions*
$\| e_1+e_3-I_{\mathbb{E}} \| < \frac{1}{\|b\|}$, $\quad e_1+e_3\succeq I_{\mathbb{E}}$, $\quad 0_{\mathbb{E}}\preceq e_2-e_4$.

$$\begin{aligned} e_1D(Ft,Fs)+(I-e_1)[D(t,Ft)+D(s,Fs)]+e_2[D(s,Ft)+D(t,Fs)] \\ \preceq e_3D(t,s)+e_4D(t,F^2t) \qquad (2) \end{aligned}$$

for all $t,s\in P$. Then, F has at least one fixed point.

Proof. Let $t_0\in P$ be arbitrary. Construct a sequence $\{t_n\}$ as follows:

$t_{n+1} := Ft_n; \quad n = 0, 1, 2, \dots$
Put $t = t_n$ and $s = t_{n+1}$ in (2), then

$$\begin{aligned} e_1 D(t_{n+1}, t_{n+2}) &+ (I - e_1)[D(t_n, t_{n+1}) + D(t_{n+1}, t_{n+2})] \\ &+ e_2[D(t_{n+1}, t_{n+1}) + D(t_n, t_{n+2})] \\ &\preceq e_3 D(t_n, t_{n+1}) + e_4 D(t_n, t_{n+2}). \end{aligned}$$

By a simple calculation, one can get

$$D(t_{n+1}, t_{n+2}) + (e_2 - e_4) D(t_n, t_{n+2}) \preceq (e_3 + e_1 - I) D(t_n, t_{n+1})$$

which implies that

$$D(t_{n+1}, t_{n+2}) \preceq k D(t_n, t_{n+1}) \tag{3}$$

where $k = e_3 + e_1 - I$. Taking account of (3), we have get inductively
$D(t_n, t_{n+1}) \preceq k D(t_{n-1}, t_n) \preceq k^2 D(t_{n-2}, t_{n-1}) \preceq \cdots \preceq k^n D(t_0, t_1)$.
In the following, we prove that $\{t_n\}_{n \in I\!N}$ is a Cauchy sequence.

$$\begin{aligned} D(t_n, t_{n+p}) &\preceq b D(t_n, t_{n+1}) + b^2 D(t_{n+1}, t_{n+2}) + \dots \\ &+ b^{p-2} D(t_{n+p-3}, t_{n+p-2}) + b^{p-1} D(t_{n+p-2}, t_{n+p-1}) \\ &+ b^p D(t_{n+p-1}, t_{n+p}) \\ &\preceq b k^n D(t_0, t_1) + b^2 k^{n+1} D(t_0, t_1) \dots \\ &+ b^{p-2} k^{n+p-3} D(t_0, t_1) + b^{p-1} k^{n+p-2} D(t_0, t_1) \\ &+ b^p k^{n+p-1} D(t_0, t_1) \\ &= (b k^n + b^2 k^{n+1} + \dots + b^{p-1} k^{n+p-2} + b^p k^{n+p-1}) D(t_0, t_1). \end{aligned}$$

So, by theorem 1.2 we have

$$\begin{aligned} \|D(t_n, t_{n+p})\| &\leq (\| b \| \| k \|^n + \| b \|^2 \| k \|^{n+1} + \dots \\ &+ \| b \|^{p-1} \| k \|^{n+p-2} + \| b \|^p \| k \|^{n+p-1}) \| D(t_0, t_1) \| \\ &= \frac{1}{\| b \|^{n-1}} \sum_{i=n}^{n+p-1} \| b \|^i \| k \|^i \| D(t_0, t_1) \| \\ &\leq \frac{1}{\| b \|^{n-1}} \sum_{i=n}^{\infty} \| b \|^i \| k \|^i \| D(t_0, t_1) \| \to 0 \end{aligned}$$

as $n \to \infty$. Thus $\{t_n\}_{n\in I\!N}$ is a Cauchy sequence. As in the proof of previous theorem, regarding the construction $t_n = F^n t_0$ together with the fact that $(P, \mathbb{E}, D)$ is F-orbitally complete, there is $z \in P$ such that $t_n \to z$. Again by the orbital continuity of F, we deduce that $t_n \to Fz$. Hence $z = Fz$. $\square$

Example 2.4. Let $P = [-1, 1]$ and $\mathbb{E} = M_{2\times 2}(I\!R)$ with $\|E\| = max_{ij}|e_{ij}|$ where e_{ij} are entries of the matrix $E \in M_{2\times 2}(I\!R)$. Then $(P, \mathbb{E}, D)$ is a C^*-algebra-valued b-metric space with

$$b = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

where the involution is given by $E^* = (\overline{E})^T$,

$$D(t,s) = \begin{bmatrix} |t-s|^2 & 0 \\ 0 & |t-s|^2 \end{bmatrix}$$

and partial ordering on $\mathbb{E}$ is given as

$$\begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \preceq \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \Longleftrightarrow e_{ij} \leq g_{ij} \quad \forall i,j = 1,2$$

Define a mapping $F : P \to P$ by $Ft = \frac{1}{2}t + \frac{1}{3}$ and let $e_1 = 1, e_2 = e_4 = 0$ and $e_3 = \frac{1}{2}$. F satisfies hypothesis of last theorem. Then F has at least one fixed point $F(\frac{2}{3}) = \frac{2}{3}$.

Let $I\!N$ be the set of positive integer and $(P, \mathbb{E}, D)$ be a C^*-algebra-valued metric space. The point $t_0 \in P$ is an *n-periodic point* of $F : P \to P$ if $t_0 = F^n t_0$ and $t_0 \neq F^k t_0$ for $k = 1, 2, 3, ..., n-1$ for some $n \in I\!N$.

