

Journal of Mathematical Extension
Vol. 16, No. 2, (2022) (3)1-21
URL: <https://doi.org/10.30495/JME.2022.1736>
ISSN: 1735-8299
Original Research Paper

Evaluating Groups of Decision Making Units in the Data Envelopment Analysis based on Cooperative Games

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Abstract. Data Envelopment Analysis is a non-parametric method for evaluating the efficiency of those Decision Making Units (DMUs) that have the same functionality and use multiple inputs to generate multiple outputs. DMUs may sometimes be divided into several groups according to a series of criteria, and it is intended to evaluate a group of similar DMUs. In this paper groups are fairly evaluated under a common platform, each group was considered as a player in a cooperative game and a subset of groups was considered as a coalition. Assuming the Production Possibility Set (PPS) is made up of DMUs belonging to the groups that are a coalition's member, a characteristic function was defined in terms of the sum of the efficiency of all units to determine the marginal effect of each group in different coalitions. The groups were then evaluated using the Shapley Value as a unique solution of the cooperative game. Some Examples were provided to describe and apply the method.

AMS Subject Classification:91A12 ;90C08.

Keywords and Phrases: Data envelopment analysis (DEA), Group evaluation, Game theory, Cooperative game, Shapley Value.

Received: July 2020; Accepted: December 2020

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1 Introduction

Data Envelopment Analysis (DEA) is used as a nonparametric instrument to evaluate the relative efficiency of homogeneous DMUs with multiple inputs so that they produce multiple outputs. In contrast to the singular selection of DMUs which provides for the evaluation of DMU's singularly, the grouping of DMUs enhances the dimensions of the evaluation process and, thus, provides for a more intelligent decision making. For example evaluating the branches of banks in different provinces of a country or evaluating the performance of chain hotels in the tourism industry and etc. A brief review of the literature indicates that there is a great need for further research in this field. The following is a brief summary of previous and current research.

Charnes et al. [8] first introduced the CCR model for evaluating the efficiency of DMUs and then in another paper [9] introduced the group analysis. Banker and Morey [4], argued that a group of DMUs in nesting mode could be defined as the categorical variable. Further, they compared the efficiency of each DMU with those of the category to which it belonged and the categories below it.

In the absence of the possibility of classifying DMU's into homogeneous groups, Cook et al. al. [10] classified the DMUs hierarchically in different levels. In this method, the efficiencies found at one level were counted in the efficiency of the higher levels. They also devised a model for keeping track of various ratings received by a DMU in different possible groupings.

Camanho and Dyson [6] concentrated on technical efficiency. Their goal was to evaluate the groups and identify each internally inefficient DMU to be compared and contrasted with inefficiencies of its group. They used Malmquist Productivity Index. Maniadakis and Thanassoulis [16] revised the model introduced by Camanho and Dyson in order to show the cost when the input prices are available. In another model, Cook and Zhu [11] introduced a goal-programming model for achieving a set of common weights between groups so that these weights minimize the maximum discrepancies between inter-group scores from their ideal level.

O'Donnell et al. [17] used the concept of meta-frontier to compare the technical efficiency of companies that could be divided into various

groups. They divided the efficiency measured by the meta-frontier into two parts: the distance of a DMU from its own frontier and the distance of its group's frontier from the meta-frontier.

Bagherzade Valami [21] used the production technology to devise a model for evaluating the performance of a group in which the efficiency of each DMU is defined as the distance of that DMU from the group's frontier. The model defines the efficiency of a group as the geometric mean of the efficiencies of all the DMUs based on that group's PPS. The higher the geometric mean of the efficiencies of all the DMUs, the higher the group efficiency. Ang et al.[1] Developed group efficiency and group cross-efficiency models to evaluate chain hotels using two definitions of average performance and weakest performance. Rahmani parchikolae et al.[18] extended the Malmquist productivity index to DMUs with interval data for groups.

