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Strong Synchronized System

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Abstract. In this paper introduced the notion of a strong synchronized system; that is a synchronized system whose there is unique finite path in Fischer cover labeled synchronizing block. We aim to introduce a class of synchronized systems containing sofics. Every irreducible sofic shift is an strong synchronized.

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1 Introduction

One of the most studied dynamical systems is a subshift of finite type (SFT). An SFT is a system whose set of forbidden blocks is finite ([6]). Equivalently, an SFT X is a subshift whose any block of length greater than a certain number M is synchronizing; that is, if m is *any* block with $|m| \geq M$ and if v_1m and mv_2 are both blocks of X, then v_1mv_2 is a block of X. If an irreducible system has at least one synchronizing block, then it is called a *synchronized system* and examples are *sofics*: factors of SFT's.

For a synchronized system, Fiebig in ([2]) prove that there is some finite path e in Fischer cover labeled m terminating in α such that

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$m \in F_{-}(\alpha)$. In the other hand, cardinal$\{e \in \mathcal{E}_{X_0^+} : t(e) = \alpha\} \geq 1$. In this note we consider shift space that cardinal$\{e \in \mathcal{E}_{X_0^+} : t(e) = \alpha\} = 1$.

2 Background and Definitions

This section is devoted to the very basic definitions in symbolic dynamics. The notations have been taken from ([6]) and the proof of the relevant claims in this section can be found there. Let $\mathcal{A}$ be an alphabet, that is a non-empty finite set. The full shift $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. A block over $\mathcal{A}$ is a finite sequences of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty block* and denoted by ε. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote a block of length $j - i$ by $x_{[i,\,j]} = x_i x f_{i+1} \ldots x_j$. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. Let $\mathcal{F}$ be the collection of all forbidden blocks over $\mathcal{A}$. The complement of $\mathcal{F}$ is the set of admissible blocks or just blocks in X. For any such $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any block in $\mathcal{F}$. A shift space is a closed subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$. For some collection $\mathcal{F}$ of forbidden blocks over $\mathcal{A}$.

Let $W_n(X)$ denote the set of all admissible n blocks. The language of X is the collection $W(X) = \cup_n W_n(X)$. A shift space X is *irreducible* if for every ordered pair of blocks $u,\, v \in W(X)$ there is a block $w \in W(X)$ so that $uwv \in W(X)$. It is *mixing* if for every ordered pair $u,\, v \in W(X)$, there is an N such that for each $n \geq N$ there is a block $w \in W_n(X)$ such that $uwv \in W(X)$. A shift space X is called a *shift of finite type* SFT if there is a finite set $\mathcal{F}$ of forbidden blocks such that $X = X_{\mathcal{F}}$. An *edge shift* denoted by $X(G)$, is a shift space that consist of all bi-infinite walks in a directed graph G. Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A *labeled graph* $\mathcal{G}$ is a pair $(G,\, \mathcal{L})$ where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. A *sofic shift* $X_{\mathcal{G}}$ is the set of sequences obtained by reading the label of walks on G.

$$X_{\mathcal{G}} = \{\mathcal{L}_{\infty}(\varepsilon) : \varepsilon \in X_G\} = \mathcal{L}_{\infty}(X_G).$$

We say $\mathcal{G}$ is a presentation of $X_{\mathcal{G}}$. The follower set of a vertex m of G

is $F_-(m) = \{\mathcal{L} - label$ of all finite paths terminating at $m\}$. Every SFT is $sofic$ ([6, Theorem 3.1.5]), but the converse is not true. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels.

Let X be a shift space and $w \in W(X)$. The follower set $F_+(w)$ of w is defined by $F_+(w) = \{v \in W(X) : wv \in W(X)\}$ (resp. $F_-(w)$). Let $x \in W(X)$. Then, $x_+ = (x_i)_{i \in Z^+}$ $(resp.\ x_- = (x_i)_{i<0})$ is called $right$ $(resp.\ left)$ infinite X-ray. For a left infinite X-ray, say x_- its follower set is $w_+(x_-) = \{x^+ \in X^+ : x_- x_+ \in X\}$. Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph X^+. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that $x_- a$ is an X-ray and $I_1 = w_+(x_-), I_2 = w_+(x_- a)$. This labeled graph is called the *Krieger graph* for X. A block $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$. An irreducible shift space X is a synchronized system if it has synchronizing block. A block $m \in W(X)$ is *half synchronizing* if there is a left transitive point $x \in X$ such that $x_{[-|m|+1], 0]} = m$ and $w_+(x_{(-\infty, 0]}) = w_+(m)$, That we denote by

$$(x,\, m). \tag{1}$$

If X is a half synchronized system with half synchronizing m, the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by X_0^+ and is called the *right Ficher cover* of X.

