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Original Research Paper

An Existence Result for Some Fractional-Integro Differential Equations in Banach Spaces via Deformable Derivative

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Abstract. In this paper we investigate further properties of the deformable derivative and use the results to study the existence of solutions to the integro-differential equation $D^\alpha y(t) = h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds, t \in [0, T]$, with initial condition $y(0) = y_0$, where $D^\alpha y(t)$ is the deformable derivative of y, $0 < \alpha < 1$. We use Weissinger's fixed point theorem and Krasnoselskii's fixed point theorem to achieve our main results. An example is provided for illustration.

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1 Introduction

In their paper [5], F. Zulfeqarr, A. Ujlayan, and P. Ahuja introduced the new concept of deformable derivative using the limit approach as in the usual derivative. They called it "deformable" as its intrinsic property of continuously deforming function to derivative. This derivative is linearly related to the usual derivative. The deformable derivative can be viewed as a derivative of the fractional order. Recently, A. Meraj, D.N. Pandey [3] used this concept to study the existence and uniqueness of solutions to the Cauchy problem $D^{\alpha}x(t) = Ax(t) + f(t, x(t)), t \in (0, T],\ x(0) = x_0$, where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X (using the Banach fixed point theorem), and the approximate controllability to the related problem (using the Schauder fixed point theorem). In our previous paper [2], we established the existence and uniqueness of solutions to Cauchy problems with non-local condition $D^{\alpha}x(t) = f(t, x(t)), t \in (0, T],\ x(0) + g(x) = x_0$, via the deformable derivative.

In the present paper, we first investigate some elementary properties of the deformable derivative(section 2), and then explore in section 3 the existence of solutions for fractional integro-differential equations of the type $D^{\alpha}y(t) = h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds, t \in [0, T]$, with initial condition $y(0) = y_0$, where $D^{\alpha}y(t)$ is the deformable derivative of the function $y(t)$. Finally, section 4 is devoted to the study of an example.
Let $J = [0, T], (X, \|.\|)$ be a Banach space and $C(J, X)$ denote the Banach space of all continuous bounded functions $g : J \to X$ equipped with the norm $\|g\|_{C(J,X)} = \sup\ \{|g(t)| : t \in J\ ,\text{for any } g(t) \in X.\}$

2 Deformable Derivative

Definition 2.1. ([5]) *Let f be a real valued function on [a,b], $\alpha + \beta = 1$*

1. *The Deformable derivative of f of order α at $t \in (a, b)$ is defined as:*

$$D^{\alpha}f(t)=\lim_{\epsilon\to 0}\frac{(1+\epsilon\beta)f(t+\epsilon\alpha)-f(t)}{\epsilon}.$$

If the limit exists, we say that f is α-differentiable at t.

2. *For $t\in[a,b]$, and $\alpha\in(0,1]$, the α-integral of f is given by:*

$$I_a^{\alpha}f(t)=\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_a^t e^{\frac{\beta}{\alpha}x}f(x)dx.$$

Remark 2.2. *If $\alpha=1$, then $\beta=0$, and we recover the usual derivative and the usual Riemann integral.This shows that the deformable derivative and the α-integral are generalizations of the usual derivative and the usual Riemann integral, respectively.*

Theorem 2.3. ([2],[5]) *The operators D^{α} and I_a^{α} possess the following properties :*
Let $\alpha,\alpha_1,\alpha_2\in(0,1]$ such that $\alpha+\beta=1$, $\alpha_i+\beta_i=1$ for $i=1,2$.

1. *A differentiable function f at a point $t\in(a,b)$ is always α-differentiable at that point for any α. Moreover we have*

$$D^{\alpha}f(t)=\beta f(t)+\alpha Df(t),$$

where $D:=\frac{d}{dt}$ is the usual derivative.