This study evaluates the groups in a common framework by considering a cooperative and competitive relationship between groups. comparison in a common platform is more comparable and reliable. Each group was considered as a player in a cooperative game and a subset of groups was considered as a coalition. Assuming PPS is made up DMUs belonging to the groups of the coalition S , a characteristic function was defined in terms of the sum of the efficiency of all units to determine the marginal effect of each group in different coalitions. Then, the groups were uniquely evaluated using the Shapley value as a cooperative game solution. The present study, however, took a different approach compared to that by Li et al. [15].

On the other hand, in the study of Bagherzadeh Valami [21], the efficiency of the group was calculated based only on the Geometric mean of all the DMUs efficiency with the frontier of that group, indicating that the DMUs have been measured only by a single-member coalition of the groups, meaning that, in fact, the marginal effect of each group has been considered a null coalition. In the present paper, all DMUs were evaluated through different coalitions in a common frontier. Therefore, the marginal effects of a group on all possible coalitions were used to evaluate the efficiency of the respective group, making the evaluation of groups more accurate and fair.

The different sections of the present study are as follows: In Section

2, some of basics of DEA are listed, the efficiency of a DMU in a group and the efficiency of a group are defined and an introduction to game theory is presented. In Section 3, a new method is used to measure the efficiency of groups based on the cooperative games. In Section 4, the proposed method is first illustrated by two simple examples, and then, a real world example is provided to indicate the applicability of the method. Section 5 contains the results of the present paper.

2 Preliminaries

2.1 Data Envelopment Analysis

Decision Making Units (DMUs) ($DMU_j, j \in \{1, \dots, n\}$) are units that uses m inputs ($x_{ij}, i \in \{1, \dots, m\}$) to generate s outputs ($y_{rj}, r \in \{1, \dots, s\}$) and their evaluation is desired. Data envelopment analysis is an useful tool to evaluate relative efficiency of DMUs that for this evaluation needs Production Possibility Set (PPS). PPS is the sets of all DMUs (X, Y) that outputs of y can produce by input of x and satisfies in five axioms: Inclusion of observations, Convexity, possibility, Minimum interpolation, Constant or Variable return to scale. Production Possibility Set (PPS) with constant return to scale (CRS) is a set of (X, Y) pairs where Input X can generate Output Y .

$$T_c = \{(X, Y) \mid X \geq \sum_{j=1}^n \lambda_j X_j, \quad Y \leq \sum_{j=1}^n \lambda_j Y_j, \quad \lambda_j \geq 0, \quad j = 1, \dots, n\}.$$

By adding the constraint $\sum_{j=1}^n \lambda_j = 1$, a PPS with a variable return to scale (VRS) is obtained.

There are two types of orientation in DEA performance evaluation based on management decisions and control conditions, input oriented and output oriented. In input oriented models, by keeping the output level constant, the minimum input that produces the same output is desirable and in output oriented models, by keeping the input level constant, the maximum output that produces by the same input is desirable.

DEA models are radial or non-radial. In radial models, all inputs and outputs decrease or increase in the same proportion, but not necessarily

in non-radial models. CCR model introduced by Charnes et al.[8] is a radial model:

CCR- input oriented model

$$\begin{aligned}
 & \min \theta \\
 \text{s. t. } & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io}, & i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro}, & r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{1}$$

In the optimal value of Formula (1), $\theta^* = 1$ means that DMU_o is efficient and located on T_c 's frontier.

CCR-output oriented model

$$\begin{aligned}
 & \max \varphi \\
 \text{s. t. } & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io}, & i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi y_{ro}, & r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{2}$$

In the optimal value of Formula (2), $\varphi^* = 1$ means that DMU_o is efficient and located on T_c 's frontier.

Another radial method is BBC [3].

