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On Strongly Ozaki λ-Pseudo Bi-Close-to-Convex Functions

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Abstract. In the current article, we introduce and investigate two new families of strongly Ozaki λ-pseudo bi-close-to-convex functions in the open unit disk. We determine upper bounds for the second and third coefficients of functions belonging to these new subclasses. Also, we point out several certain special cases for our results.

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1 Introduction

Denote by $\mathcal{A}$ the collection of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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Further, let $\mathcal{S}$ indicate the subclass of $\mathcal{A}$ consisting of the form (1) which are univalent in $\mathbb{U}$. One of the important and well examined subclass of $\mathcal{S}$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha, (0 \leq \alpha < 1)$, defined by the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \ , z \in \mathbb{U}$$

and the class $\mathcal{C}(\alpha)$ of convex functions of order $\alpha, (0 \leq \alpha < 1)$, is defined by the condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \ \ , z \in \mathbb{U}.$$

Especially, for $\alpha = 0$, the above classes reduce well-known classes $\mathcal{S}^*$ and $\mathcal{C}$ which are the class of starlike functions and the class of convex functions, respectively.

Also, a function $f \in \mathcal{A}$ belongs to $\mathcal{K}$, the class of close-to-convex functions, if and only if there exists $g \in \mathcal{S}^*$ such that

$$Re\left(e^{i\theta}\frac{zf'(z)}{g(z)}\right) > 0 \ \ , z \in \mathbb{U}$$

for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Especially, for $\theta = 0$, the class $\mathcal{K}$ reduce to the class of close-to-convex functions and defined by $\mathcal{K}_0$.

Further, a function $f \in \mathcal{S}$ is said to be strongly starlike of order α if

$$\left|arg\frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2} \ \ , z \in \mathbb{U}$$

and is said to be strongly convex of order α if

$$\left|arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \alpha\frac{\pi}{2} \ \ , z \in \mathbb{U}$$

for some α, $(0 < \alpha \leq 1)$. Also we denote these classes $\tilde{\mathcal{S}}^*(\alpha)$ and $\tilde{\mathcal{C}}(\alpha)$, respectively. Clearly, $\tilde{\mathcal{S}}^*(1) = \mathcal{S}^*$ and $\tilde{\mathcal{C}}(1) = \mathcal{C}$. In 1966, Stankiewicz [27] and in 1969, Brannan and Kirwan [5] introduced these classes, independently. It is interesting that the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ become smaller but $\tilde{\mathcal{S}}^*(\alpha)$ and $\tilde{\mathcal{C}}(\alpha)$ become larger as α increases.

The class $\tilde{\mathcal{K}}(\alpha)$ of strongly close-to-convex functions defined by

$$\left| arg\frac{zf'(z)}{g(z)} \right| < \alpha\frac{\pi}{2} \ \ , z \in \mathbb{U}$$

for $(0 < \alpha \leq 1)$ and $g \in \mathcal{S}^*$.

Although the class $\mathcal{K}$ was firstly formally introduced by Kaplan [10] in 1952, in 1941 Ozaki [14],(also see [13]) had already considered the functions in $\mathcal{A}$ satisfying the following condition for the class of close-to-convex functions:

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \ \ , z \in \mathbb{U}. \tag{2}$$

The functions satisfying the inequality (2) are close-to-convex and therefore these functions are in $\mathcal{S}$ by the definition of Kaplan [10].

In 2017, Ozaki's condition was generalized by Kargar and Ebadian [11] as follows:

Definition 1.1. [11] *Let $\mathcal{F}(\delta)$ denote the class of locally univalent normalized analytic functions f in the unit disk satisfying the condition*

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \delta \ \ , z \in \mathbb{U}. \tag{3}$$

for some $-\frac{1}{2} < \delta \leq 1$.