Let X be a shift space. The *entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |W_n(X)|.$$

For any synchronized system X, we define the *synchronized entropy* h_{syn} to be

$$h_{\text{syn}}(X) = \limsup_n \frac{1}{n} \log(\text{cardinal}\{a \in W_n(Y) : mam \in W(X)\}),$$

where m is an arbitrary synchronizing block in $W(X)$. A shift space X is *almost sofic* if there are sofic shifts $X_n \subseteq X$ such that $\lim_{n \to \infty} h(X_n) = h(X)$.

3 Strong Synchronized Systems

We aim to introduce a class of synchronized systems containing sofics. Let X be a shift space and $R(X) = \overline{\mathrm{Per}X}$ and set $S(X)$ to denote the set of synchronizing blocks for $R(X)$. For $s, t \in S(X)$ we write $s \sim t$ whenever there are blocks $u, v \in W(R(X))$ such that sut, $tvs \in W(R(X))$. Then, $\sim$ is an equivalence relation in $S(X)$. Note that $s \sim t$ if only if there is an $x \in R(X) = \overline{\mathrm{Per}X}$ such that $s, t \subseteq x$. Let $x \in R(X)$ and for integers p and s with $p \leq s$, set

$$\mathrm{gap}(x_{[p,q]}, x_{[s,t]}) = \begin{cases} 0 & s \leq q \\ s-q & \text{otherwise} \end{cases}$$

and call it the *gap* between two blocks $x_{[p,q]}$ and $x_{[s,t]}$. Pick $\alpha \in S(X)/\sim$, $p < s$, $q \leq t$ and $\{u = x_{[p,q]}, v = x_{[s,t]}\} \subseteq \alpha$. If the only minimal synchronizing blocks in $x_{[p,t]}$ are u and v, then call u and v the *consecutive minimal pairs of* α *in* $x_{[p,t]} \subset x$. In ([2]) $X_{(\alpha,0)}$ was defined to be the set of elements of $x \in R(X)$ satisfying the following.

i. If $i \in \mathbb{Z}$, then there are $m, m' \in \alpha$ such that $m \subseteq x_{(-\infty,i]}$, $m' \subseteq x_{[i,+\infty)}$.

ii. $|\{u \in \alpha : u \subseteq x\}| < \infty$.

iii. There is $M > 0$ such that if u and v are the consecutive minimal pairs of α in x, then $\mathrm{gap}(u, v) \leq M$.

Example 3.1. Let G be the graph in Figure 1 and $X = X(G)$. Then, $X_{(\alpha,0)} = \{0, 1\}^{\mathbb{Z}}$.

We associate to $X_{(\alpha,0)}$ an irreducible graph Γ_α. To do so for each $m \in \alpha$, let $F(m) = \{u \in W(X_{(\alpha,0)}) : \ mu \in W(X_{(\alpha,0)})\}$ and let the vertices of Γ_α to be $\{F(m) : m \in \alpha\}$. Assign an edge labeled a from $F(m)$ to $F(m')$ when $a \in F(m)$ and $F(ma) = F(m')$. Then, Γ_α is a minimal right resolving cover and is called *Fischer cover* of $\overline{X_{(\alpha,0)}}$.

Definition 3.2. *A block* $m \in W(X)$ *is called* strong synchronizing *if there is an element* $\alpha \in S(X)/\sim$ *such that whenever there are two finite paths* e, e' *in* Γ_α *labeled* m, *then* $e = e'$.

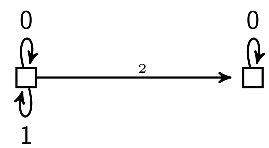

Figure 1: The graph G.