SBM [20] and enhanced Russell measure (ERM) [5, 13] are non-radial methods that have been provided to calculate the efficiency of DMUs in

DEA. The ERM model is as follow:

$$\begin{aligned}
 R_o^* &= \min \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{r=1}^s \varphi_r} \\
 \text{s. t. } & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{io}, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi_r y_{ro}, \quad r = 1, \dots, s \\
 & \theta_i \leq 1, \quad i = 1, \dots, m \\
 & \varphi_r \geq 1, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned}$$

DMU_o is ERM efficient if and only if $R_o^* = 1$, that is equal to $\theta_i^* = 1, \forall i$ and $\varphi_r^* = 1, \forall r$. But there is no distinction between efficient units. The super efficiency method based on Modified ERM was provided by Izadikhah et al. [14] for ranking efficient DMUs. As indicated in the study of Izadikhah et al. [14], when a DMU leaves the reference set and is located outside of the PPS, the ERM model does not get the correct super-efficiency value, because the direction of movement should actually be towards the frontier. However, this model moves away from the frontier instead of moving from the eliminated DMU located beyond the frontier towards the frontier. Therefore, Izadikhah et al. [14] introduced the following super efficiency method based on modified ERM model, which allows both movement directions simultaneously and obtains the right super-efficiency value for the DMUs both inside and outside the

frontiers:

$$\begin{aligned}
R_o^* &= \min \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{r=1}^s \varphi_r} \\
\text{s. t. } &\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j x_{ij} \leq \theta_i x_{io}, \quad i = 1, \dots, m \\
&\sum_{\substack{j=1 \\ j \neq 0}}^n \lambda_j y_{ij} \geq \varphi_r y_{ro}, \quad r = 1, \dots, s \\
&\theta_i - 1 \leq M\delta, \quad i = 1, \dots, m \\
&-\theta_i + 1 \leq M(1 - \delta), \quad i = 1, \dots, m \\
&-\varphi_r + 1 \leq M\delta, \quad r = 1, \dots, s \\
&\varphi_r - 1 \leq M(1 - \delta), \quad r = 1, \dots, s \\
&\delta \in \{0, 1\} \\
&\theta_i, \lambda_j \geq 0, \quad \forall i, j,
\end{aligned} \tag{3}$$

where M is a number that is large enough and δ is a binary variable that establishes only one of the following conditions:

$$\begin{aligned}
\text{(I)} &\begin{cases} \theta_i \leq 1, & i = 1, \dots, m, \\ \varphi_r \geq 1, & r = 1, \dots, s. \end{cases} \\
\text{(II)} &\begin{cases} \theta_i \geq 1, & i = 1, \dots, m, \\ \varphi_r \leq 1, & r = 1, \dots, s. \end{cases}
\end{aligned}$$

R_o^* is the efficiency of DMU_o , when the PPS is made up of all DMUs except DMU_o . If $R_o^* < 1$, then condition (I) holds true, that is DMU_o is inside the PPS. If $R_o^* = 1$, then $\theta_i = 1$ and $\varphi_r = 1$, that is, DMU_o is on the PPS frontier, and if $R_o^* > 1$, then condition (II) holds true, that is DMU_o is outside the PPS. Although the values of θ_i and φ_r may not be unique in Formula (3), the value of R_o^{At} is always unique.

2.2 Efficiency of a DMU with the frontier of a group

In this section, the definition presented by Izadikhah et al. [14] of super efficiency is extended to the case where a subset of the DMU is removed.

Consider n number of DMUs where each DMU_j ($j \in \{1, \dots, n\}$) uses m inputs (x_{ij} , $i \in \{1, \dots, m\}$) to generate s outputs (y_{rj} , $r \in \{1, \dots, s\}$) Moreover, suppose these n DMUs are divided into q groups as A_1 , A_2 , ..., and A_q . The efficiency of DMU_o with A_t frontier is defined in such a way that pps is made only from DMUs belonging to group A_t ($t \in \{1, 2, \dots, q\}$):

$$\begin{aligned}
R_o^{A_t} &= \min \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{r=1}^s \varphi_r} \\
\text{s. t. } &\sum_{j \in A_t} \lambda_j x_{ij} \leq \theta_i x_{io}, \quad i = 1, \dots, m \\
&\sum_{j \in A_t} \lambda_j y_{rj} \geq \varphi_r y_{ro}, \quad r = 1, \dots, s \\
&\theta_i - 1 \leq M\delta, \quad i = 1, \dots, m \\
&-\theta_i + 1 \leq M(1 - \delta), \quad i = 1, \dots, m \\
&-\varphi_r + 1 \leq M\delta, \quad r = 1, \dots, s \\
&\varphi_r - 1 \leq M\delta, \quad r = 1, \dots, s \\
&\delta \in \{0, 1\} \\
&\theta_i, \lambda_j \geq 0, \quad \forall i, j,
\end{aligned}$$