2. *Let f be differentiable at a point t for some α. Then it is continuous there.*

3. *Let f be defined in (a,b). For any α, f is α-differentiable iff it is differentiable.*

4. *Suppose f and g are α-differentiable. Then*

$$\begin{aligned}D^{\alpha}(f\circ g)(t)&=\beta(f\circ g)(t)+\alpha D(f\circ g)(t)\\&=\beta(f\circ g)(t)+\alpha f'(g(t))g'(t).\end{aligned}$$

5. *Let f be continuous on [a,b]. Then $I_a^\alpha f$ is α- differentiable in (a,b), and we have*

$$D^\alpha(I_a^\alpha f(t)) = f(t), and$$
$$I_a^\alpha(D^\alpha f(t) = f(t) - e^{\frac{\beta}{\alpha}(a-t)} f(a).$$

6. $D^\alpha \left(\frac{f}{g}\right) = \frac{gD^\alpha(f) - \alpha f}{g^2}$.
7. *Linearity :* $D^\alpha(af + bg) = aD^\alpha f + bD^\alpha g$.
8. *Commutativity :* $D^{\alpha_1} \cdot D^{\alpha_2} = D^{\alpha_2} \cdot D^{\alpha_1}$.
9. *For a constant* $c, D^\alpha(c) = \beta c$.
10. $D^\alpha(fg) = (D^\alpha f)g + \alpha f Dg$.
11. *Linearity :* $I_a^\alpha(bf + cg) = bI_a^\alpha f + cI_a^\alpha g$.
12. *Commutativity :* $I_a^{\alpha_1} I_a^{\alpha_2} = I_a^{\alpha_2} I_a^{\alpha_1}$.

Let's now recall some tools which we will use in the sequel.

Theorem 2.4. *(Krasnoselskii fixed point theorem). Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be two operators such that:*
(1)$Ax + By \in M$ whenever $x, y \in M$.
(2) A is compact and continuous.
(3) B is a contraction mapping.
Then, there exists $z \in M$ such that $z = Az + Bz$.

Theorem 2.5. ([1])*(Weissinger fixed point theorem) Assume (E, d) to be a non empty complete metric space and let $\beta_j \geq 0$ for every $j \in \mathbb{N}$ such that $\sum_{j=0}^{n-1} \beta_j$ converges. Further more let the mapping $\mathbb{T} : E \to E$ satisfy the inequality $d(\mathbb{T}^j u, \mathbb{T}^j v) \leq \beta_j d(u, v)$, for every $j \in \mathbb{N}$ and every $u, v \in E$. Then, $\mathbb{T}$ has a unique fixed point u^*. Moreover, for any $v_0 \in E$, the sequence $\{\mathbb{T}^j v_0\}_{j=1}^{\infty}$ converges to this fixed point u^*.*

3 Application to Evolution Equations

We consider the following Cauchy problem with initial condition

$$D^{\alpha}y(t) = h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds, \; t \in J \tag{1}$$

$$y(0) = y_0, \tag{2}$$

where D^{α} is the deformable derivative of order $\alpha \in (0, 1)$, and
$f : J \times X \to X, \; K : J \times J \times X \to X$ are continuous functions.

Lemma 3.1. *The system (1)-(2) is equivalent to the following nonlinear integral equation:*

$$y(t) = e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, y(s))ds$$
$$+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau, s, y(s))d\tau ds. \tag{3}$$

Proof. Assume (1)-(2). Then

$$I^{\alpha}D^{\alpha}y(t) = I^{\alpha}\left(h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds\right).$$

Using Theorem 2.3, we get,

$$I^{\alpha}\left(h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds\right) = I^{\alpha}(h(y(t)) + I^{\alpha}f(t, y(t))$$
$$+ \; I^{\alpha}\left(\int_0^t K(t, s, y(s))ds\right).$$

Using Theorem 2.3 and Definition 2.1, we get

$$y(t) - e^{\frac{-\beta}{\alpha}t}y(0) = \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, y(s))ds$$

$$+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau, s, y(s))d\tau ds.$$

Finally, using (2),

$$y(t) = e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,y(s))ds$$
$$+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau,s,y(s))d\tau ds.$$

Conversely, assuming (3) and taking D^α of both sides of the equation, we get

$$\begin{aligned}
D^\alpha y(t) = \beta e^{\frac{-\beta}{\alpha}t}y_0 &+ \frac{\beta}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{\beta}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,y(s))ds \\
&+ \frac{\beta}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau,s,y(s))d\tau ds \\
&+ \alpha D(e^{\frac{-\beta}{\alpha}t}y_0) + \alpha D\left(\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds\right) \\
&+ \alpha D\left(\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,y(s))ds\right) \\
&+ \alpha D\left(\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau,s,y(s))d\tau ds\right) \\
&= h(y(t)) + f(t,y(t)) + \int_0^t K(t,s,y(s))ds.
\end{aligned}$$

The proof is now complete. □

Definition 3.2. *A function $y \in C(J,X)$ is said to be a mild solution to (1)-(2) if*

$$y(t) = e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,y(s))ds$$
$$+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau,s,y(s))d\tau ds.$$

Now let's consider the following hypotheses:
(H1) $h : C(J,X) \to X$ is continuous, bounded and there exists $M \in (0,1)$ such that $\|h(u) - h(v)\| \leq M\|u - v\|_\infty$, for $u, v \in C(J,X)$.