Call an irreducible shift space X *strong synchronized* if it has a strong synchronizing block. In the sequel, we will show that

$$\text{sofics} \subsetneq \text{strong synchronized systems} \subsetneq \text{synchronized systems}. \quad (2)$$

Clearly, any strong synchronizing block is a synchronizing block. The next example shows that the second inclusion in (2) is not equality.

Example 3.3. Let X be the Dyke system. Add a new symbol $*$ (which will be a synchronizing block) to the set of four brackets and let S be the subshift which consists of all bi-infinite sequences of these five symbols such that any finite subblock which does not contain a $*$ obeys the standard brackets rules ([2]). We claim that S is not strong synchronized system.

Let u be a strong synchronizing block of S. So that is a synchronizing block and $* \subseteq u$. We can write $u = a * b$ where $a, b \in W(S)$. Since $*$ is a synchronizing block, there is a unique vertex β of Fischer cover S_0^+ with this properties $* \in F_-(\beta)$ and $w_+(\beta) = w_+(*)$ ([2, Page 146]). Since $*b \in W(S)$, so there is a cycle C containing b and passing through $w_+(*)$ and $w_+(u)$. See Figure 2.

Pick $a' \in W(S)$ such that $[a'$, $(a' \in W(S)$ and for each $x_- \in S^-$, $x_- a' a \in S^-$. Set $x_- := \cdots]]][a'$, $y_- := \cdots]]](a'$. Since $w_+(x_-) \neq w_+(y_-)$ and

$$w_+(x_- a*) = w_+(y_- a*),\ w_+(*[a') = w_+(x_-),\ w_+(*(a') = w_+(y_-),$$

so there are finite pathes e, e' labeled $a*b = u$ and terminating at $w_+(u)$. This show that synchronized system S is not strong synchronized.

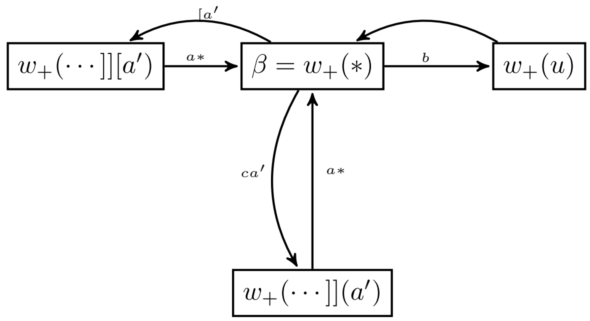

Figure 2: A subgraph of Fischer cover S.

If m is a synchronizing block of X, then there is a unique vertex I of Γ_α with the property $m \in F_-(I)$ as in $w_+(u)$ in Figure 2. But m is an strong synchronizing block if and only if there are two unique vertexes I and J of Γ_α such that $m \in F_-(I)$ and $m \in F_+(J)$. Note that if X is a synchronized system, then $\{\alpha\} = S(X)/\sim$ and $\Gamma_\alpha = X_0^+$.

Proposition 3.4. *Every irreducible sofic shift is an strong synchronized.*

Proof. Let m be a synchronizing block of X. There is a finite path π in Fischer cover X_0^+ labeled m. Set $i(\pi) := I$ and $t(\pi) := J$. Since labeling of finite graph X_0^+ is right resolving, so the labeling of X_0^+ is left closing ([2, Theorem 3.2]) and so m is a strong synchronizing block of X. $\square$
The next example shows that the converse of the above proposition is not necessarily true and so (2) is sort out compeletly.

Example 3.5. Set $1_\beta := 21010^2 10^3 1\ldots.$ Since 1_β is not eventually periodic so by ([3, Theorem 2.4.2]) β-shift is not sofic. Since 2 is a strong synchronizing block for β-shift, so it is a strong synchronized.

Set $S_t(X)$ to denote the set of strong synchronizing blocks for X. If $m \in S_t(X)$ and $\alpha = [m]$, then for each $a \in W(X_{(\alpha,0)})$, let C'_a be a cycle in Γ_α containing a and passing through $w_+(m)$. Let u, $v \in W(X_{(\alpha,0)})$. Set the *distance* between two blocks u and v, to be 0 when $u = v$ and

$$\min\{\frac{1}{2}(|u'mu''|+|v'mv''|) : \mathcal{L}(C'_{uu'mu''}) = uu'mu'', \mathcal{L}(C'_{vv'mv''}) = vv'mv''\}$$