The class $\mathcal{F}(1)$ was studied by Ponnusamy et al.[16]. It is clear that

$$\mathcal{F}(\frac{1}{2}) = \mathcal{C}$$

and

$$\mathcal{F}(\delta) \subset \mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$$

for all $\delta \in (-\frac{1}{2}, \frac{1}{2})$. Although the definition of the class $\mathcal{K}$ involve an independent starlike function g, the definition of the class $\mathcal{F}(\delta)$ doesn't involve such a function. But members of $\mathcal{F}(1)$ have coefficients which grow at the same rate as those in $\mathcal{K}$ [16].

In 2019, Allu et.al. [3] extended the class $\mathcal{F}(\delta)$ as follows:

Definition 1.2. [3] *The function $f \in \mathcal{A}$ is called strongly Ozaki close-to-convex if and only if*

$$\left| arg\left(\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\right| < \alpha\frac{\pi}{2} \tag{4}$$

for $0<\alpha\le 1,\ \frac{1}{2}\le\delta\le 1$ *and* $z\in\mathbb{U}$. *This class denoted by* $\mathcal{F}_{\mathcal{O}}(\delta,\alpha)$.

In [4], Babalola defined the class $\mathcal{L}_\lambda$ of $\lambda-$pseudo starlike functions as follows:
Let $f \in \mathcal{A}$ and $\lambda \ge 1$ is real. Then f belongs to the class $\mathcal{L}_\lambda$ of $\lambda-$pseudo starlike functions in the unit disc $z \in \mathbb{U}$ if and only if

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > 0, z\in\mathbb{U}.$$

He also proved that all pseudo starlike functions are univalent in the unit disc $\mathbb{U}$. Especially, for $\lambda = 1$, one can obtain the class of starlike functions.
The class of starlike functions includes the class of f belongs to the class $\mathcal{L}_\lambda$ of $\lambda-$pseudo convex functions in the unit disc $z\in\mathbb{U}$ if and only if

$$\Re\left(\frac{[(zf'(z))']^\lambda}{f'(z)}\right) > 0, z\in\mathbb{U}.$$

According to the Koebe One-Quarter Theorem [15] every function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \ge \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \tag{5}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ stands for the class of bi-univalent functions in $\mathbb{U}$ given by (1). In their pioneering work, Srivastava et al. [20] have apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Ali [2], Adegani and et al. [1], Caglar et al. [7] and others (see, for example [8, 9, 19, 21, 22, 23, 24, 25, 26, 28]). We notice that the class Σ is not empty. For example, the functions z, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1}{2}\log\frac{1+z}{1-z}$ are members of Σ. However, the familiar Koebe function is not a member of Σ. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n = 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem. Also,Brannan and Taha [6] introduced certain subclasses of bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$. These are $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{C}_\Sigma(\alpha)$ which are named bi-starlike function of order $\alpha, (0 \leq \alpha < 1)$ and bi-convex function of order $\alpha, (0 \leq \alpha < 1)$,respectively. Furthermore, we know that it is not true: A function f is bi-convex in $\mathbb{U}$ if and only if zf' is bi-starlike in $\mathbb{U}$. This is clear from that the function $f(z) = \frac{z}{1-z}$ which is bi-convex, however for that the function zf' is the Koebe function which is not bi-starlike because of not bi-univalent.

Motivated by the above mentioned papers, by using $\lambda-$pseudo convex functions,we introduce two new subclasses of bi-univalent function class Σ namely, strongly Ozaki $\lambda-$pseudo bi-close-to-convex function classes. Also, we derive estimates on the initial coefficients of these new subclasses of the bi-univalent function class Σ.

2 The Function Class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta, \alpha)$

We begin by defining the subclass $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta, \alpha)$ as follows:

Definition 2.1. *The function f given by (1) is said to be in the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta, \alpha)$ if the following conditions are satisfied: $f \in \Sigma$ and*

$$\left| arg \left(\frac{2\delta - 1}{2\delta + 1} + \frac{2}{2\delta + 1} \left(\frac{[(zf'(z))']^{\lambda}}{f'(z)} \right) \right) \right| < \alpha \frac{\pi}{2} \tag{6}$$

and

$$\left| arg \left(\frac{2\delta - 1}{2\delta + 1} + \frac{2}{2\delta + 1} \left(\frac{[(wg'(w))']^{\lambda}}{g'(w)} \right) \right) \right| < \alpha \frac{\pi}{2} \tag{7}$$

for $0 < \alpha \leq 1$; $\frac{1}{2} \leq \delta \leq 1$; $\lambda \geq 1$; $z, w \in \mathbb{U}$ and the function g is given by (5).