(H2) $f : J \times X \to X$ is of Carathéodory, i.e. for any $u \in X$, $f(t, v)$ is strongly measurable with respect to $t \in J$ and for any $t \in J$, $f(t, u)$ is continuous with respect to $u \in X$. Moreover, there exists $\theta \in$ (0,1], $L > 0$ such that
$\|f(t, u) - f(t, v)\| \leq L\|u - v\|^{\theta}, t \in J, u, v \in X.$
(H3) $K : D \times X \to X$, is continuous on D and there exists $\gamma \in$ (0,1], $\rho \in L^1(J)$ such that
$\|K(\tau, s, u(s)) - K(\tau, s, v(s))\| \leq \rho(\tau)\|u - v\|^{\gamma}, (\tau, s) \in D, u, v \in X,$
where $D = \{(t, s) : 0 \leq s \leq t \leq 1\}$.
Now we state and prove the following result.

Theorem 3.3. *Assume (H1),(H2) and H(3) hold. Then equation (1)-(2) has a solution in C(J,X) on J .*

Proof. We transform problem (1)-(2) into a fixed point problem.
Define $F : C(J, X) \to C(J, X)$ by

$$Fy(t) = e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, y(s))ds \\ + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau, s, y(s))d\tau ds.$$

Let F be the sum of two operators P and Q, defined as follows:

$$Py(t) = e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds, \tag{4}$$

and

$$Qy(t) = \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s, y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} \\ \times \int_s^t K(\tau, s, y(s))d\tau ds. \tag{5}$$

For any function $z \in \mathcal{C}$ and for some $j \in \mathbb{N}$, we define as in [4] the following norm:
$\|z\|_j = sup\{e^{-jt}\|z(t)\| : t \in J\}.$

Step 1 : Consider $B_r = \{z \in \mathcal{C} : \|z\|_j \leq r\}$. We first prove that $Pz + Qz^* \in B_r \subset \mathcal{C}$, for every $z, z^* \in B_r$.
Let us set

$$\sigma = \sup_{(s,z^*)\in J\times B_r} \|f(s, z^*(s))\|,$$

$$\sigma^* = \sup_{(\tau,s,z^*)\in D\times B_r} \int_s^t \|K(\tau, s, z^*(s))\|d\tau, \eta = \sup_{z\in B_r} \|h(z)\|,$$

and there exists

$$r = \|z_0\| + \frac{\eta + \sigma + \sigma*}{\beta} + 1.$$

For $z, z^* \in B_r$ and $t \in J$, we have

$$\begin{aligned}
&\|Pz(t) + Qz^*(t)\| \\
&\quad \leq \|z_0\| + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\|h(z(s))\|ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s} \\
&\quad\quad \times \|f(s, z^*(s)\|ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t \|K(\tau, s, z(s))\|d\tau ds \\
&\quad \leq \|z_0\| + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\ sup_{z\in B_r}\|h(z(s))\|ds \\
&\quad\quad + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\ sup_{(s,z^*)\in J\times B_r}\|f(s, z^*(s)\|ds \\
&\quad\quad + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\ sup_{(\tau,s,z^*)\in D\times B_r}\int_s^t \|K(\tau, s, z^*(s))\|d\tau ds \\
&\quad \leq \|z_0\| + \frac{\eta}{\beta} + \frac{\sigma}{\beta} + \frac{\sigma^*}{\beta} \\
&\quad = \|z_0\| + \frac{\eta + \sigma + \sigma^*}{\beta}.
\end{aligned}$$

Finally,

$$\|Pz + Qz^*\|_j \leq e^{-j}\left(\|z_0\| + \frac{\eta + \sigma + \sigma*}{\beta}\right) < r.$$

This means that $Pz + Qz^* \in B_r$.
Step 2 : We prove that the operator P is a contraction map on B_r.