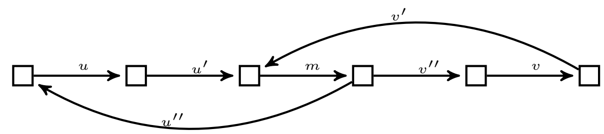

Figure 3: The distance of u and v.

otherwise. Denote this distance by $d_m(u, v)$ see Figure 3. It is not hard to see that $(W(X_{(\alpha,0)}), d_m)$ is a metric space. Note that the definition of d_m is dependent on the particular choice of m. Let $m_1, m_2 \in \alpha \cap S_t(X)$. Then, for each $u \in W(X_{(\alpha,0)})$ $d_{m_1}(m_2, u) = d_{m_1}(m_1, u) + d_{m_2}(m_1, m_2) + |m_1'| + |m_2| - |m_2'|$.

Lemma 3.6. *If $uv \in W(X_{(\alpha,0)})$ and $m \in S_t(X)$, then*

i. $d(v, m) \leq \frac{1}{2}|u| + d(uv, m)$.

ii. $d(u, m) \leq \frac{1}{2}|v| + d(uv, m)$.

Proof. Let $d(v, m) = \frac{1}{2}(|v'| + |v''| + 2|m| + 2|m'|)$ and

$$d(uv, m) = \frac{1}{2}(|a| + |b| + 2|m| + 2|m'|) \tag{3}$$

for some $v', v'', a, b \in W(X_{([m],0)})$. By definition, $|a| + |m| + |b| + |u| \geq |v'| + |m| + |v''|$ So $d(v, m) \leq \frac{1}{2}(|a| + |b| + |u| + 2|m| + 2|m'|)$ and by (3), $d(v, m) \leq \frac{1}{2}|u| + d(uv, m)$. Similar reasoning works for (ii). □

Lemma 3.7. *labelaub*

i. Let $m \in S_t(X)$ and $d(u, m) = \frac{1}{2}(|u'| + |u''| + 2|m| + 2|m'|)$.

(a) If $u' = a'a''$, $u'' = b'b''$, then,

$$d(b''ua', m) = \frac{1}{2}(|b'| + |a''| + 2|m| + 2|m'|) \leq d(u, m).$$

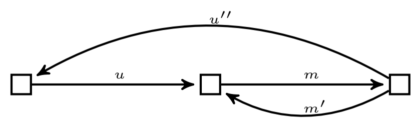

Figure 4: Distance of u and m when $um \in W(X_{(\alpha,0)})$.

(b) If $u' = \varepsilon$ and $u'' \neq \varepsilon$, then

$$d(u''umm', m) = |m| + |m'| = d(u, m) - \frac{1}{2}|u''|.$$

(c) If $u' \neq \varepsilon$ and $u'' = \varepsilon$, then

$$d(m'muu', m) = |m| + |m'| = d(u, m) - \frac{1}{2}|u'|.$$

(d) If $u' = u'' = \varepsilon$, then $d(m'mumm') = d(u, m) = |m| + |m'|$

ii. There are nonempty blocks a and b such that $d(aub, m) \leq d(u, m)$.

Proof. (a) Let C be a cycle containing $b''ua'$ and m. It suffice to show that

$$|C| \geq |b''| + |u| + |a'| + |a''| + |m| + |b'|.$$

Let $\mathcal{L}(C) = b''ua'w'mw''$. Then, by definition,

$$|a'|+|w'|+|m|+|w''|+|b''| \geq |u'|+|m|+|u''| = |a'|+|a''|+|m|+|b'|+|b''|$$

and we are done.

(b) Let π_a be a path in Γ_α and labeled a. There is a finite path $\pi_{u''umm'}$ terminating at $i(\pi_m)$ and starting at $t(\pi_m)$, so (b) is trivial. See Figure 4.

Similar reasoning works for (c) and (d).

The part of (ii) follows from (i). □

Proposition 3.8. *Let $m \in S_t(X)$ and $r > 0$. Then,*

$$N_{m,r} := N_r(m) \cup \{u \subseteq v : v \in N_r(m)\}$$

is a languages of a shift space.

Proof. We prove the proposition by showing that if $u \in N_{m,r}$, then

i. every subblock of u belongs to $N_{m,r}$.

ii. there are nonempty blocks a and b in $N_{m,r}$ so that $aub \in N_{m,r}$.

