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Original Research Paper

On Relative Deficiencies of Difference Polynomials from the View Point of Integrated Moduli of Logarithmic Derivative

S. Kumar Datta
University of Kalyani

S. Sarkar*
Dukhulal Nibaran Chandra College

G. Chakraborty
Sidho-Kanho-Birsha University

A. Manna
Mousini Co-operative High School(H.S.)

Abstract.
Let f be a transcendental entire function defined in the open complex plane $\mathbb{C}$. A difference-monomial generated by f is an expression of the form

$$F = f^n(f^m - 1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j},$$

where n, m and ν_j are all non-negative integers. Now for the sake of definiteness let us take,

$$M_i[f] = f^n(f^m - 1)\prod_{j=1}^{i}(f(z+c_j))^{\nu_j},$$

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*Corresponding Author

where $1 \leq i \leq d$. If $M_1[f], M_2[f], \ldots, M_n$[f] are such monomials in f as defined above, then $\psi[f] = a_1M_1[f] + a_2M_2[f] + \ldots + a_nM_n[f]$ where $a_i \neq 0$ $(i = 1, 2, \ldots, n)$ is called a difference-polynomial generated by f. In this paper, we compare the Valiron defect with the relative Nevanlinna defect of a particular type of differential-difference polynomial generated by a transcendental entire function with respect to integrated moduli of logarithmic derivative. Some examples are provided in order to justify the results obtained.

AMS Subject Classification: 30D35; 30D30
Keywords and Phrases: Entire function, meromorphic function, relative Nevanlinna defect, relative Valiron defect, difference polynomial, integrated moduli of logarithmic derivative

1 Introduction

Let f be a transcendental entire function defined in the open complex plane $\mathbb{C}$. A difference-monomial generated by f , is an expression of the form

$$F = f^n(f^m - 1)\prod_{j=1}^{d}(f(z + c_j))^{\nu_j} ,$$

where m, n and ν_j are all non-negative integers.

Now for the sake of definiteness let us take,

$$M_i[f] = f^n(f^m - 1)\prod_{j=1}^{i}(f(z + c_j))^{\nu_j} ,$$

where $1 \leq i \leq d$.

If $M_1[f], M_2[f], \ldots, M_n[f]$ are such monomials in f as defined above, then

$$\Psi[f] = a_1M_1[f] + a_2M_2[f] + \ldots + a_nM_n[f]$$

where $a_i \neq 0$ $(i = 1, 2, \ldots, n)$ is called a *difference-polynomial* generated by f .

For $a \in \mathbb{C} \cup \{\infty\}$, the quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty}\frac{N(r, a; f)}{T(r, f)} = \liminf_{r\to\infty}\frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna's deficiency of the value a. Similarly, the Valiron defect of a is defined as

$$\Delta(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r\to\infty} \frac{m(r,a;f)}{T(r,f)}.$$

The term

$$\delta_R^{(k)}(a;f) = 1 - \limsup_{r\to\infty} \frac{N\left(r,a;f^{(k)}\right)}{T(r,f)},\ k = 1,2,3,\ldots$$

is called the relative Nevanlinna's deficiencies of a with respect to $f^{(k)}$. In a like manner relative Valiron's deficiencies of a is defined. Xiong [6] has shown various relations between the usual deficiencies and the relative deficiencies for meromorphic functions.

The following definition is also obvious.

Definition 1.1. *The order ρ_f and the lower order λ_f of a meromorphic function f are defined as follows*

$$\rho_f = \limsup_{r\to\infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{T(r,f)}{\log r}.$$

If $\rho_f < \infty$ then f is of finite order.

We may now recall the following definition.

If f be a meromorphic function in $\mathbb{C}$ then its integrated moduli of the logarithmic derivative $I(r,f)$ is defined by

$$I(r,f) = \frac{r}{2\pi} \int_0^{2\pi} \left| \frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)} \right| d\theta$$

where $0 < r < +\infty$ $\{cf.[4]\}$.

