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Original Research Paper

Evolution of the First Eigenvalue of the (p, q)-Laplacian System Under Rescaled Yamabe Flow

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Abstract. Consider the triple $(M, g, d\mu)$ as a smooth metric measure space and M is an n-dimensional compact Riemannian manifold without boundary, also $d\mu = e^{-f(x)}dV$ is a weighted measure. We are going to investigate the evolution problem for the first eigenvalue of the weighted (p, q)-Laplacian system along the rescaled Yamabe flow and we hope find some monotonic quantities.

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1 Introduction

Consider (M, g) as a compact n-dimensional manifold without boundary, the Yamabe problem which was studied first by Yamabe in [18], is to find a metric g conformal to g_0 such that it's scalar curvature R_g is constant.

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Generally two metrics g and g_0 were called conformal if $g = e^{-2u}g_0$ where u is positive and smooth function in M. In special case if we write $g = u_{\frac{4}{n-2}}g_0$, then the scalar curvature R_g of g can be written as

$$R_g = u^{-\frac{n+2}{n-2}}\Delta_{g_0}u + R_{g_0}u. \tag{1}$$

Therefore, the Yamabe problem is to solve (1) such that R_g is constant. The above equation solved by Trudinger [16], Aubin [2], and Schoen [14]. Yamabe flow was introduced by Hamilton in [6] for the first time. The Yamabe flow defined as the evolution of the metric $g = g(t)$

$$\frac{\partial}{\partial t}g_{ij} = -Rg_{ij}, \quad g(0) = g_0 \tag{2}$$

and normalized Yamabe flow was also defined as well as

$$\frac{\partial}{\partial t}g_{ij} = -\left(R - r\right)g_{ij}, \quad g(0) = g_0 \tag{3}$$

where R is the scalar curvature. Also $r = \frac{\int_M RdV}{\int_M dV}$ is the average of the scalar curvature of the Riemannian metric g.
First of all, Schwetlick and Struwe in [15] proved the convergence of the Yamabe flow for the case when $3 \leq n \leq 5$ with the assumption that the initial metric has large energy. Finally Brandle in [4] has shown that the Yamabe flow convergence to a metric with constant scalar curvature.
In this paper, we are going to try to find some evolution equations and some monotonic quantities of rescaled Yamabe flow, coupled with harmonic flow which is defined as

$$\begin{cases} \frac{\partial}{\partial t}g_{ij} = -\left(R - s(t)\right)g_{ij} \quad g|_{t=t_0} = g_0, \\ \frac{\partial f}{\partial t} = \Delta f, \quad f(0,x) = f_0(x), \end{cases} \tag{4}$$

where $s(t)$ is constant only depended on time variable t, easily we can see the system (4) in different cases gives us systems (2) or (3).
It has been known before that there is a one-to-one relationship between Yamabe flow (2), and rescaled Yamabe flow (4), in which if we consider $g(t)$ as a solution for the flow (2) on $[0, T)$ such that T is the maximum value of t where the flow has solution on $[0, t)$, then we can find the

function $\psi(t)$ as

$$\psi(t) = \left(1 - \int_0^t s(\nu)d\nu\right)^{-1},$$

and also $\bar{t} = \int_0^t \psi(\nu)d\nu$. In this case $\bar{g}(\bar{t}) = \psi(t)g(t)$ will be the solution for the rescaled Yamabe flow (4).
Let $u : M \longrightarrow \mathbb{R}$, $u \in W_0^{1,p}(M)$ where $W_0^{1,p}(M)$ is Sobolev space, for $p \in [1, \infty)$ we have seen before the introduction of p-Laplacian of u as below

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}(Hess\, u)(\nabla u, \nabla u),$$

where

$$(Hess\, u)(X, Y) = \nabla(\nabla u)(X, Y) = X.(Y.u) - (\nabla_X Y).u, \qquad X, Y \in \chi(M).$$