Definition of $N_{m,r}$ implies that (i) is trivial. To prove (ii), let $u \in N_r(m)$. By Lemma **??**, there are nonempty blocks a, b such that $d(aub, m) \leq d(u, m)$. So $aub \in N_r(m)$ and we are done. □ Let $X_{(m,r)}$ be a shift space such that $N_{m,r}$ is its language. We can associate to $X_{(m,r)}$ an irreducible graph Γ_m. If $u \in N_r(m)$, then $d(u, m) = \frac{1}{2}(|u'| + |u''| + 2|m| + 2|m'|)$ for some u', $u'' \in W(X_{(\alpha,0)})$.

Let C_u be the cycle in Γ_α labeled $uu'mu''$ and passing through $w_+(m)$. Then, Γ_m consist of all C_u when $d(u, m) < r$. It is easy to see that $W(X_{(m,r)}) \subseteq W(\mathcal{L}(\Gamma_m))$. Now suppose π_u be a finite path in Γ_m labeled u. So there is a cycle C_v in Γ_m labeled $vv'mv''$ such that $u \subseteq vv'mv''$. Since $d(v''vv', m) = |m| + |m'| \leq d(v, m) < r$, thus $v''vv' \in N_r(m)$ and so $u \in N_{m,r}$ or $W(\mathcal{L}(\Gamma_m)) \subseteq W(X_{(m,r)})$. Thus $X_{(m,r)} = \mathcal{L}(\Gamma_m)$. Also by definition of Γ_m, it is a right resolving graph and follower separated. Which set over claim by ([9, page 3563]) and so Γ_m is Fischer cover of $X_{(m,r)}$.

Proposition 3.9. *Let X be a shift space and $m \in S_t(X)$. Then, $X_{(m,r)}$ is a sofic.*

Proof. Set $\alpha := [m]$ and $Z := X_{(\alpha,0)}$. Let

$$\{(i, j) \in \mathbb{Z}^2 : i, j \geq 0 \text{ and } i+j < r-2|m|+2|m'|\} = \{(i_1, j_1), \dots, (i_k, j_k)\}$$

and for each $1 \leq l \leq k$ let $|\{u' \in W(Z) : u'm \in W(Z) \text{ and } |u'| = i_l\}| = n'_l$, $|\{u'' \in W(Z) : mu'' \in W(Z) \text{ and } |u'| = j_l\}| = n''_l$. Then, by the fact that if $u \in N_r(m)$ then $|u'| + |u''| < r - 2|m| - 2|m'|$ for some u', $u'' \in W(X_{(\alpha,0)})$, so

$$|\{C_u : u \in N_r(m)\}| \leq n'_1 n''_1 + \cdots + n'_k n''_k.$$

Thus Γ_m s a finite graph or $X_{(m,r)}$ is a sofic. □

Remark 3.10. Note that as in half synchronized case in ([2]), we can define $X_{(m,r)}$. For this let X be a shift space. For $m, m' \in S_t(X)$ we write $m \sim' m'$ when there are $u \in W(R(X))$ such that $m, m' \subseteq \mathcal{L}(C_u)$. Then, $\sim'$ is an equivalence relation in $S_t(X)$.

Corollary 3.11. *i. Every sofic shift is a bounded metric space.*

ii. Let X be an irreducible bounded metric space and $m \in S_t(X)$. Then, there is $r > 0$ such that $X = X_{(\alpha,0)} = X_{(m,r)}$.

Proof. (i) Let X be a sofic and $m \in S_t(X)$. There is a cycle C in X_0^+ such that for any $I \in \mathcal{V}_{X_0^+}$, C passing through I. Let length of C be M and $u \in W(X) = W(X_{(\alpha,0)})$ where $\alpha = [m]$. Then, $|u'|, |u''|, |m'| \leq M$. So $d(u, m) < 4|M| + |m|$ or $u \in N_r(m)$ where $r := 4|M| + |m|$. So $X \subseteq N_r(m)$.

(ii) There is $r > 0$ such that $X \subseteq N_r(m)$. Thus $W(X) \subseteq W(X_{(m,r)}) \subseteq W(X_{(\alpha,0)})$ and since $W(X_{(\alpha,0)}) \subseteq W(X)$, so $X = X_{(\alpha,0)} = X_{(m,r)}$. □
Note that $X_{(\alpha,0)}$ is clearly a subshift of $R(X)$, but generally not closed. see Example 3.3.