In this paper by using the concept of $I(r,f)$, we call the following four terms as

$$\delta_I(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{I(r,f)},$$

$$\Delta_I(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;f)}{I(r,f)},$$

$$\delta_I^F(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;F)}{I(r,f)} \quad \text{and}$$

$$\Delta_I^F(a;f) = 1 - \liminf_{r\to\infty} \frac{N(r,a;F)}{I(r,f)}.$$

In this paper, we consider $F = f^n(f^m - 1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}$ and find some of these relationship between relative Nevanlinna's deficiencies and relative Valiron deficiencies under the flavour of integrated moduli of logarithmic derivative of *difference-polynomial* generated by transcendental entire functions in the direction of [1], [2] and [7]. Also relevant examples are provided in order to justify the sharper estimation of the results obtained. The term $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r\to\infty$ through all values of r if f is of finite order and except possibly for a set of r of finite linear measure otherwise. We do not explain the standard definitions and notations of the value distribution and the Nevanlinna theory of entire and meromorphic functions as those are available in [5] and [3].

2 Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1. *Let f be a transcendental entire function and*

$$F = f^n(f^m - 1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}.$$

Then

$$\lim_{r\to\infty}\frac{T(r,F)}{T(r,f)} = (n+m+\nu).$$

Lemma 2.2. $\{$*p.*41, *[3]*$\}$ *Let f be meromorphic and non-constant in $|z| \le R_0$. Then*

$$\frac{S(r,f)}{T(r,f)} \to 0 \tag{1}$$

as $r \to R_0$ *with the following provisions :*

(a) (1) *holds without restrictions if* $R_0 = +\infty$ *and* f *is of finite order in the plane.*

(b) If f *has infinite order in the plane,* (1) *still holds as* $r \to \infty$ *outside a certain exceptional set* E *of finite length. Here* E *depends only on* f*.*

(c) If $R_0 < +\infty$ *and*

$$\limsup_{r\to\infty} \frac{T(r,f)}{\log\{\frac{1}{R_0 - r}\}} = +\infty ,$$

then (1) *holds as* $r \to R_0$ *through a suitable sequence* r_n*, which depends on* f *only.*

Lemma 2.3. *[4] Let* f *be an entire function of finite order* ρ *with no zeros in* $\mathbb{C}$*. Then*

$$\lim_{r\to\infty} \frac{I(r,f)}{T(r,f)} = \pi\rho.$$

Lemma 2.4. *Let* f *be a non-constant meromorphic function of finite order in* $\mathbb{C}$*. Then*

$$\lim_{r\to\infty} \frac{S(r,f)}{I(r,f)} = 0.$$

Proof. In view of Lemma 2.2, we get that

$$\lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} = 0.$$

Now,

$$\begin{aligned} \lim_{r\to\infty} \frac{S(r,f)}{I(r,f)} &= \lim_{r\to\infty} \left\{ \frac{S(r,f)}{T(r,f)} \cdot \frac{T(r,f)}{I(r,f)} \right\} \\ &= \lim_{r\to\infty} \frac{S(r,f)}{T(r,f)} \cdot \lim_{r\to\infty} \frac{T(r,f)}{I(r,f)} = 0. \end{aligned}$$

This completes the proof of the lemma. $\square$

Lemma 2.5. *Let* f *be a transcendental entire function of non-zero finite order having the maximum deficiency sum. Also* f *has no zeros in* $\mathbb{C}$*. Then*

$$\delta_I(a;f) = \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)}$$

and

$$\Delta_I(a;f) = \left(1 - \frac{1}{\pi\rho}\right) + \limsup_{r\to\infty} \frac{m(r,a;f)}{I(r,f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I(a;f) &= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{I(r,f)} \\
&= 1 - \limsup_{r\to\infty} \left\{ \frac{N(r,a;f)}{T(r,f)} \cdot \frac{T(r,f)}{I(r,f)} \right\} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} \cdot \lim_{r\to\infty} \frac{T(r,f)}{I(r,f)} \\
&= 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} \cdot \frac{1}{\pi\rho} \\
&= \frac{1}{\pi\rho} \left\{ 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r,a;f)}{T(r,f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)} \cdot \frac{I(r,f)}{T(r,f)} \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \frac{1}{\pi\rho} \left\{ \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)} \cdot \pi\rho \right\} + \left(1 - \frac{1}{\pi\rho}\right) \\
&= \left(1 - \frac{1}{\pi\rho}\right) + \liminf_{r\to\infty} \frac{m(r,a;f)}{I(r,f)}.
\end{aligned}$$