We note that for $\lambda = 1$, the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\alpha)$ reduces to the class $\mathcal{F}_{O,\Sigma}(\delta,\alpha)$ introduced and studied by Tezelci and Eker [28].

Now, we continue by finding the upper bounds for the initial coefficient of the functions in the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\alpha)$ as follows:

Theorem 2.2. *For $0 < \alpha \leq 1$; $\frac{1}{2} \leq \delta \leq 1$; $\lambda \geq 1$, let a function f be in the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\alpha)$. Then*

$$|a_2| \leq \frac{\alpha(2\delta+1)}{\sqrt{(2\delta+1)\alpha(8\lambda^2-7\lambda+1)+4(2\lambda-1)^2(1-\alpha)}} \tag{8}$$

and

$$|a_3| \leq \frac{(2\delta+1)\alpha\left[(4\lambda^2+\lambda-1)+2\left|2\lambda^2-4\lambda+1\right|\right]}{3(8\lambda^2-7\lambda+1)(3\lambda-1)}. \tag{9}$$

Proof. Let $f \in \mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\alpha)$. From (6) and (7), we have

$$\frac{2\delta-1}{2\delta+1} + \frac{2}{2\delta+1}\left(\frac{[(zf'(z))']^{\lambda}}{f'(z)}\right) = [p(z)]^{\alpha} \tag{10}$$

and

$$\frac{2\delta-1}{2\delta+1} + \frac{2}{2\delta+1}\left(\frac{[(wg'(w))']^{\lambda}}{g'(w)}\right) = [q(w)]^{\alpha} \tag{11}$$

where $p(z)$ and $q(w)$ are the functions with positive real part i.e are Caratheódory functions which have the following series expansions:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 \cdots, \tag{12}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + +q_3 w^3 \cdots. \tag{13}$$

Comparing the corresponding coefficients in (10) and (11), after simplifying, we have

$$\frac{4}{2\delta+1}(2\lambda-1)a_2 = \alpha p_1 \tag{14}$$

$$\frac{2}{2\delta+1}\left[4(2\lambda^2-4\lambda+1)a_2^2 + 3(3\lambda-1)a_3\right] = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 \tag{15}$$

and

$$-\frac{4}{2\delta+1}(2\lambda-1)a_2=\alpha q_1 \tag{16}$$

$$\frac{2}{2\delta+1}\left[2(4\lambda^2+\lambda-1)a_2^2-3(3\lambda-1)a_3\right]=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{17}$$

It follows from (14) and (16) that

$$p_1=-q_1 \tag{18}$$

and

$$\frac{32}{(2\delta+1)^2}(2\lambda-1)^2a_2^2=\alpha^2(p_1^2+q_1^2). \tag{19}$$

From(15), (17) and (19), we find that

$$a_2^2=\frac{(2\delta+1)^2\alpha^2(p_2+q_2)}{4(2\delta+1)\alpha(8\lambda^2-7\lambda+1)+16(2\lambda-1)^2(1-\alpha)}. \tag{20}$$

Since $p(z)$ and $q(w)$ are Caratheódory functions, considering Caratheódory lemma we have $|p_n|\leq 2$ and $|q_n|\leq 2$ for $n\in\mathbb{N}=\{1,2,\cdots\}$ (see [15]). Hence, by further computations, we obtain

$$|a_2|^2\leq\frac{(2\delta+1)^2\alpha^2}{(2\delta+1)\alpha(8\lambda^2-7\lambda+1)+4(2\lambda-1)^2(1-\alpha)} \tag{21}$$

which is the desired inequality (24).