Proceeding from the assumptions and B_r as in Step 1, for $z, z^* \in B_r$ and $t \in J$, we have

$$\begin{aligned}
\|Pz(t) - Pz^*(t)\| &\leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} \|h(z(s)) - h(z^*(s)\| ds \\
&\leq \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} M \|z(s) - z^*(s)\| ds \\
&\leq M \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} e^{js} \sup_{s \in J} e^{-js} \|z(s) - z^*(s)\| ds \\
&= M[I_0^\alpha e^{jt}] \|z(s) - z^*(s)\|_j.
\end{aligned}$$

Since

$$I_0^\alpha e^{jt} = \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} e^{js} ds,$$

we have

$$\begin{aligned}
\|Pz(t) - Pz^*(t)\| &= M \left[\frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} e^{js} ds \right] \|z - z^*\|_j \\
&= M \left[\frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{(\frac{\beta}{\alpha}+j)s} ds \right] \|z - z^*\|_j \\
&= M \left[\frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{(\frac{\beta+\alpha j}{\alpha})s} ds \right] \|z - z^*\|_j \\
&= M \frac{e^{\frac{-\beta}{\alpha}t}}{(\alpha j + \beta)} \left[e^{(\frac{\beta+\alpha j}{\alpha})t} - 1 \right] \|z - z^*\|_j \\
&\leq M e^{\frac{-\beta}{\alpha}t} [e^{jt} e^{\frac{\beta}{\alpha}t} - 1] \|z - z^*\|_j \\
&\leq M e^{jt} \|z - z^*\|_j.
\end{aligned}$$

Thus,

$$\|Pz - Pz^*\|_j \leq M \|z - z^*\|_j.$$

Since $M < 1$, we conclude that P is a contraction map on B_r.

Step 3: We show that the operator Q is completely continuous on B_r.

Now we prove that (QB_r) is uniformly bounded, $(\overline{QB}_r)$ is equicontinuous, and $Q : B_r \to B_r$ is continuous.
First, we show that (QB_r) is uniformly bounded.
For $z \in B_r$ and $t \in J$, we have

$$
\begin{aligned}
&\|Qz(t)\| \\
\leq& \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\|f(s,z(s)) - f(s,0)\|ds \\
&+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\|f(s,0)\|ds \\
&+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t \|K(\tau,s,z(s)) - K(\tau,s,0)\|d\tau ds \\
&+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t \|K(\tau,s,0)\|d\tau ds \\
&\leq \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}Le^{\theta js}\|z\|_j^{\theta}ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}R\ ds \\
&+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t \rho(\tau)d(\tau)e^{\gamma js}\|z\|_j^{\gamma}ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}R^*\ ds \\
&\leq (L\|z\|_j^{\theta} + \|\rho\|_{L^1}\|z\|_j^{\gamma})\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}e^{js}ds + [R+R^*]\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}ds \\
&\leq (Lr^{\theta} + \|\rho\|_{L^1}\ r^{\gamma})e^{jt} + \frac{[R+R^*]}{\beta}.
\end{aligned}
$$

Thus,

$$\|Qz\|_j \leq Lr^{\theta} + \|\rho\|_{L^1}\ r^{\gamma} + \frac{[R+R^*]}{\beta e^j} := l,$$

where

$$R = \sup_{s\in J}\|f(s,0)\|,$$

and

$$R^* = \sup_{(\tau,s)\in D}\int_t^s \|K(\tau,s,0)\|d\tau.$$

Therefore, $QB_r \subset B_l$ for any $z \in B_r$, i.e., the set $\{Qz : z \in B_r\}$ is uniformly bounded.
Next, we prove that $(\overline{QB_r})$ is equicontinuous. For $z \in B_r$ and for $t_1, t_2 \in J$, with $t_1 \leq t_2$,