The *orbit* of F at a point t is defined by $O_F(t) = \{F^n t : n \geq 0\}$. Suppose that Φ is the set of all nondecreasing $\varphi : \mathbb{E}^+ \to \mathbb{E}^+$ such that $\varphi(t) \prec t$ for all $t \in \mathbb{E}^+$.
Obviously $\varphi(t) = \gamma^* t \gamma \in \Phi$ for all $\gamma \in \mathbb{E}$, where $\|\gamma\| < 1$.
We discuss the existence of fixed points for above mappings F where

$$D(Ft, Fs) + D(Fs, Fz) \preceq \varphi(D(t,s) + D(s,z)) \tag{4}$$

for $t, s, z \in P$ with $t \neq s \neq z \neq t$, and $\varphi \in \Phi$; By definition of n-periodic point we obtain the next result.

Lemma 2.5. *Let $F : P \to P$ be a mapping on a C^*-algebra-valued metric space $(P, \mathbb{E}, D)$. Then $F^i t_0 \neq F^j t_0$ for any n-periodic point $t_0 \in P$ of F, where $0 \leq i \leq j \leq n-1$.*

Theorem 2.6. *Suppose that F is a self mapping on the complete C^*-algebra-valued metric space $(P, \mathbb{E}, D)$ and satisfying in (4).*
Then

(1) *There are at most two distinct fixed points for F, in P;*
(2) *The number of 2-periodic points for F, in P is zero or two;*
(3) *If $n \geq 3$ then F has any n-periodic points in P;*
(4) *If F has an orbit without 2-periodic points then F has a fixed point in P.*

Proof. Suppose that F has three distinct fixed points, t, s, z, in P. Then

$$D(t,z) + D(z,s) \preceq \varphi(D(t,z) + D(z,s)) \prec D(t,z) + D(z,s),$$

which is a contradiction.

If $s \in P$ is a 2-periodic point of F. Then Fs is also a 2-periodic point of F distinct from s. We assert that F has the only two 2-periodic points s and Fs.

Now suppose that $z \in P$ is another 2-periodic point of F such that $s \neq z \neq Fs$. Thus we have $Fs \neq Fz \neq F^2 s \neq Fs$. Then

$$\begin{aligned} D(s,z) + D(z,Fs) &= D(F^2 s, F^2 z) + D(F^2 z, F^3 s) \\ &\preceq \varphi^2(D(F^2 s, F^2 z) + D(F^2 z, Fs)) \\ &\prec D(s,z) + D(z,Fs), \end{aligned}$$

which is a contradiction. Thus F has exactly two 2-periodic points. Now we show that F has any n-periodic point for $n \geq 3$. If $t_0 \in P$ is an n-periodic point of F for $n \geq 3$, then
$t_k = F^k t_0, D_k = D(t_k, t_{k+1}) + D(t_{k+1}, t_{k+2})$ for all $0 \leq k \leq n$. By Lemma 2.5, we infer that

$$D_k \preceq \varphi(D(t_{k-1}, t_k) + D(t_k, t_{k+1})) \prec D_{k-1},$$

for all $1 \leq k \leq n$. We have that

$$D_0 = D_n \preceq \varphi(D_{n-1}) \prec D_{n-1} \preceq ... \prec D_0,$$

which is a contradiction.

Suppose that, there exists a point $t_0 \in P$ which is F has no 2-periodic points in $O_F(t_0)$. Set $t_n = F_n t_0, D_n = D(t_n, t_{n+1}) + D(t_{n+1}, t_{n+2})$ for any $n \geq 0$. If there exists some $n \geq 0$ with $t_n = t_{n+1}$, then t_n is a fixed point of F; if $t_n \neq t_{n+1}$ for any $n \geq 0$, we get

$$D_n \preceq \varphi(D_{n-1}) \preceq \varphi^2(D_{n-2}) \preceq ... \preceq \varphi^n(D_0).$$

For each $i, j, l \in I\!N$ such that $i > j \geq l$, by the triangular inequality, we have

$$D(t_i, t_j) \preceq \sum_{n=l}^{i-1} D_n \preceq \sum_{n=l}^{i-1} \varphi^n(D_0) \preceq \sum_{n=1}^{\infty} (\gamma^*)^n D_0 \gamma^n$$

$$\|D(t_i, t_j)\| \leq \sum_{n=1}^{\infty} \|\gamma\|^{2n} \|D_0\|.$$

Since $\|\gamma\| < 1$ then $\{t_n\}_{n \geq 0}$ is a Cauchy sequence in P. It follows from completeness of $(P, \mathbb{E}, D)$ that there exists a point $a \in P$ such that $lim_{n\to\infty} t_n = a$. Obviously, there exists some integer $k \in I\!N$ with $t_n \neq a$ for all $n \geq k$. Therefore

$$\begin{aligned} D(t_{n+1}, Fa) + D(Fa, Ft_{n+2}) &\preceq \varphi(D(t_n, a) + D(a, t_{n+2})) \\ &\prec D(t_n, a) + D(a, t_{n+2}) \to 0 \end{aligned}$$

as $n \to \infty$, which implies that $lim_{n\to\infty} t_n = Fa$. Hence $Fa = a$. This completes the proof. $\square$

Corollary 2.7. *Let $F : P \to P$ be a mapping on a complete C^*-algebra-valued metric space $(P, \mathbb{E}, D)$ such that*

$$D(Ft, Fs) + D(Fs, Fz) + D(Fz, Ft) \preceq \varphi(D(t, s) + D(s, z) + D(z, t))$$

for all $t, s, z \in P$ with $t \neq s \neq z \neq t$, where $\varphi \in \Phi$. Then the conclusions of Theorem 2.6 holds.

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