Next, by using (15),(17) and (18) we can easily see that

$$\begin{aligned}\frac{12}{2\delta+1}(8\lambda^2-7\lambda+1)(3\lambda-1)a_3&=2\alpha(4\lambda^2+\lambda-1)p_2\\&-4\alpha(2\lambda^2-4\lambda+1)q_2+3\alpha(\alpha-1)(3\lambda-1)p_1^2.\end{aligned} \tag{22}$$

If we take $\alpha=1$, applying Caratheódory lemma, we obtain

$$|a_3|\leq\frac{(2\delta+1)\left[(4\lambda^2+\lambda-1)+2\left|2\lambda^2-4\lambda+1\right|\right]}{3(8\lambda^2-7\lambda+1)(3\lambda-1)}.$$

Consider $0<\alpha<1$. From (22), we can write

$$\begin{aligned}&\frac{12}{2\delta+1}(8\lambda^2-7\lambda+1)(3\lambda-1)Re(a_3)\\&=\alpha Re\left\{2(4\lambda^2+\lambda-1)p_2-4\alpha(2\lambda^2-4\lambda+1)q_2+3(\alpha-1)(3\lambda-1)p_1^2\right\}.\end{aligned} \tag{23}$$

From Herglotz's representation formula [15] for the functions $p(z)$ and $q(w)$, we have

$$p(z) = \int_0^{2\pi} \frac{1+e^{-it}z}{1-e^{-it}z} d\mu_1(t)$$

and

$$q(w) = \int_0^{2\pi} \frac{1+e^{-it}w}{1-e^{-it}w} d\mu_2(t)$$

where $\mu_i(t)$ are increasing on $[0, 2\pi]$ and $\mu_i(2\pi) - \mu_i(0) = 1$, $i = 1, 2$.

Also, we have

$$p_n = 2\int_0^{2\pi} e^{-int} d\mu_1(t)$$

$$q_n = 2\int_0^{2\pi} e^{-int} d\mu_2(t)$$

for $n \in \mathbb{N}$. Now, we can write (22) as follows:

$$\begin{aligned}
\frac{12}{2\delta+1}(8\lambda^2 - 7\lambda + 1)(3\lambda - 1)Re(a_3) &= 4\alpha(4\lambda^2 + \lambda - 1)\int_0^{2\pi} cos2td\mu_1(t) \\
&+ 8\alpha(-2\lambda^2 + 4\lambda - 1)\int_0^{2\pi} cos2td\mu_2(t) \\
&- 12\alpha(1-\alpha)(3\lambda - 1)\left[\left(\int_0^{2\pi} costd\mu_1(t)\right)^2 - \left(\int_0^{2\pi} sintd\mu_1(t)\right)^2\right]
\end{aligned}$$

$$\begin{aligned}
&\leq 4\alpha(4\lambda^2 + \lambda - 1)\int_0^{2\pi} cos2td\mu_1(t) + 8\alpha|2\lambda^2 - 4\lambda + 1|\int_0^{2\pi} cos2td\mu_2(t) \\
&\qquad + 12\alpha(1-\alpha)(3\lambda - 1)\left(\int_0^{2\pi} sintd\mu_1(t)\right)^2
\end{aligned}$$

$$\begin{aligned}
&= 4\alpha\left\{(4\lambda^2 + \lambda - 1)\int_0^{2\pi}(1 - 2sin^2t)d\mu_1(t)\right. \\
&\qquad + 2|2\lambda^2 - 4\lambda + 1|\int_0^{2\pi}(1 - 2sin^2t)d\mu_2(t) \\
&\qquad \left. + 3(1-\alpha)(3\lambda - 1)\left(\int_0^{2\pi} sintd\mu_1(t)\right)^2\right\}.
\end{aligned}$$

By Jensen's inequality [17], we know

$$\left(\int_0^{2\pi} |sint| d\mu(t)\right)^2 \leq \int_0^{2\pi} sin^2 t d\mu(t).$$