$$
\begin{aligned}
&\|Qz(t_2) - Qz(t_1)\| \\
&\leq \frac{1}{\alpha}\int_0^{t_2} |e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}| e^{\frac{\beta}{\alpha}s}\|f(s,z(s))\|ds \\
&\quad + \frac{1}{\alpha}\int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}\|f(s,z(s))\|ds \\
&\quad + \frac{1}{\alpha}\int_0^{t_1} |e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1}| e^{\frac{\beta}{\alpha}s}\int_s^{t_2} \|K(\tau,s,z(s))\|d\tau ds \\
&\quad + \frac{1}{\alpha}\int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}\int_s^{t_2} \|K(\tau,s,z(s))\|d\tau ds \\
&\quad + \frac{1}{\alpha}\int_0^{t_1} e^{\frac{-\beta}{\alpha}t_1} e^{\frac{\beta}{\alpha}s}\int_{t_1}^{t_2} \|K(\tau,s,z(s))\|d\tau ds \\
&\leq \frac{1}{\alpha}\int_0^{t_1} (e^{\frac{-\beta}{\alpha}t_1} - e^{\frac{-\beta}{\alpha}t_2}) e^{\frac{\beta}{\alpha}s}\|f(s,z(s))\|ds \\
&\quad + \frac{1}{\alpha}\int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}\|f(s,z(s))\|ds \\
&\quad + \frac{1}{\alpha}\int_0^{t_1} (e^{\frac{-\beta}{\alpha}t_1} - e^{\frac{-\beta}{\alpha}t_2}) e^{\frac{\beta}{\alpha}s}\int_s^{t_2} \|K(\tau,s,z(s))\|d\tau ds \\
&\quad + \frac{1}{\alpha}\int_{t_1}^{t_2} e^{\frac{-\beta}{\alpha}t_2} e^{\frac{\beta}{\alpha}s}\int_s^{t_2} \|K(\tau,s,z(s))\|d\tau ds \\
&\quad + \frac{1}{\alpha}\int_0^{t_1} e^{\frac{-\beta}{\alpha}t_1} e^{\frac{\beta}{\alpha}s}\left[\int_s^{t_2} \|K(\tau,s,z(s))\|d\tau - \int_s^{t_1} \|K(\tau,s,z(s))\|d\tau\right] ds \\
&\leq \left[\frac{2 + e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1} - 2e^{\frac{\beta}{\alpha}(t_1-t_2)}}{\beta}\right]\sigma \\
&\quad + \left[\frac{2 + e^{\frac{-\beta}{\alpha}t_2} - e^{\frac{-\beta}{\alpha}t_1} - 2e^{\frac{\beta}{\alpha}(t_1-t_2)}}{\beta}\right]\sigma^* \\
&\leq 2\left[\frac{1 - e^{\frac{\beta}{\alpha}(t_1-t_2)}}{\beta}\right]\sigma + 2\left[\frac{1 - e^{\frac{\beta}{\alpha}(t_1-t_2)}}{\beta}\right]\sigma^*.
\end{aligned}
$$

We notice that $\|Qz(t_2) - Qz(t_1)\| \to 0$ as $t_1 \to t_2$. Therefore, $(\overline{QB_r})$ is equicontinuous.

And because f and K are continuous, we can conclude that $Q : B_r \to B_r$. As a consequence of step 3 with Arzela-Ascoli's theorem, (QB_r) is a relatively compact set. Therefore, the operator Q is completely continuous. We now conclude the result of this theorem based on Krasnoselskii's theorem. $\square$

To prove the existence of a unique solution to Eq(1)-(2) via Weissinger's fixed point theorem, we define an operator

$$\begin{aligned} Ay(t) = e^{\frac{-\beta}{\alpha}t}y_0 + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}h(y(s))ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}f(s,y(s))ds \\ + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t K(\tau,s,y(s))d\tau ds. \end{aligned}$$

Given $y, x \in$ C(J,$\mathbb{R}$)$, t \in J$,

$$\begin{aligned} &|A(y)(t) - A(x)(t)| \\ &\leq \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|h(y(s)) - h(x(s))|ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}|f(s,y(s)) \\ &- f(s,x(s))|ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t |K(\tau,s,y(s)) - K(\tau,s,x(s))|d\tau ds \\ \leq &\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}M|y(s) - x(s)|ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}L|y(s) - x(s)|ds \\ &+ \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t \|\rho\|_{L^1}|y(s) - x(s)|d\tau ds \\ \leq &\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}M\|y - x\|_\infty ds + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}L\|y - x\|_\infty ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\int_0^t e^{\frac{\beta}{\alpha}s}\int_s^t \|\rho\|_{L^1}\|y-x\|_\infty d\tau ds\\
\leq&\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}M\|y-x\|_\infty\int_0^t e^{\frac{\beta}{\alpha}s}ds+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}L\|y-x\|_\infty\int_0^t e^{\frac{\beta}{\alpha}s}ds\\
&+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\|\rho\|_{L^1}\|y-x\|_\infty\int_0^t e^{\frac{\beta}{\alpha}s}ds\\
\leq&\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}M\|y-x\|_\infty\frac{\alpha}{\beta}[e^{\frac{\beta}{\alpha}t}-1]+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}L\|y-x\|_\infty\frac{\alpha}{\beta}[e^{\frac{\beta}{\alpha}t}-1]\\
&+\frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\|\rho\|_{L^1}\|y-x\|_\infty\frac{\alpha}{\beta}[e^{\frac{\beta}{\alpha}t}-1]\\
\leq&\frac{M\|y-x\|_\infty}{\beta}+\frac{L\|y-x\|_\infty}{\beta}+\frac{\|\rho\|_{L^1}\|y-x\|_\infty}{\beta}\\
=&\frac{M+L+\|\rho\|_{L^1}}{\beta}\|y-x\|_\infty.
\end{aligned}
$$