Also weighted p-Laplacian can be introduced as

$$\Delta_{p,f} u = e^f div\left(e^{-f}|\nabla u|^{p-2}\nabla u\right) = \Delta_p u - |\nabla u|^{p-2}\nabla f.\nabla u,$$

where $p \in [1, \infty)$ and u is any smooth function on M.
Now consider (M^n, g) as a closed Riemannian manifold we are going to define weighted (p, q)-Laplacian system as

$$\begin{cases} \Delta_{p,f} u = -\lambda|u|^\alpha|v|^\beta v & \text{in M}, \\ \Delta_{q,f} v = -\lambda|u|^\alpha|v|^\beta u & \text{in M}, \\ u = v = 0 & \text{on } \partial\text{M}, \end{cases} \tag{5}$$

where $p > 1$, $q > 1$ and α, β are real numbers such that

$$\alpha > 0,\ \beta > 0, \qquad \frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1.$$

In this case λ is an eigenvalue of such system. Also the existence and uniqueness of solution of the system (5) was studied in [19] expansively.
A first positive eigenvalue of a system (5) is defined as

$$inf\{A(u, v) : (u, v) \in W_0^{1,p}(M) \times W_0^{1,q}(M), B(u, v) = 1\},$$

where the pair of (u, v) is the eigenfunctions corresponding to eigenvalue λ and

$$A(u,v) = \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu,$$
$$B(u,v) = \int_M |u|^\alpha |v|^\beta uv d\mu,$$

where $d\mu = e^{-f}dV$. The lemma below gives us the continuousness of $A(u, v, t)$ in C^1-topology which has been mentioned before.

Lemma 1.1. *If g_1 and g_2 are two metrics on Riemannian manifold M^n which satisfy $(1+\varepsilon)^{-1} < g_2 < (1+\varepsilon)g_1$ then for any $p \geq q > 1$, we have*

$$\lambda(g_2) - \lambda(g_1) \leq \left((1+\varepsilon)^{\frac{p+n}{2}} - (1+\varepsilon)^{-\frac{n}{2}}\right)\lambda(g_1),$$

which means, $\lambda(t)$ is a continues function respect to t-variable.

Proof. In local coordinate we have $dv = \sqrt{detg}dx^1 \wedge ... \wedge dx^n$, therefore

$$(1+\varepsilon)^{-\frac{n}{2}} d\mu_{g_1} < d\mu_{g_2} < (1+\varepsilon)^{\frac{n}{2}} d\mu_{g_1}.$$

Assume that

$$G(g,u,v) = \frac{\alpha+1}{p}\int_M |\nabla u|_g^p d\mu_g + \frac{\beta+1}{q}\int_M |\nabla v|_g^q d\mu_g,$$

then it implies

$$\begin{aligned}
&\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1} G(g_2,u,v) - \int_M |u|^\alpha|v|^\beta uv d\mu_{g_2} G(g_1,u,v)\\
&= \frac{\alpha+1}{p}\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}\left(\int_M |\nabla u|_{g_2}^p d\mu_{g_2} - \int_M |\nabla u|_{g_1}^p d\mu_{g_1}\right)\\
&+ \frac{\alpha+1}{p}\left(\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1} - \int_M |u|^\alpha|v|^\beta uv d\mu_{g_2}\right)\int_M |\nabla u|_{g_1}^p d\mu_{g_1}\\
&+ \frac{\beta+1}{q}\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1}\left(\int_M |\nabla v|_{g_2}^q d\mu_{g_2} - \int_M |\nabla v|_{g_1}^q d\mu_{g_1}\right)\\
&+ \frac{\beta+1}{q}\left(\int_M |u|^\alpha|v|^\beta uv d\mu_{g_1} - \int_M |u|^\alpha|v|^\beta uv d\mu_{g_2}\right)\int_M |\nabla v|_{g_1}^q d\mu_{g_1},
\end{aligned}$$

then by applying the lemma's assumption we get

$$\begin{aligned}
&\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1} G(g_2,u,v) - \int_M |u|^\alpha |v|^\beta uv d\mu_{g_2} G(g_1,u,v) \\
&\le \frac{\alpha+1}{p}\left((1+\varepsilon)^{\frac{p+n}{2}} - (1+\varepsilon)^{-\frac{n}{2}}\right)\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}\int_M |\nabla u|^p_{g_1} d\mu_{g_1} \\
&+ \frac{\beta+1}{q}\left((1+\varepsilon)^{\frac{q+n}{2}} - (1+\varepsilon)^{-\frac{n}{2}}\right)\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}\int_M |\nabla v|^q_{g_1} d\mu_{g_1} \\
&\le \left((1+\varepsilon)^{\frac{p+n}{2}} - (1+\varepsilon)^{-\frac{n}{2}}\right) G(g_1,u,v)\int_M |u|^\alpha |v|^\beta uv d\mu_{g_1}.
\end{aligned}$$