Hence, we have

$$\begin{aligned}\frac{12}{2\delta+1}(8\lambda^2-7\lambda+1)(3\lambda-1)Re(a_3) \leq \\ 4\alpha\left\{\left((4\lambda^2+\lambda-1)+2|2\lambda^2-4\lambda+1|\right)\right. \\ -4|2\lambda^2-4\lambda+1|\int_0^{2\pi} sin^2 t d\mu_2(t) \\ \left.-\left[2(4\lambda^2+\lambda-1)-3(1-\alpha)(3\lambda-1)\right]\int_0^{2\pi} sin^2 t d\mu_1(t)\right\}.\end{aligned}$$

and so

$$Re(a_3) \leq \frac{(2\delta+1)\alpha\left[(4\lambda^2+\lambda-1)+2\left|2\lambda^2-4\lambda+1\right|\right]}{3(8\lambda^2-7\lambda+1)(3\lambda-1)}$$

which implies

$$|a_3| \leq \frac{(2\delta+1)\alpha\left[(4\lambda^2+\lambda-1)+2\left|2\lambda^2-4\lambda+1\right|\right]}{3(8\lambda^2-7\lambda+1)(3\lambda-1)}$$

(for details, see [12]). This completes proof of the theorem. □

By taking $\lambda = 1$ in Theorem (2.2) we state

Corollary 2.3. *[28] Let f given by 1 be in the class $\mathcal{F}^1_{O,\Sigma}(\delta,\alpha) = \mathcal{F}_{O,\Sigma}(\delta,\alpha)$*

$$|a_2| \leq \frac{\alpha(2\delta+1)}{\sqrt{2\alpha(2\delta+1)+4(1-\alpha)}} \tag{24}$$

and

$$|a_3| \leq \frac{\alpha(2\delta+1)}{2}. \tag{25}$$

Remark 2.4. We consider a function $f(z)$ given by

$$f(z) = \frac{z}{1-az}$$

where $0 < a \leq 1$.It is easy to see that $f(z) \in \mathcal{S}$. For this function $f(z)$, we have that

$$g(w) = f^{-1}(z) = \frac{w}{1+aw} \in \mathcal{S}.$$

For such $f(z)$ and $g(w)$, we have that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+az}{1-az}$$

and

$$1 + \frac{wg''(w)}{g'(w)} = \frac{1-aw}{1+aw}.$$

It follows that $f(z)$ satisfies

$$\left|\frac{2}{2\delta+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{2}{2\delta+1}\left(\frac{1+a^2}{1-a^2}\right)\right| < \frac{2}{2\delta+1}\left(\frac{2a}{1-a^2}\right)$$

for $z \in \mathbb{U}$. Also, this gives us that

$$\left|\frac{2\delta-1}{2\delta+1} + \frac{2}{2\delta+1}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \frac{2}{2\delta+1}\left(\frac{1+a^2}{1-a^2}\right)\right| < \frac{2}{2\delta+1}\left(\frac{2a}{1-a^2}\right).$$

From the above inequality, we have that

$$\left|arg\left(\frac{2\delta-1}{2\delta+1} + \frac{2}{2\delta+1}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right| < sin^{-1}\left(\frac{2a}{1+a^2}\right) \quad , z \in \mathbb{U}.$$

Next, we consider the function $g(w)$. It is easy to see that $g(w)$ satisfies

$$\left|\frac{2}{2\delta+1}\left(1 + \frac{wg''(w)}{g'(w)}\right) - \frac{2}{2\delta+1}\left(\frac{1+a^2}{1-a^2}\right)\right| < \frac{2}{2\delta+1}\left(\frac{2a}{1-a^2}\right)$$

for $w \in \mathbb{U}$. This implies that

$$\left|arg\left(\frac{2\delta-1}{2\delta+1} + \frac{2}{2\delta+1}\left(1 + \frac{wg''(w)}{g'(w)}\right)\right)\right| < sin^{-1}\left(\frac{2a}{1+a^2}\right) \quad , w \in \mathbb{U}.$$

Therefore, if we consider a real number α such that

$$\alpha = \frac{2}{\pi} sin^{-1}\left(\frac{2a}{1+a^2}\right),$$

then $f(z)$ belongs to the class $\mathcal{F}^1_{O,\Sigma}(\delta, \alpha)$.