Therefore,

$$\|Ay-Ax\|_\infty \leq \frac{M+L+\|\rho\|_{L^1}}{\beta}\|y-x\|_\infty.$$

Theorem 3.4. [2] *Assume (H1),(H2), and (H3) hold, and in addition, assume that* $M+L<\frac{1}{2}, \|\rho\|_{L^1}\leq\frac{1}{2}$. *Then*

$$\|A^n y - A^n x\|_\infty \leq \frac{(M+L+\|\rho\|_{L^1})^n T^n}{\beta n!}\|y-x\|_\infty. \tag{6}$$

Proof. We prove using the principle of mathematical induction. For this, we assume that $n\in\{0,1,2,...\}, t\in J$.
For $n=0$, the statement is trivially true. Assuming (6) is true for $n=k$,

we prove the inequality for $n = k + 1$.

$$
\begin{aligned}
&|A^{k+1}y(t) - A^{k+1}x(t)| \\
&\qquad = |A(A^k y(t)) - A(A^k x(t))| \\
&\qquad \le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} |h(A^k y(s)) - h(A^k x(s))| ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} |f(s, A^k y(s)) - f(s, A^k x(s)| ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} \int_s^t |K(\tau, s, A^k y(s)) - K(\tau, s, A^k x(s))| d\tau ds \\
&\qquad \le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} M |(A^k y(s)) - (A^k x(s))| ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} L |A^k y(s)) - A^k x(s)| ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} \int_s^t \rho(\tau) |A^k y(s)) - A^k x(s))| d\tau ds \\
&\qquad \le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} \frac{M^{k+1} s^k}{k!} \|y - x\|_\infty ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} \frac{L^{k+1} s^k}{k!} \|y - x\|_\infty ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \int_0^t e^{\frac{\beta}{\alpha}s} \int_s^t \frac{\|\rho\|_{L^1}^{k+1} s^k}{k!} \|y - x\|_\infty d\tau ds \\
&\qquad \le \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \frac{M^{k+1}}{k!} \|y - x\|_\infty \int_0^t e^{\frac{\beta}{\alpha}s} s^k ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \frac{L^{k+1}}{k!} \|y - x\|_\infty \int_0^t e^{\frac{\beta}{\alpha}s} s^k ds \\
&\qquad\quad + \frac{1}{\alpha} e^{\frac{-\beta}{\alpha}t} \frac{\|\rho\|_{L^1}^{k+1}}{k!} \|y - x\|_\infty \int_0^t e^{\frac{\beta}{\alpha}s} s^k ds.
\end{aligned}
$$

We let $\alpha \ge \beta$. Then, we get

$$\begin{aligned}
|A^{k+1}y(t) - A^{k+1}x(t)| &\leq \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\frac{M^{k+1}}{k!}\|y-x\|_\infty \frac{\alpha}{\beta}e^{\frac{\beta}{\alpha}t}\int_0^t s^k ds \\
&\quad + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\frac{L^{k+1}}{k!}\|y-x\|_\infty \frac{\alpha}{\beta}e^{\frac{\beta}{\alpha}t}\int_0^t s^k ds \\
&\quad + \frac{1}{\alpha}e^{\frac{-\beta}{\alpha}t}\frac{\|\rho\|_{L^1}^{k+1}}{k!}\|y-x\|_\infty \frac{\alpha}{\beta}e^{\frac{\beta}{\alpha}t}\int_0^t s^k ds \\
&\leq \frac{M^{k+1}T^{k+1}}{\beta(k+1)!}\|y-x\|_\infty + \frac{L^{k+1}T^{k+1}}{\beta(k+1)!}\|y-x\|_\infty \\
&\quad + \frac{\|\rho\|_{L^1}^{k+1}T^{k+1}}{\beta(k+1)!}\|y-x\|_\infty \\
&= \frac{(M^{k+1}+L^{k+1}+\|\rho\|_{L^1}^{k+1})T^{k+1}}{\beta(k+1)!}\|y-x\|_\infty \\
&\leq \frac{(M+L+\|\rho\|_{L^1})^{k+1}T^{k+1}}{\beta(k+1)!}\|y-x\|_\infty \ .
\end{aligned}$$