This proves the first part of the lemma.
Similarly, we can prove the second part of the lemma. □

Lemma 2.6. *Let f be a transcendental entire function of non-zero finite order ρ having no zeros in $\mathbb{C}$ and*

$$F = f^n(f^m - 1) \prod_{j=1}^{d} (f(z + c_j))^{\nu_j}.$$

Then

$$\lim_{r\to\infty} \frac{T(r,F)}{I(r,f)} = \frac{(n+m+\nu)}{\pi\rho}.$$

Proof. In view of Lemma 2.1, we get that

$$\lim_{r\to\infty}\frac{T(r,F)}{T(r,f)}=(n+m+\nu).$$

Now,

$$\begin{aligned}\lim_{r\to\infty}\frac{T(r,F)}{I(r,f)} &= \lim_{r\to\infty}\frac{T(r,F)}{T(r,f)}\cdot\frac{T(r,f)}{I(r,f)}\\ &= \frac{(n+m+\nu)}{\pi\rho}.\end{aligned}$$

This completes the proof of the lemma. $\square$

3 Theorems

In this section, we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental entire function of non-zero finite order ρ with no zeros in $\mathbb{C}$ and*

$$F=f^n(f^m-1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j}.$$

If $n\geq 1$ then for any α,

$$\frac{(n+m+\nu)}{\pi\rho}+\delta_I^F(\alpha;f)=1+\lim\inf_{r\to\infty}\frac{m(r,\alpha;F)}{I(r,f)}$$

and

$$\frac{(n+m+\nu)}{\pi\rho}+\Delta_I^F(\alpha;f)=1+\lim\sup_{r\to\infty}\frac{m(r,\alpha;F)}{I(r,f)}.$$

Proof. We know that

$$\begin{aligned}
\delta_I^F(\alpha;f) &= 1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{I(r,f)}\\
&= 1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,F)}.\lim_{r\to\infty}\frac{T(r,F)}{I(r,f)}\\
&= 1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,F)}\cdot\frac{(n+m+\nu)}{\pi\rho}\\
&= \frac{(n+m+\nu)}{\pi\rho}+\left[1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,F)}\cdot\frac{(n+m+\nu)}{\pi\rho}\right]-\frac{(n+m+\nu)}{\pi\rho}\\
&= \frac{(n+m+\nu)}{\pi\rho}\left[1-\limsup_{r\to\infty}\frac{N(r,\alpha;F)}{T(r,F)}\right]+\left\{1-\frac{(n+m+\nu)}{\pi\rho}\right\}\\
&= \frac{(n+m+\nu)}{\pi\rho}\liminf_{r\to\infty}\frac{m(r,\alpha;F)}{T(r,F)}+\left\{1-\frac{(n+m+\nu)}{\pi\rho}\right\}\\
&= \frac{(n+m+\nu)}{\pi\rho}\liminf_{r\to\infty}\frac{m(r,\alpha;F)}{I(r,f)}\cdot\lim_{r\to\infty}\frac{I(r,f)}{T(r,F)}+\left\{1-\frac{(n+m+\nu)}{\pi\rho}\right\}\\
&= \frac{(n+m+\nu)}{\pi\rho}\liminf_{r\to\infty}\frac{m(r,\alpha;F)}{I(r,f)}\cdot\frac{\pi\rho}{(m+n+\nu)}+\left\{1-\frac{(n+m+\nu)}{\pi\rho}\right\}\\
&= \limsup_{r\to\infty}\frac{m(r,\alpha;F)}{I(r,f)}+\left\{1-\frac{(n+m+\nu)}{\pi\rho}\right\}.
\end{aligned}$$

$$i.e., \frac{(n+m+\nu)}{\pi\rho}+\delta_I^F(\alpha;f)=1+\lim\inf_{r\to\infty}\frac{m(r,\alpha;F)}{I(r,f)}.$$

This proves the first part of the theorem.
Similarly, we can prove the second part of the theorem. □