3 The Function Class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$

We define the subclass $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$ as follows:

Definition 3.1. *The function $f \in \mathcal{A}$ given by (1) is said to be in the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$ if the following conditions are satisfied: $f \in \Sigma$ and*

$$Re\left(\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(\frac{[(zf'(z))']^{\lambda}}{f'(z)}\right)\right)>\beta \tag{26}$$

and

$$Re\left(\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(\frac{[(wg'(w))']^{\lambda}}{g'(w)}\right)\right)>\beta \tag{27}$$

for $0 \le \beta < 1;\ \frac{1}{2} \le \delta \le 1;\ \lambda \ge 1,\ \ z,w \in \mathbb{U}$ and the function g is given by (5).

We note that for $\lambda = 1$, the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$ reduces to the class $\mathcal{F}_{O,\Sigma}(\delta,\beta)$ introduced and studied by Tezelci and Eker [28] and for $\lambda = 1, \delta = \frac{1}{2}$, the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$ reduces to the class $\mathcal{C}_{\Sigma}(\beta)$ introduced and studied by Brannan and Taha [6].

Now, we continue by finding the upper bounds for the initial coefficient of the functions in the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$ as follows:

Theorem 3.2. *For $0 \le \beta < 1;\ \frac{1}{2} \le \delta \le 1;\ \lambda \ge 1$, let f be in the class $\mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$. Then*

$$|a_2| \le \sqrt{\frac{(1-\beta)(2\delta+1)}{8\lambda^2-7\lambda+1}} \tag{28}$$

and

$$|a_3| \le \frac{(1-\beta)(2\delta+1)\left[(4\lambda^2+\lambda-1)+2\left|2\lambda^2-4\lambda+1\right|\right]}{3(8\lambda^2-7\lambda+1)(3\lambda-1)}. \tag{29}$$

Proof. Let $f \in \mathcal{F}^{\lambda}_{O,\Sigma}(\delta,\beta)$. From (26) and (27), we have

$$\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(\frac{[(zf'(z))']^{\lambda}}{f'(z)}\right)=\beta+(1-\beta)p(z) \tag{30}$$

and

$$\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(\frac{[(wg'(w))']^{\lambda}}{g'(w)}\right)=\beta+(1-\beta)q(w) \tag{31}$$

where $p(z)$ and $q(w)$ are the functions with positive real part i.e are Carathéodory functions which have the same notation in Theorem 2.2.

Comparing the corresponding coefficients in (30) and (31), after simplifying, we have

$$\frac{4}{2\delta+1}(2\lambda-1)a_2=(1-\beta)p_1 \tag{32}$$

$$\frac{2}{2\delta+1}\left[4(2\lambda^2-4\lambda+1)a_2^2+3(3\lambda-1)a_3\right]=(1-\beta)p_2 \tag{33}$$

and

$$-\frac{4}{2\delta+1}(2\lambda-1)a_2=(1-\beta)q_1 \tag{34}$$

$$\frac{2}{2\delta+1}\left[2(4\lambda^2+\lambda-1)a_2^2-3(3\lambda-1)a_3\right]=(1-\beta)q_2. \tag{35}$$

It follows from (32) and (34) that

$$p_1=-q_1 \tag{36}$$

and

$$\frac{32}{(2\delta+1)^2}(2\lambda-1)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{37}$$

From(33) and (35), we find that

$$a_2^2=\frac{(2\delta+1)(1-\beta)}{4(8\lambda^2-7\lambda+1)}(p_2+q_2). \tag{38}$$

Hence, by further computations, we obtain

$$|a_2|^2\leq\frac{(2\delta+1)(1-\beta)}{(8\lambda^2-7\lambda+1)}. \tag{39}$$

which is the desired inequality (28).