Corollary 3.12. *Let $m \in S_t(X)$ and every cycle in Γ_m containing m. Then, X is a sofic if and only if X is an SFT.*

Proof. Let X be a sofic system. So there is $r > 0$ such that $X = X_{(m,r)}$. Similar to Proposition 3.9, there is $k \in \mathbb{N}$ such that

$$\{C_u : u \in W(X_{(m,r)})\} := \{C_{u_1}, C_{u_2}, \ldots, C_{u_k}\}.$$

Set $M := \max\{|C_{u_1}|, |C_{u_2}|, \ldots, |C_{u_k}|\}$ and let $u \in W(X_{(m,r)})$ where $|u| \geq 2M$. By definition of Γ_m, there must be at least one $m \subseteq u$ and such u is essentially synchronizing. As a result, any block of length $2M$ in $W(X_{(m,r)})$ is a synchronizing block and so $X = X_{(m,r)}$ is an SFT ([6, Theorem 2.1.8]). □ The conclusion of Corollary 3.12 is not true when there is a cycle in Γ_m labeled m.

Thomsen in ([9]) considers a synchronized component X of a general subshift and proves that

$$\sup\{h(A) : A \subseteq X \text{ is an irreducible SFT}\} = h(X_0^+) = h_{\text{syn}}(X). \quad (4)$$

This was extended to half synchronized component in ([6]) by showing that

$$\sup\{h(A) : A \subseteq X \text{ is an irreducible sofic}\} = h(X_0^+) = h_{\text{hsyn}}(X).$$

Now we will introduce this notion to strong synchronized.

Recall that if $m \in S_t(X)$, $\alpha := [m]$ and $u \in W(X)$, then C_u is a cycle in Γ_α labeled $uu'mu''$ and passing through $w_+(m)$.

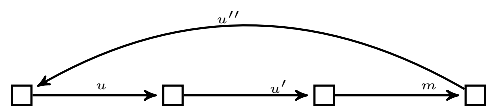

Figure 5: The subgraph of Γ_m.

Proposition 3.13. *Let* $m \in S_t(X)$. *Set* $\alpha := [m]$, $Z := X_{(\alpha,0)}$. *Set*

$$\{C_u : u \in W(Z)\} := \{C_{u_1}, C_{u_2}, \cdots\}$$

and

$$A_{(m,1)} := X(C_{u_1}),\ A_{(m,2)} := X(C_{u_1} \cup C_{u_2}),\ \ldots.$$

Then, $\overline{X_{(\alpha,0)}} = \overline{\cup_{n\in\mathbb{N}} A_{(m,n)}}$ *and*

i. Every $A_{(m,n)}$ *is a sofic.*

ii. $\lim_{n\to\infty} h(A_{(m,n)}) = h(\Gamma_\alpha)$.

Proof. By the fact that for each $n \in \mathbb{N}$, $C_{u_1} \cup \ldots \cup C_{u_n}$ is a finite graph, (i) is trivial.

For (ii), let $\epsilon > 0$. So there is $n \geq 1$ such that

$$h(\Gamma_\alpha) - \epsilon < \frac{1}{n} \log |\{C : C \text{ is a cycle in } \Gamma_\alpha \text{ passing through } t(\pi_m), |C| = n\}|.$$

Let $\{C : C \text{ is a cycle in } \Gamma_\alpha \text{ passing through } t(\pi_m), |C| = n\} = \{C_1, \ldots, C_N\}$. For each $1 \leq i \leq N$, set $v_i := \mathcal{L}(C_i)m'$ Figure 6. So there is a cycle C'_i in Γ_α labeled $\mathcal{L}(C_i)m'm$ and so $C'_i = C_{u_{j_i}}$ for some $j_i \in \mathbb{N}$. Thus $C_1 \cup C_2 \cup \cdots \cup C_N \subseteq \Gamma_{(m,t)}$ where $t := \max\{j_1, j_2, \ldots, j_N\}$. Thus

$$N \leq |\{C : C \text{ is a cycle in } \Gamma_{(m,t)} \text{ passing through } t(\pi_m), |C| = n\}|$$

and so $h(\Gamma_\alpha) - \epsilon < h(A_{(m,t)})$. Hence $h(\Gamma_\alpha) < \lim h(A_{(m,n)})$ and we are done. □ Let G be generating graph for a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$. Fix $\{a_1, a_2 \ldots, a_r\} \subseteq \mathcal{A}$ and $\{u_1, u_2, \ldots, u_r\} \subseteq W(X)$. We construct a new graph from G denoted by $G_{u_i \hookrightarrow a_i}$ by replacing u_i for a_i whenever there is a path in G labeled a_i for all $1 \leq i \leq r$.