Theorem 3.2. *Let f be a transcendental entire function of non-zero finite order ρ having no zeros in $\mathbb{C}$ and a be any non-zero finite complex number. Then*

$$\delta_I(0;f)+\Delta_I^F(\infty;f)+\delta_I(a;f)+\frac{1}{\pi\rho}\le\Delta_I(\infty;f)+\Delta_I^F(0;f)+1.$$

Proof. Let us consider the following identity

$$\frac{a}{f}=1-\frac{f-a}{F}\frac{F}{f}.$$

In view of Lemma 2.2 and $m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{a}{f}\right) + O(1)$, we get from the above identity that

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f-a}{F}\right) + m\left(r, \frac{F}{f}\right)$$

$$i.e., \quad m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{f-a}{F}\right) + S(r, f). \tag{2}$$

Now by Nevanlinna's first fundamental theorem, it follows from Inequality (2) that

$$m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{f-a}{F}\right) - N\left(r, \frac{f-a}{F}\right) + S(r, f)$$

$$i, e., \ m\left(r, \frac{1}{f}\right) \leq T\left(r, \frac{F}{f-a}\right) - N\left(r, \frac{f-a}{F}\right) + S(r, f)$$

$$i.e., \ m\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{F}{f-a}\right) + m\left(r, \frac{F}{f-a}\right) - N\left(r, \frac{f-a}{F}\right) + S(r, f)$$

$$i.e., \ m\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{F}{f-a}\right) - N\left(r, \frac{f-a}{F}\right) + S(r, f). \tag{3}$$

In view of Lemma 2.4 and Lemma 2.5, we obtain from Inequality (3) that

$$m\left(r, \frac{1}{f}\right) \leq N(r, F) + N\left(r, \frac{1}{f-a}\right) - N(r, f-a) - N\left(r, \frac{1}{F}\right) + S(r, f)$$

$$\begin{aligned} i.e., \ \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f}\right)}{I(r, f)} \leq \liminf_{r\to\infty} &\left\{ \frac{N(r, F)}{I(r, f)} - \frac{N(r, f)}{I(r, f)} - \frac{N\left(r, \frac{1}{F}\right)}{I(r, f)} \right\} \\ &+ \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{I(r, f)} \end{aligned}$$

$$\begin{aligned} i.e., \ \liminf_{r\to\infty} \frac{m\left(r, \frac{1}{f}\right)}{I(r, f)} \leq \liminf_{r\to\infty} \frac{N(r, F)}{I(r, f)} &- \liminf_{r\to\infty} \frac{N(r, f)}{I(r, f)} \\ &- \liminf_{r\to\infty} \frac{N\left(r, \frac{1}{F}\right)}{I(r, f)} + \limsup_{r\to\infty} \frac{N\left(r, \frac{1}{f-a}\right)}{I(r, f)} \end{aligned}$$

$$i.e.,\ \delta_I(0;f)-\left(1-\frac{1}{\pi\rho}\right)\le\left\{1-\Delta_I^F(\infty;f)\right\}-\left\{1-\Delta_I(\infty;f)\right\}$$
$$-\left\{1-\Delta_I^F(0;f)\right\}+\left\{1-\delta_I(a;f)\right\}$$

$$i.e.,\ \delta_I(0;f)+\Delta_I^F(\infty;f)+\delta_I(a;f)+\frac{1}{\pi\rho}\le\Delta_I(\infty;f)+\Delta_I^F(0;f)+1.$$

This proves the theorem. □

Remark 3.3. The condition that a is any non-zero finite complex number in Theorem 3.2 is essential as is evident from the following example.