Therefore,

$$\|A^{K+1}y - A^{K+1}x\|_\infty \leq \frac{(M+L+\|\rho\|_{L^1})^{k+1}T^{k+1}}{\beta(k+1)!}\|y-x\|_\infty.$$

In order to show that the operator A satisfies the assumptions of Weissinger's fixed point theorem with

$$\Omega_n = \frac{(M+L+\|\rho\|_{L^1})^n T^n}{\beta n!},$$

we just need to show that the series $\sum_{n=0}^{\infty}\Omega_n$ converges. So,

$$\sum_{n=0}^{\infty}\Omega_n = \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{((M+L+\|\rho\|_{L^1})T)^n}{\Gamma(n+1)}.$$

Fixing $\beta \in (0,1)$ so that $\frac{1}{\beta} = \Phi$, we get the power series representation of the Mittag-Leffler function

$$\Phi\mathbb{E}_{1,1}((M+L+\|\rho\|_{L^1})T).$$

Therefore, the series converges and thus A has a unique fixed point. Therefore Eq (1)-(2) has a unique solution due to Weissinger's fixed point theorem. $\square$

4 An Example

Consider the following nonlinear fractional integro differential equation

$$D_{0+}^{\frac{1}{2}}y(t)=\frac{1}{2}\left(\frac{(y(t))^2}{(y(t))^2+1}\right)+(e^{-t}-t+1)(y(t))^2+\int_0^t t\left(\frac{y(s)}{y(s)+1}\right)^{\frac{1}{4}}ds \tag{7}$$

with the initial condition

$$y(0)=0 \tag{8}$$

Here $\alpha=\frac{1}{2}, h(y(t))=\frac{1}{2}\left(\frac{(y(t))^2}{(y(t))^2+1}\right), f(t,y(t))=(e^{-t}-t+1)(y(t))^2$,

and $K(t,s,y(s))=t\left(\frac{y(s)}{y(s)+1}\right)^{\frac{1}{4}}$.

For $u,v\in X=R^+$ and $t\in J$, one can see that

$$\begin{aligned}|h(u)-h(v)| &= \frac{1}{2}\left|\frac{u^2}{u^2+1}-\frac{v^2}{v^2+1}\right| \\ &= \frac{1}{2}\left|\frac{u^2-v^2}{(u^2+1)(v^2+1)}\right| \\ &= \frac{1}{2}\left|(u-v)\frac{u+v}{(u^2+1)(v^2+1)}\right| \\ &\leq \frac{1}{2}\left|u-v\right|,\end{aligned}$$

Thus (**H1**)is satisfied with $M = \frac{1}{2}$,

$$\begin{aligned}\|f(t,u) - f(t,v)\| =& \|(e^{-t} - t + 1)[u]^{\frac{1}{2}} - (e^{-t} - t + 1)[v]^{\frac{1}{2}}\| \\ =& \|(e^{-t} - t + 1)(u^{\frac{1}{2}} - v^{\frac{1}{2}})\| \\ \leq& 2\|u^{\frac{1}{2}} - v^{\frac{1}{2}}\| \\ \leq& 2\|u - v\|^{\frac{1}{2}}\end{aligned}$$

Thus (**H2**) is satisfied with $\theta = \frac{1}{2}, L = 2$. Finally,

$$\begin{aligned}\|K(t,s,u) - K(t,s,v)\| =& \|t[\frac{u}{u+1}]^{\frac{1}{4}} - t[\frac{v}{v+1}]^{\frac{1}{4}}\| \\ \leq& t\|u^{\frac{1}{4}} - v^{\frac{1}{4}}\| \\ \leq& t\|u - v\|^{\frac{1}{4}}\end{aligned}$$

Thus, (**H3**) is satisfied with $\gamma = \frac{1}{4}, t \in L^1[0,1]$.
By applying Theorem 3.3, we conclude that the problem (1)-(2) has a solution on J.

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