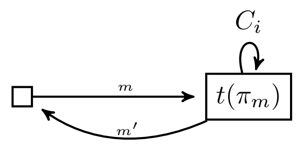

Figure 6: The graph G.

Proposition 3.14. *Let $\mathcal{A} = \{a_1, a_2, \ldots, a_r\}$ and X_0^+ be the Fischer cover of $\mathcal{A}^{\mathbb{Z}}$ at the base point I_0. Pick $\{u_1, u_2, \ldots, u_r\} \subseteq W(X)$ such that for each $1 \le i \le r$,*

$$u_i \notin \{u : u \text{ is a finite concatenation of } \{u_j : 1 \le j \le r \text{ and } j \ne i\}\}.$$

For each $N \in \mathbb{N}$, set

$$\mathcal{N} := \{(n_1, n_2, \ldots, n_r) \in (\mathbb{N} \cup \{0\})^r : k_1 n_1 + k_2 n_2 + \cdots + k_r n_r = N\}$$

where $k_i := |u_i|$. Then,

$$h((X_0^+)_{u_i \hookrightarrow a_i}) = \lim_{N\to\infty} \frac{1}{N} \log \Sigma_{(n_1, n_2, \ldots, n_r)\in\mathcal{N}} \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

where $n = n_1 + n_2 + \cdots + n_r$.

Proof. For each $1 \le i \le r$, there is exactly one cycle labeled u_i. So

$$\Sigma_{(n_1, n_2, \ldots, n_r)\in\mathcal{N}} \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

is the number of cycles of length N based at I_0 and we are done. □

Corollary 3.15. *Let $m \in S_t(X)$. For each $u \in W(X_{(\alpha,0)})$, set $k_i := |u_i u_i' m u_i''|$. If all $A_{(m,n)}$ is an SFT, then*

$$h(\Gamma_\alpha) = \lim_{N\to\infty} \frac{1}{N} \log \Sigma_{(n_1, n_2, \ldots, n_r)\in\mathcal{N}} \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}.$$

For instance let X be the golden mean shift that formed of $\{0, 1\}^{\mathbb{Z}}$ by replacing 00 instead 0 . Then, $h(X) = \lim_{N\to\infty} \frac{1}{N} \log \Sigma_{n+2m=N} \binom{n+m}{n}$.

4 Strong Synchronized Derived

Suppose $\alpha \in S(X)/\sim$. Let $X_{(\alpha_s,0)}$ denote the set of elements $x \in X_{(\alpha,0)}$ satisfying the following.

i. If $i \in \mathbb{Z}$, then there are $m, m' \in \alpha \cap S_t(X)$ such that $m \subseteq x_{(-\infty,i]}$, $m' \subseteq x_{[i,+\infty)}$.

ii. There is $M > 0$ such that if u and v are the consecutive minimal pairs of $\alpha \cap S_t(X)$ in x, then $\mathrm{gap}(u,v) \leq M$.

$X_{(\alpha_s,0)}$ is clearly a subshift of $X_{(\alpha,0)}$, but generally not closed. Clearly $\overline{X_{(\alpha_s,0)}} = \overline{X_{(\alpha,0)}}$ and

$$X_{(\alpha_s,0)} \subseteq X_{(\alpha,0)}. \tag{5}$$

The next example shows that the inclusion in (5) is not equality.

Example 4.1. In the golden mean shift X, $X = X_{(\alpha,0)}$ and $X_{(\alpha,0)} - X_{(\alpha_s,0)} = \{0^\infty\}$.

If X is a synchronized (resp. strong synchronized) system, then there is exactly one $\alpha \in S(X)/\sim$ such that $\overline{X_{(\alpha,0)}} = X$ (resp. $\overline{X_{(\alpha_s,0)}} = X$) ([9, Lemma 3.5]) and call it the *top component* (*resp. top strong component*).