So $R_o^{A_t}$ is the efficiency of DMU_o assuming that the PPS has been made by DMUs belonging to the group A_t . If $R_o^{A_t} = 1$, then the DMU_o is located on the frontier of the PPS which is made up of the DMUs belonging to group A_t . Therefore, although DMU_o is efficient with respect to this frontier, it may not be efficient with respect to the general frontier belonging to the PPS made up of all the DMUs. This means that the DMU_o may achieve the best efficiency in its group, but it may not perform well compared with the other groups. If $R_o^{A_t} < 1$, then the DMU_o is located inside the PPS made by the DMUs belonging to group A_t and, therefore, is inefficient in relation to this frontier. If $R_o^{A_t} > 1$, then the DMU_o is located outside the PPS made up of the DMUs belonging to group A_t and is, therefore, super-efficient with respect to this frontier.

2.3 Group and its efficiency

The efficiency of group A_t is defined as the sum of the efficiency of all DMUs based on that group's frontier, as follows:

$$E(A_t) = \sum_{j=1}^n R_j^{A_t},$$

where n is the number of all DMUs.

2.4 Game theory

The game theory is used in decision-making problems where multiple decision makers have conflicting interests. Consider a competitive condition in which $N = \{1, 2, \dots, n\}$ is the set of players in the game. Players can compete in two ways:

1. Non-cooperative game: Players act individually and have personal accomplishments. In this game, it is tried to predict the strategies adopted by each player to obtain the most impressive achievement.

2. Cooperative game: The players are expected to form a coalition to boost their achievement. Cooperative games are usually characterized by the players in the game and a characteristic function $\mathcal{C}(S)$ as $\langle N, \mathcal{C}(S) \rangle$. Assuming that the coalition S is a subset of the players. The characteristic function $\mathcal{C}(S)$, which is the achievement of the players in S from the game, is what the members of the S coalition can be sure to gain together in the coalition. Evidently, what players have gained in the S Coalition must be fairly divided between the players of the coalition S . Suppose the prize vector $X = \{x_1, x_2, \dots, x_n\}$ is the proportion of n players (e.g. x_i is the prize of the i^{th} player). This prize vector must comply with the two following conditions:

1. $\mathcal{C}(N) = \sum_{i=1}^n x_i$ Group rationality.
2. $x_i \geq \mathcal{C}(\{i\}), \forall i \in N$ Individual rationality.

Equation 1 states that in any reasonable prize vector, the total player prize must be equal to the amount that the players receive in the grand coalition (The coalition consist of all players). Equation 2 states that player i 's award must be at least as large as the award he receives alone. There are several solutions to obtain the prize vector, including the core, stable set, kernel, nucleolus, and Shapley Value [12]. In the meantime,

Shapley[19]showed that there is a unique prize vector that satisfies in three following axioms and fairly allocate overall prize among players:

1. Relabeling of players interchange the player's reward.
2. If $\mathcal{C}(S \cup A_k) = \mathcal{C}(S)$ for all coalitions S , i.e. player i adds no value to any coalition, then the reward of player i from Shapley Value is zero, $x_i = 0$.
3. If \mathcal{C}_1 and \mathcal{C}_2 be two characteristic functions for games, with the same payers, for the game with the characteristic function $\mathcal{C}_1 + \mathcal{C}_2$, the reward is equal to the sum of reward for \mathcal{C}_1 and \mathcal{C}_2 .