Example 3.4. Let $f=\exp z$, $n=0$, $m=1$, $\nu=0$ and $a=0,\infty$. Then we see that $N(r,f)=0$,

$$\begin{aligned}F&=f^n(f^m-1)\prod_{j=1}^{d}\left(f(z+c_j)\right)^{\nu_j}\\&=e^z-1,\end{aligned}$$

$$\begin{aligned}I(r,f)&=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{re^{i\theta}}\cdot re^{i\theta}\cdot i}{e^{re^{i\theta}}}\right|d\theta\\&=\frac{r}{2\pi}\int_0^{2\pi}\left|re^{i\theta}\cdot i\right|d\theta\\&=\frac{r}{2\pi}\int_0^{2\pi}(r)\,d\theta=\frac{r^2}{2\pi}\int_0^{2\pi}d\theta=\frac{r^2}{2\pi}\cdot 2\pi=r^2\neq 0\end{aligned}$$

and

$$\rho=\limsup_{r\to\infty}\frac{\log T(r;f)}{\log r}=\limsup_{r\to\infty}\frac{\log\frac{r}{\pi}}{\log r}=1.$$

Now,

$$\delta_I(0;f)=\Delta_I^F(\infty;f)=\delta_I(\infty;f)=\Delta_I(\infty;f)=\Delta_I^F(0;f)=1$$

Hence

$$\delta_I(0;f)+\Delta_I^F(\infty;f)+\delta_I(a;f)+\frac{1}{\pi\rho}=3+\frac{1}{\pi}$$

and

$$\Delta_I(\infty;f)+\Delta_I^F(0;f)+1=3,$$

which contradicts Theorem 3.2.

Theorem 3.5. *Let $a, b \neq 0, \infty$ be any two distinct complex numbers. Then for any transcendental entire function f of non-zero finite order ρ with no zeros in $\mathbb{C}$,*

$$2\delta_I(a; f) + \delta_I(b; f) + 2\Delta_I^F(\infty; f) + \frac{1}{\pi\rho} \leq 2\Delta_I(\infty; f) + 2\Delta_I^F(0; f) + 1.$$

Proof. Considering the identity

$$\frac{b-a}{f-a} = \frac{F}{f-a}\left\{\frac{f-a}{F} - \frac{f-b}{F}\right\}$$

and in view of Lemma 2.1, we obtain that

$$m\left(r, \frac{b-a}{f-a}\right) \leq m\left(r, \frac{f-a}{F}\right) + m\left(r, \frac{f-b}{F}\right) + m\left(r, \frac{F}{f-a}\right)$$

$$\begin{aligned} i.e.,\ m\left(r, \frac{b-a}{f-a}\right) \leq\ & T\left(r, \frac{f-a}{F}\right) - N\left(r, \frac{f-a}{F}\right) + T\left(r, \frac{f-b}{F}\right) \\ & - N\left(r, \frac{f-b}{F}\right) + S(r, f). \end{aligned} \tag{4}$$

Since $m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{b-a}{f-a}\right) + O(1)$ and $T(r, f) = T\left(r, \frac{1}{f}\right) + O(1)$, it follows from Inequality (4) that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) \leq\ & T\left(r, \frac{F}{f-a}\right) - N\left(r, \frac{f-a}{F}\right) + T\left(r, \frac{F}{f-b}\right) \\ & - N\left(r, \frac{f-b}{F}\right) + S(r, f) + O(1) \end{aligned}$$

$$\begin{aligned} i.e., m\left(r, \frac{1}{f-a}\right) \leq\ & N\left(r, \frac{F}{f-a}\right) + m\left(r, \frac{F}{f-a}\right) - N\left(r, \frac{f-a}{F}\right) \\ & + N\left(r, \frac{F}{f-b}\right) + m\left(r, \frac{F}{f-b}\right) - N\left(r, \frac{f-b}{F}\right) \\ & + S(r, f) + O(1) \end{aligned}$$

$$i.e., m\left(r,\frac{1}{f-a}\right) \leq N\left(r,\frac{F}{f-a}\right) - N\left(r,\frac{f-a}{F}\right) + N\left(r,\frac{F}{f-b}\right) - N\left(r,\frac{f-b}{F}\right) + S\left(r,f\right) + O\left(1\right). \quad (5)$$