Let X be a shift space and $\alpha \in S(X)/\sim$. Suppose $(\overline{X_{(\alpha,0)}})_c$ be the top component of the synchronized system $\overline{X_{(\alpha,0)}}$. Thomsen in ([9]) prove that

$$X_{(\alpha,0)} \subseteq (\overline{X_{(\alpha,0)}})_c. \tag{6}$$

The next Proposition shows that for strong synchronized systems the inclusion in (6) is equality.

Proposition 4.2. *Let X be a shift space and $\alpha \in S(X)/\sim$. Suppose $(\overline{X_{(\alpha_s,0)}})_c$ be the strong top component of the strong synchronized system $\overline{X_{(\alpha_s,0)}}$. Then, $X_{(\alpha_s,0)} = (\overline{X_{(\alpha_s,0)}})_c$.*

Proof. By definition 3.2, $S_t(\overline{X_{(\alpha,0)}}) \subseteq S_t(X)$ and so $X_{(\alpha_s,0)} = (\overline{X_{(\alpha_s,0)}})_c$. $\square$

Let X be an strong synchronized system. We set

$$\partial_{\mathrm{s}} X = \{x \in X \mid \; u \subseteq x \;\Rightarrow\; u \notin S_t(X)\}.$$

and call it the *strong derived* shift space of X. Since $\partial_s X$ is a shift space we can continue, and consider $\partial_s(\partial_s X) = \partial_s^2 X$, $\partial_s(\partial_s^2 X) = \partial_s^3 X$, etc. Of course, it can happen that these constructions give nothing interesting; it can be that there are no synchronizing blocks for X, in which case $\partial_s X = X$. For convenience we set $\partial_s^0 X = X$. We define the *strong depth* of X to be

$$Depth_s(X) = \sup\{n \in \mathbb{N} : \partial_s^n X \neq \emptyset\}.$$

Thus a minimal shift space with infinitely many points as well as an SFT have depth 0, but for different reasons.

Example 4.3. i. In example 3.3, $\partial_s S = S$ and $\partial S = X$.

ii. Let Y be a sofic. Then, by 3.4 $\partial_s Y = \partial Y$ and so by ([9, Theorem 6.6]), $\partial_s Y$ is sofic.

iii. Suppose Z be synchronized system. Then, by ([8, Theorem 6.16]), $h(Z) = \max\{h(X_0^+), h(\partial_s Z)\}$.

Proposition 4.4. *If $m \in S(X)$ and $m \notin S_t(X)$, then m is a synchronizing block of $\partial_s X$.*

Proof. Let $am, mb \in W(\partial_s X)$. So there are $x, y \in \partial_s X$ such that $am = x_{[i,0]}$ and $mb = y_{(0,j]}$ for some $i, j \in \mathbb{Z}$. Since $m \in S(X)$, so $x_{(-\infty,i)}amby_{(j,+\infty)} \in X$. It suffice to show that

$$S_t(X) \cap \{x_{[i_0,j_0]} : i_0 \leq 0, j_0 > 0\} = \emptyset.$$

Pick $i_0 \leq 0$ and $j_0 > 0$. There is unique vertex I of Fischer cover $\Gamma_{[m]}$ with this properties $m \in F_-(I)$ and $w_+(I) = w_+(m)$. Since $x_{[i_0,0]} \notin S_t(X)$, so there are two finite paths in $\Gamma_{[m]}$ labeled $x_{[i_0,0]}$ and terminating at I and so $x_{[i_0,j_0]} \notin S_t(X)$. □ The next example shows that there are systems X such that $m \in S(X) - S_t(X)$ which may not be a synchronized.

Example 4.5. Let G be the graph in Figure 7 and $X = X(G)$. Then, $\partial_s X = \{0^\infty, 1^\infty\}$.

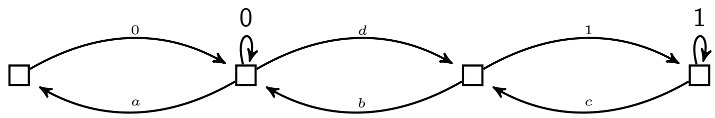

Figure 7: The graph G.

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