Next, if we subtract (35) from (33), we can easily see that

$$\begin{aligned}\frac{12}{2\delta+1}(8\lambda^2-7\lambda+1)(3\lambda-1)a_3 =&2(1-\beta)(4\lambda^2+\lambda-1)p_2\\ &-4(1-\beta)(2\lambda^2-4\lambda+1)q_2\end{aligned} \tag{40}$$

which implies

$$|a_3| \leq \frac{(2\delta+1)(1-\beta)\left[(4\lambda^2+\lambda-1)+2\left|2\lambda^2-4\lambda+1\right|\right]}{3(8\lambda^2-7\lambda+1)(3\lambda-1)}.$$

This completes proof of the theorem. □

Putting $\lambda = 1$ in Theorem 3.2, we obtain the following result:

Corollary 3.3. *[28] Let f given by (1) be in the class $\mathcal{F}^1_{O,\Sigma}(\delta,\beta) = \mathcal{F}_{O,\Sigma}(\delta,\beta)$*

$$|a_2| \leq \sqrt{\frac{(1-\beta)(2\delta+1)}{2}} \tag{41}$$

and

$$|a_3| \leq \frac{(1-\beta)(2\delta+1)}{2}. \tag{42}$$

Remark 3.4. Let us consider functions $f(z)$ and $g(w)$ given in Remark 2.4. Then $f(z)$ and $g(w)$ satisfy

$$\left|\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{2}{2\delta+1}\left(\frac{1+a^2}{1-a^2}\right)\right| < \frac{2}{2\delta+1}\left(\frac{2a}{1-a^2}\right)$$

and

$$\left|\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{wg''(w)}{g'(w)}\right)-\frac{2}{2\delta+1}\left(\frac{1+a^2}{1-a^2}\right)\right| < \frac{2}{2\delta+1}\left(\frac{2a}{1-a^2}\right)$$

where $z, w \in \mathbb{U}$. Therefore, $f(z)$ satisfies

$$Re\left(\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{zf''(z)}{f'(z)}\right)\right) > \frac{2(1-a)}{(2\delta+1)(1+a)}\ , z \in \mathbb{U}$$

and

$$Re\left(\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{wg''(w)}{g'(w)}\right)\right) > \frac{2(1-a)}{(2\delta+1)(1+a)}\ , w \in \mathbb{U}.$$

Therefore, if we consider a real number β such that

$$\beta = \frac{2(1-a)}{(2\delta+1)(1+a)},$$

then $f(z)$ is in the class $\mathcal{F}^1_{O,\Sigma}(\delta,\beta)$.

Taking $\lambda = 1$ and $\delta = \frac{1}{2}$ in Theorem 3.2, we obtain the following result:

Corollary 3.5. *[6] Let f given by (1) be in the class $\mathcal{F}^1_{O,\Sigma}(\frac{1}{2}, \beta) = \mathcal{C}_\Sigma(\beta)$*

$$|a_2| \leq \sqrt{1-\beta} \tag{43}$$

and

$$|a_3| \leq 1-\beta. \tag{44}$$

Corollary 3.6. *[6] Taking $\lambda = 1$, $\delta = \frac{1}{2}$ and $\beta = 0$ in Theorem 3.2 we have*

$$\mathcal{F}^1_{O,\Sigma}(\frac{1}{2}, 0) = \mathcal{C}_\Sigma(0) \subset \mathcal{C},$$

where $\mathcal{C}$ is the class of all normalized convex function in $\mathbb{U}$, which implies that $|a_n| \leq 1$, $n = 2, 3, \cdots$ which is sharp.

4 Conclusion

This paper aims to derive the two initial Taylor–Maclaurin coefficient estimates of functions within two new families of strongly Ozaki λ-pseudo bi-close-to-convex functions in the open unit disk. The results of this article will encourage other researchers to find more general results using different operators, especially the $q-$ differential operator. The researchers can be used a recent survey-cum-expository review article [18] which involves a wide variety of operators of basic (or $q-$) calculus and fractional $q-$ calculus and their widespread applications in Geometric Function Theory of Complex Analysis.

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