In the Shapley Value solution, the unique share of i^{th} player from the prize, i.e. x_i is obtained as follows:

$$x_i = \sum_{\substack{S \subset N \\ i \notin S}} \frac{(s)!(n-s-1)!}{n!} (\mathcal{C}(S) - \mathcal{C}(S \cup \{i\})).$$

3 Evaluation of groups based on the cooperative game

Groups are considered as the players of a cooperative game in order to be able to compare them within a common framework. Let G be the set of g groups and the coalition S is a subset of s groups, then the characteristic function of the coalition S , $\mathcal{C}(S) : 2^g \rightarrow R$ is defined as the sum of the efficiency of all DMUs, assuming that PPS has been formed by all the DMUs belonging to the s group of the coalition S :

$$\mathcal{C}(S) = \sum_{j=1}^n R_j^S,$$

Since the values of $R_j^S, j = 1, \dots, n$ are unique, so their sum, the characteristic function, is also unique. If $A_k \notin S$, then the characteristic function $\mathcal{C}(S \cup A_k)$ is defined as the sum of the efficiency of all DMUs, assuming that the PPS has been generated by all DMUs of the s groups of coalition S and group A_k :

$$\mathcal{C}(S \cup A_k) = \sum_{j=1}^n R_j^{S \cup \{A_k\}}.$$

Given these definitions, the marginal effect of group A_k to the coalition S , which is the alteration in the sum of efficiency of DMUs due to the addition of group A_k to coalition S , is defined as follows:

$$ME^S(A_k) = \mathcal{C}(S) - \mathcal{C}(S \cup A_k).$$

$ME^S(A_k)$ is unique because of uniqueness of $\mathcal{C}(S)$ and $\mathcal{C}(S \cup A_k)$. The characteristic function $\mathcal{C}(S)$ is an achievement that the members of coalition S are expected to achieve together. What is gained by the players participating in a coalition should be fairly divided between them. There are different solutions to divide the prize, from which Shapley Value [19] is understandable and easy to interpret. The Shapley Value [19] is used to obtain the solution of this cooperative game, i.e. the share of players participating in the Coalition S :

$$\begin{aligned} \varphi_{A_k}(\mathcal{C}) &= \sum_{\substack{S \subseteq \{A_1, A_2, \dots, A_q\} \\ A_k \notin S}} \frac{(s)!(q-s-1)!}{q!} (\mathcal{C}(S) - \mathcal{C}(S \cup \{A_k\})) \\ &= \sum_{\substack{S \subseteq \{A_1, A_2, \dots, A_q\} \\ A_k \notin S}} \frac{(s)!(q-s-1)!}{q!} (ME^S(A_k)), \end{aligned}$$

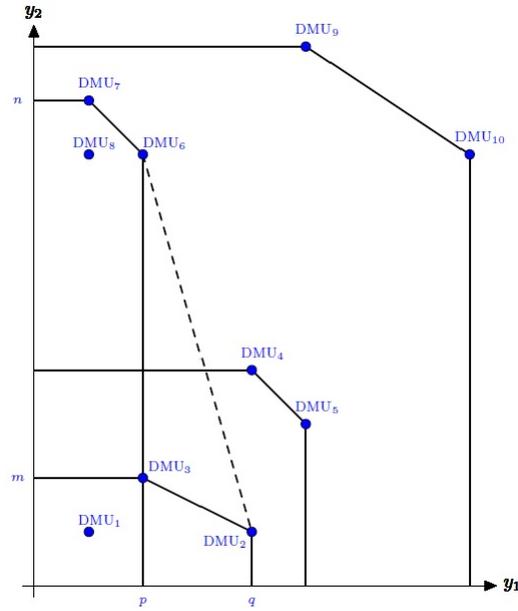
where q is the total number of groups and s is the number of groups participating in the coalition S . $\varphi_{A_k}(\mathcal{C})$ is the unique amount of achievement expected by the A_k player in this cooperative game through participating in the Coalition S . The higher the Shapley Value, the better the rank of the group. Due to the uniqueness of the shapley value [19], the group rankings are also unique. In the next section, the method is first described using a simple example, a comparative example is provided and then a real example is used to demonstrate the applicability of the method.