In view of Lemma 2.4 and Lemma 2.5, we get from Inequality (5) that

$$m\left(r,\frac{1}{f-a}\right) \leq N\left(r,F\right) + N\left(r,\frac{1}{f-a}\right) - N\left(r,f-a\right) - N\left(r,\frac{1}{F}\right) + N\left(r,F\right) + N\left(r,\frac{1}{f-b}\right) - N\left(r,f-b\right) - N\left(r,\frac{1}{F}\right) + S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f-a}\right) \leq 2N\left(r,F\right) - 2N\left(r,f\right) - 2N\left(r,\frac{1}{F}\right) + N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) + S\left(r,f\right) + O(1)$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I\left(r,f\right)} \leq 2\liminf_{r\to\infty}\left\{\frac{N\left(r,F\right)}{I\left(r,f\right)} - \frac{N\left(r,f\right)}{I\left(r,f\right)} - \frac{N\left(r,\frac{1}{F}\right)}{I\left(r,f\right)}\right\} + \limsup_{r\to\infty}\left\{\frac{N\left(r,\frac{1}{f-a}\right)}{I\left(r,f\right)} + \frac{N\left(r,\frac{1}{f-b}\right)}{I\left(r,f\right)}\right\}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I\left(r,f\right)} \leq 2\left\{\liminf_{r\to\infty}\frac{N\left(r,F\right)}{I\left(r,f\right)} - \liminf_{r\to\infty}\frac{N\left(r,f\right)}{I\left(r,f\right)} - \liminf_{r\to\infty}\frac{N\left(r,\frac{1}{F}\right)}{I\left(r,f\right)}\right\} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-a}\right)}{I\left(r,f\right)} + \limsup_{r\to\infty}\frac{N\left(r,\frac{1}{f-b}\right)}{I\left(r,f\right)}$$

$$i.e.,\ \delta_I\left(a;f\right) - \left(1-\frac{1}{\pi\rho}\right) \leq 2\left\{1-\Delta_I^F\left(\infty;f\right)\right\} - 2\left\{1-\Delta_I\left(\infty;f\right)\right\} - 2\left\{1-\Delta_I^F\left(0;f\right)\right\} + \left\{1-\delta_I\left(a;f\right)\right\} + \left\{1-\delta_I\left(b;f\right)\right\}$$

i.e., $2\delta_I(a;f)+\delta_I(b;f)+2\Delta_I^F(\infty;f)+\dfrac{1}{\pi\rho} \leq 2\Delta_I(\infty;f)+2\Delta_I^F(0;f)+1.$

Thus the theorem is established. $\square$

Remark 3.6. The condition that a and b are two distinct complex numbers in Theorem 3.5 is necessary as we see in the following example.

Example 3.7. Let $f = \exp(2z)$, $n = 0$, $m = 1$, $\nu = 0$, $a = 0$, ∞ and $b = 0$, ∞. Then we see that $N(r,f) = 0$,

$$\begin{aligned} F &= f^n(f^m - 1)\prod_{j=1}^{d}(f(z+c_j))^{\nu_j} \\ &= e^{2z} - 1, \end{aligned}$$

$$\begin{aligned} I(r,f) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right| d\theta = \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{e^{2re^{i\theta}}\cdot 2re^{i\theta}\cdot i}{e^{re^{i\theta}}}\right| d\theta \\ &= \frac{r}{2\pi}\int_0^{2\pi}\left|2re^{i\theta}\cdot i\right| d\theta \\ &= \frac{r}{\pi}\int_0^{2\pi}(r)\,d\theta = \frac{r^2}{\pi}\int_0^{2\pi}d\theta = \frac{r^2}{\pi}\cdot 2\pi = 2r^2 \neq 0 \end{aligned}$$

and

$$\rho = \limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} = \limsup_{r\to\infty}\frac{\log 2r}{\log r} = 1.$$

Also,

$$\delta_I(a;f) = \delta_I(b;f) = \Delta_I^F(\infty;f) = \Delta_I(\infty;f) = \Delta_I^F(0;f) = 1.$$

Hence,

$$2\delta_I(a;f) + \delta_I(b;f) + 2\Delta_I^F(\infty;f) + \frac{1}{\pi\rho} = 5 + \frac{1}{\pi}$$

and

$$2\Delta_I(\infty;f) + 2\Delta_I^F(0;f) + 1 = 5,$$

which is contrary to Theorem 3.5.