4 Numerical examples

Example 4.1. A set of 10 DMUs, which includes an input of 1 for all units and two outputs, is divided into four groups A , B , C and D (Table 1). The frontier of groups are depicted in Figure 1.

Table 1: data of 10 DMUs

DMU	Group	Input1	output1	output2
DMU ₁	<i>A</i>	1	1	1
DMU ₂	<i>A</i>	1	4	1
DMU ₃	<i>A</i>	1	2	2
DMU ₄	<i>B</i>	1	4	4
DMU ₅	<i>B</i>	1	5	3
DMU ₆	<i>C</i>	1	2	8
DMU ₇	<i>C</i>	1	1	9
DMU ₈	<i>C</i>	1	1	8
DMU ₉	<i>D</i>	1	5	10
DMU ₁₀	<i>D</i>	1	8	8

**Figure 1:** frontier of groups

The presented method was used to calculate the marginal effect of these four groups in different coalitions as shown in Table 2, where their

average is the Shapley Value of that group. As an illustration, the marginal effect of adding group C to the coalition $\{A\}$, which appears in the second row of the fourth column of Table 2, is as follow:

If the PPS is made of the DMUs belonging to the Group A (Figure 1), it denotes line that connects the point q , DMU_2 , and DMU_3 to the point m . With this PPS, the characteristic function of coalition $\{A\}$, which is the sum of efficiency of all DMUs, is equal to 16.02. By adding the DMUs of Group C to the coalition $\{A\}$, PPS made by the DMUs of the Group A and C is denoted by the line that connects the point q , DMU_2 , DMU_6 , and DMU_7 to the point n . In this PPS, the characteristic function of the Coalition $\{A, C\}$, i.e. the sum of efficiency of all DMUs, is equal to 9.84. Therefore, the marginal effect of C in the coalition $\{A\}$ is obtained as follows:

$$ME^{\{A\}}(C) = \mathcal{C}(A) - \mathcal{C}(A \cup C) = 16.02 - 9.84 = 6.18.$$

Table 2: Marginal effect of groups in various coalitions

Possible coalitions of groups	A	B	C	D
$\{A\}$	0	5.28	6.18	11.62
$\{B\}$	0	0	2.06	6.34
$\{C\}$	0.79	1.95	0	6.23
$\{D\}$	0	0	0	0
$\{A, B\}$	0	0	2.06	6.34
$\{A, C\}$	0	1.16	0	5.44
$\{A, D\}$	0	0	0	0
$\{B, C\}$	0	0	0	4.28
$\{B, D\}$	0	0	0	0
$\{C, D\}$	0	0	0	0
$\{A, B, C\}$	0	0	0	4.28
$\{A, B, D\}$	0	0	0	0
$\{A, C, D\}$	0	0	0	0
$\{B, C, D\}$	0	0	0	0

In Table 3, the obtained Shapley Values are seen in the second column, according to which the groups are ranked (column 3 of Table 3).

As expected, Group D , with groups B , C and D as its subgroups, has the most outputs and achieved best rank. Group A that has less outputs achieved worst rank.

Table 3: Ranking groups by two proposed method

Group	Shapley Value	Ranking by Shapley Value
A	0.06	4
B	0.60	3
C	0.74	2
D	3.18	1

In the next section, with an example, we compare the presented method with the method of Bagherzade Valami [21] article.

Example 4.2. A set of 8 DMUs, which Bagherzade Valami [21] has used includes an input of 1 for all units and two outputs, is divided into three groups A , B , and C (Table 4), $A = \{DMU_4, DMU_7\}$, $B = \{DMU_7, DMU_8\}$ and $C = \{DMU_2, DMU_4\}$ The frontier of three groups are depicted in Fig 2.