Theorem 3.8. *Let f be a transcendental entire function of non-zero finite order ρ having no zeros in $\mathbb{C}$ and a be a finite complex number and b, c be two distinct non-zero complex numbers. Then*

$$\delta_I(a;f) + \delta_I^F(b;f) + \delta_I^F(c;f) \leq \frac{2(m+n+\nu)}{\pi\rho}.$$

Proof. Let

$$\frac{1}{f-a} = \frac{F}{f-a}\frac{1}{F}.$$

In view of Lemma 2.1, we obtain that

$$m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{1}{F}\right) + m\left(r, \frac{F}{f-a}\right)$$

$$i.e.,\ m\left(r, \frac{1}{f-a}\right) \leq m\left(r, \frac{1}{F}\right) + S(r,f). \tag{6}$$

Applying Nevanlinna's first fundamental theorem, we get from Inequality (6) that

$$m\left(r, \frac{1}{f-a}\right) \leq T\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{F}\right) + S(r,f).$$

Now by Nevanlinna's second fundamental theorem, it follows from Lemma 2.1 that

$$\begin{aligned} m\left(r, \frac{1}{f-a}\right) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{F-b}\right) + \bar{N}\left(r, \frac{1}{F-c}\right) \\ - N\left(r, \frac{1}{F}\right) + S(r,f). \end{aligned} \tag{7}$$

In view of Lemma 2.4, 2.5, 2.6 and $\bar{N}\left(r, \frac{1}{F}\right) - N\left(r, \frac{1}{F}\right) \leq 0$, we obtain from Inequality (7) that

$$m\left(r, \frac{1}{f-a}\right) \leq \bar{N}\left(r, \frac{1}{F-b}\right) + \bar{N}\left(r, \frac{1}{F-c}\right) + S(r,f)$$

$$i.e.,\ m\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{F-b}\right) + N\left(r, \frac{1}{F-c}\right) + S(r,f)$$

$$i.e.,\ m\left(r,\frac{1}{f-a}\right)\leq T\left(r,\frac{1}{F-b}\right)+T\left(r,\frac{1}{F-c}\right)-m\left(r,\frac{1}{F-b}\right)$$
$$-m\left(r,\frac{1}{F-c}\right)+S\left(r,f\right)$$

$$i.e.,\ m\left(r,\frac{1}{f-a}\right)\leq 2T\left(r,F\right)-m\left(r,\frac{1}{F-b}\right)$$
$$-m\left(r,\frac{1}{F-c}\right)+S\left(r,f\right)$$

$$i.e.,\ \liminf_{r\to\infty}\frac{m\left(r,\frac{1}{f-a}\right)}{I\left(r,f\right)}\leq 2\liminf_{r\to\infty}\frac{T\left(r,F\right)}{I\left(r,f\right)}-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-b}\right)}{I\left(r,f\right)}$$
$$-\liminf_{r\to\infty}\frac{m\left(r,\frac{1}{F-c}\right)}{I\left(r,f\right)}$$

$$i.e.,\ \delta_I\left(a;f\right)\leq 2\cdot\frac{(m+n+\nu)}{\pi\rho}-\delta_I^F\left(b;f\right)-\delta_I^F\left(c;f\right)$$

$$i.e.,\ \delta_I\left(a;f\right)+\delta_I^F\left(b;f\right)+\delta_I^F\left(c;f\right)\leq\frac{2(m+n+\nu)}{\pi\rho}.$$

Thus the theorem is established. □

Remark 3.9. The conditions that b and c are two distinct non zero complex numbers in Theorem 3.8 are essential as is evident from the following examples.

Example 3.10. Let $f=\exp\left(z^2\right)$, $n=0$, $m=1$, $\nu=0$, $a=0$ and $b=c=\infty$. Then we see that $F=e^{z^2}-1$, $N(r,f)=N(r,F)=0$,

$$\begin{aligned}
T\left(r,f\right) &= m\left(r,f\right)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left|e^{r^2e^{2i\theta}}\right|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|e^{r^2(\cos 2\theta+i\sin 2\theta)}\right|d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log^+\left(e^{r^2\cos 2\theta}\right)d\theta=\frac{1}{2\pi}\int_0^{2\pi}r^2\cos 2\theta d\theta=\frac{r^2}{\pi},
\end{aligned}$$

$$\rho=\limsup_{r\to\infty}\frac{\log^{[2]}M\left(r,f\right)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[2]}e^{r^2}}{\log r}=\limsup_{r\to\infty}\frac{2\log r}{\log r}=2$$

and

$$\begin{aligned}I\left(r,f\right) &= \frac{r}{2\pi}\int_0^{2\pi}\left|\frac{f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right|d\theta=\frac{r}{2\pi}\int_0^{2\pi}\frac{\left|e^{r^2e^{2i\theta}}\right|\cdot\left|2ir^2e^{2i\theta}\right|}{\left|e^{r^2e^{2i\theta}}\right|}d\theta\\ &= \frac{r}{2\pi}\cdot 2r^2\int_0^{2\pi}\frac{e^{r^2\cos 2\theta}\cdot e^{c\cos 2\theta}}{e^{r^2\cos 2\theta}}d\theta=\frac{r^3}{\pi}\int_0^{2\pi}e^{\cos 2\theta}d\theta\\ &= \frac{r^3}{\pi}\cdot\frac{1}{2}\int_0^{4\pi}e^{\cos\eta}d\eta=\frac{r^3}{2\pi}\cdot 4\pi I_0\left(1\right)=\frac{2r^3}{\pi}\cdot I_0\left(1\right)\neq 0,\end{aligned}$$

where $I_n\left(z\right)$ is the modified Bessel function of the first kind such that

$$I_n\left(z\right)=\frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\cdot\cos n\theta\, d\theta.$$

Therefore,

$$\delta_I\left(a;f\right)=\delta_I^F\left(b;f\right)=\delta_I^F\left(c;f\right)=\delta_I\left(0;f\right)=\delta_I^F\left(\infty;f\right)=1.$$

Hence,

$$\delta_I\left(a;f\right)+\delta_I^F\left(b;f\right)+\delta_I^F\left(c;f\right)=3,$$

and

$$\frac{2(m+n+\nu)}{\pi\rho}=\frac{2}{2\pi}=\frac{1}{\pi}.$$

Since $3\nleq\frac{1}{\pi}$, we arrive at a contradiction to Theorem 3.8.

Example 3.11. Let $f=\exp\left(z^2\right)$, $n=0$, $m=1$, $\nu=0$, $a=0$ and $b=c=0$. Then we see that $F=e^{z^2}-1$, $N(r,f)=N(r,F)=0$, $T\left(r,f\right)=\frac{r^2}{\pi}$, $\rho=1$ and $I\left(r,f\right)=\frac{2r^3}{\pi}\cdot I_0\left(1\right)\neq 0$, where $I_n\left(z\right)$ is the modified Bessel function of the first kind such that

$$I_n\left(z\right)=\frac{1}{\pi}\int_0^{\pi}e^{z\cos\theta}\cdot\cos n\theta\, d\theta.$$

Therefore,

$$\delta_I\left(a;f\right)=\delta_I^F\left(b;f\right)=\delta_I^F\left(c;f\right)=\delta_I\left(0;f\right)=\delta_I^F\left(0;f\right)=1.$$

Hence,

$$\delta_I\left(a;f\right)+\delta_I^F\left(b;f\right)+\delta_I^F\left(c;f\right)=3$$

and

$$\frac{2(m+n+\nu)}{\pi\rho}=\frac{2}{2\pi}=\frac{1}{\pi}.$$

Since $3 \nleq \frac{1}{\pi}$, this contradicts Theorem 3.8.

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Sanjib Kumar Datta
Department of Mathematics
Professor of Mathematics
University of Kalyani
Kalyani, India
E-mail: sanjibdatta05@gmail.com

Sukalyan Sarkar
Department of Mathematics
Assistant Professor of Mathematics
Dukhulal Nibaran Chandra College
Murshidabad, India
E-mail: sukalyanmath.knc@gmail.com

Gorachand Chakraborty
Department of Mathematics
Assistant Professor of Mathematics
Sidho-Kanho-Birsha University
Purulia, India
E-mail: gorachand11@gmail.com

Arghyatanu Manna
Department of Mathematics
Assistant Teacher of Mathematics
Mousini Co-operative High School (H.S.)
Kakdwip, India
E-mail: arghyatanu.manna2011@gmail.com