Table 4: Data of 8 DMUs

DMU	Input1	output1	output2
DMU_1	1	4	0.5
DMU_2	1	3	6
DMU_3	1	5	5
DMU_4	1	4.5	2
DMU_5	1	2	3
DMU_6	1	6	3
DMU_7	1	6	1
DMU_8	1	5	4

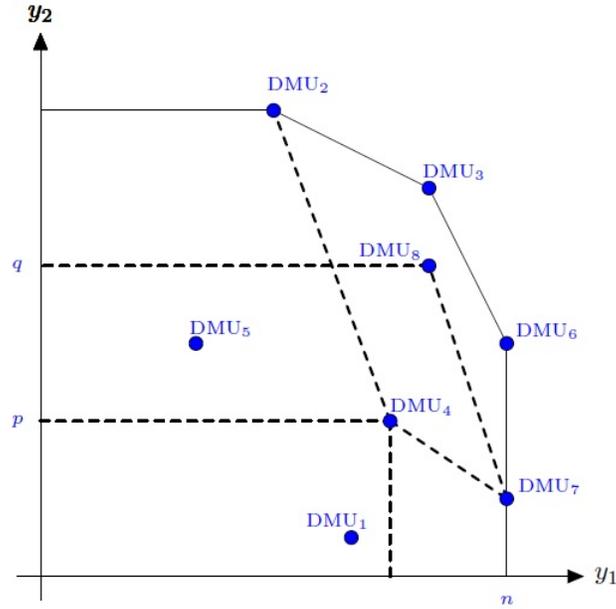


Figure 2: production possibility sets made by various groups

The presented method was used to calculate the marginal effect of these three groups in different coalitions as shown in Table 5, where their average is the Shapley Value of that group. To illustrate this method, the marginal effect of group B to the coalition $\{A\}$, which appears in the second row of the second column of Table 5, is as follow:

If the PPS is made of the DMUs belonging to the Group A (Figure 2), it denotes a line that connects the point n , DMU_7 , and DMU_4 to the point p . With this PPS, the characteristic function of coalition $\{A\}$ which is the sum of efficiency of all DMUs, is equal to 9.63:

$$\mathcal{C}(A) = 0.40 + 1.50 + 1.53 + 1.05 + 1.20 + 1.50 + 1 + 1.45 = 9.63.$$

By adding the DMUs of Group B to the collation $\{A\}$, PPS made by the DMUs of the Group A and B denoted by the line that connects point n , DMU_7 , DMU_8 to point q . In this PPS, the characteristic function of the Coalition $\{A, B\}$, i.e. sum of efficiency of all DMUs, is equal to

6.73:

$$\mathcal{C}(A \cup B) = 0.21 + 1.20 + 1.11 + 0.64 + 0.52 + 1.05 + 1 + 1 = 6.73.$$

Therefore, the marginal effect of adding B to the coalition $\{A\}$ is obtained as follows:

$$ME^{(A)}(B) = \mathcal{C}(A) - \mathcal{C}(A \cup B) = 9.63 - 6.73 = 2.90.$$

In Table 6, the obtained Shapley Values are seen in the second column, according to which the groups are ranked (column 3 of Table 6). As expected, groups B that its PPS includes the group A 's PPS and has more outputs, ranked better. The ranking of the proposed method in this example are as same as Bagherzade Valami's [21] ranking. But the ranking of the proposed method are unique, comparable and fair, because in Bagherzade Valami's method DMUs are measured by different frontiers, but in the proposed method evaluation is based on a common frontier.

Table 5: Marginal effect of groups in various coalitions

Possible coalitions of groups	A	B	C
$\{A\}$	0	2.90	2.79
$\{B\}$	0	0	0.26
$\{C\}$	0.64	1.01	0
$\{A, B\}$	0	0	0.26
$\{A, C\}$	0	0.37	0
$\{B, C\}$	0	0	0

Table 6: Ranking groups by two proposed methods

Group	Shapley Value	Ranking by Shapley Value
A	0.106	3
B	0.713	1
C	0.551	2

In the following, a real word example is presented to identify the applicability of the proposed method.

Example 4.3. To demonstrate the applicability of this method, a database consisting of 20 branches of an Iranian bank, borrowed from Amirteimoori et al. [2], were supposed in four groups. Relevant data included in Table 7: three inputs (number of employees, number of computers, space of the branch); three outputs (amount of deposits, amount of loans and amount of charges). These data, which are divided into four groups, are presented in Table 7.

Table 7: The input and output of 20 bank branches divided in 4 groups

Group	DMU	x_1	x_2	x_3	y_1	y_2	y_3	θ_{BBC}
A_1	1	0.950	0.700	0.155	0.190	0.521	0.293	1.00
A_1	2	0.796	0.600	1.000	0.227	0.627	0.462	0.90
A_1	3	0.798	0.750	0.513	0.228	0.970	0.261	0.99
A_1	4	0.865	0.550	0.210	0.193	0.632	1.000	1.00
A_2	5	0.815	0.850	0.268	0.233	0.722	0.246	0.90
A_2	6	0.842	0.650	0.500	0.207	0.603	0.569	0.75
A_2	7	0.719	0.600	0.350	0.182	0.900	0.716	1.00
A_2	8	0.785	0.750	0.120	0.125	0.234	0.298	0.80
A_2	9	0.476	0.600	0.135	0.080	0.364	0.244	0.79
A_2	10	0.678	0.550	0.510	0.082	0.184	0.049	0.29
A_3	11	0.711	1.000	0.305	0.212	0.318	0.403	0.60
A_3	12	0.811	0.650	0.255	0.123	0.923	0.628	1.00
A_3	13	0.659	0.850	0.340	0.176	0.645	0.261	0.82
A_3	14	0.976	0.800	0.540	0.144	0.514	0.243	0.47
A_3	15	0.685	0.950	0.450	1.000	0.262	0.098	1.00
A_4	16	0.613	0.900	0.525	0.115	0.402	0.464	0.64
A_4	17	1.000	0.600	0.205	0.090	1.000	0.161	1.00
A_4	18	0.634	0.650	0.235	0.059	0.349	0.068	0.47
A_4	19	0.372	0.700	0.238	0.039	0.190	0.111	0.41
A_4	20	0.583	0.550	0.500	0.110	0.615	0.764	1.00

Table 8: Ranking by two presented methods

Group	Shapley Value	Ranking by Shapley Value
A_1	1.04	2
A_2	0.77	3
A_3	1.66	1
A_4	0.15	4

In the second column of Table 8, the Shapley Values obtained using the above-mentioned method are presented and in the third column, the ranking performed by this method is presented. It has been seen that in comparing with a common platform A_3 has a better ranking than A_1 and A_2 .

5 Conclusion

In DEA problems and under most conditions, group evaluation of DMUs is much more valuable, resulting in better management decisions. In this paper, to evaluate the groups in a common platform, the groups were evaluated from a cooperative game perspective, in a way that each group was considered as a player, and a subset of the groups was assumed as a coalition. By defining the characteristic function of the coalition S as the sum of the efficiency of all DMUs when the PPS is made up of groups belonging to the coalition S , the marginal effect of each group was specified and the Shapley Value of the groups was obtained using that value. The higher the Shapley Value, the better the performance. Groups were evaluated with respect to a common platform and thus the evaluation is fair and comparable. Moreover, the marginal effects of a group on all possible coalitions were investigated in order to evaluate the efficiency of that group, which makes the evaluation of groups more precise. Ranking with the presented method is unique because of using shapley value. In this way, when efficiency with different frontier is desired, we may run into infeasible circumstances where the efficiencies cannot be correctly calculated. To prevent this, the Modified ERM can be used for the evaluation of efficiencies.

This research shows that this subject is a dynamic one and requires

more research. In future group studies, other methods for obtaining cooperative game solution and non-coalition cooperative methods such as Bargaining Game may be considered. In some circumstances, a large number of groups may cause delay in analysis and the algorithm is time-consuming. In such situations meta-heuristic algorithms (e.g., genetic algorithms) or approximate estimation methods like Castreo et al.[7] are suggested.

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