Since the eigenfunctions corresponding to $\lambda(t)$ are normalized, then we have

$$\lambda(g_2) - \lambda(g_1) \le \left((1+\varepsilon)^{\frac{p+n}{2}} - (1+\varepsilon)^{-\frac{n}{2}}\right)\lambda(g_1).$$

□

In case which is not assumed that $\lambda(t)$ is C^1-differentiable under (4) in the interval $[0,T)$, the first non-zero eigenvalue of weighted (p,q)-Laplacian system is not known to be C^1-differentiable anymore. For this problem we are going to apply techniques of Cao [5] and Wu [17] to study the evolution and monotonicity of $\lambda(t)$, where u and v are supposed to be smooth.

Consider $(M^n, g(t))$ as a solution of the rescaled Yamabe flow on the smooth manifold (M^n, g_0) in the interval $[0,T)$ then $A(u,v)$ defines the evolution of an eigenvalue of the system (5), under the variation of $g(t)$ where for the eigenfunctions associated to $\lambda(t)$ we have

$$\begin{aligned}
&\int_M |u|^\alpha |v|^\beta uv d\mu = 1,\ \int_M |u|^\alpha |v|^\beta u d\mu = 0, \qquad (6)\\
&\int_M |u|^\alpha |v|^\beta v d\mu = 0.
\end{aligned}$$

First of all, let $t_0 \in [0,T)$, $(u_0, v_0) = (u(t_0), v(t_0))$ be the eigenfunctions for the eigenvalue $\lambda(t_0)$ of weighted (p,q)-Laplacian system (5). We consider the smooth functions as below

$$h(t) = u_0\Big[\frac{det[g_{ij}(t)]}{det[g_{ij}(t_0)]}\Big]^{\frac{1}{2(\alpha+\beta+1)}},\ l(t) = v_0\Big[\frac{det[g_{ij}(t)]}{det[g_{ij}(t_0)]}\Big]^{\frac{1}{2(\alpha+\beta+1)}},$$

along the rescaled Yamabe flow. Now let

$$u(t)=\frac{h(t)}{\left(\int_M |h(t)|^{\alpha}|l(t)|^{\beta}h(t)l(t)d\mu\right)^{\frac{1}{p}}},$$

and

$$v(t)=\frac{l(t)}{\left(\int_M |h(t)|^{\alpha}|l(t)|^{\beta}h(t)l(t)d\mu\right)^{\frac{1}{q}}}.$$

Where $u(t)$ and $v(t)$ are smooth functions under the rescaled Yamabe flow and also satisfy in (6), and at time t_0, $(u(t_0),v(t_0))$ is the eigenfunctions for $\lambda(t_0)$ of weighted (p,q)-Laplacian system (5), $\lambda(t_0)=A(u(t_0),v(t_0))$ and if $(M^n,g(t),f)$ be a solution of the (4) on the smooth manifold (M^n,g_0,f_0) in the interval $[0,T)$ then we can write the smooth eigenvalue function $\lambda(u,v,t)$ along the flow (4) as below

$$\lambda(u,v,t)=\frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu+\frac{\beta+1}{q}\int_M |\nabla v|^q d\mu, \tag{7}$$

where

$$\lambda(u,v,t)|_{t=t_0}=\lambda(t_0).$$

In recent years, studying the evolution equations under geometric flows became a hot topic in understanding the geometry of manifolds. Perelman in [11], studying the functional:

$$F\left(g(t),f(t)\right)=\int_M \left(R+|\nabla f|^2\right)e^{-f}dV,$$

and showed that this functional is non-decreasing along the Ricci flow coupled to a backward heat-type equation. There are some other works in variational formulas. Second author in [3] has studied the eigenvalues problem of p-Laplace operator acting on the space of functions under the Yamabe flow, also P. Ho in [7] studied the first non-zero eigenvalue of Laplacian of g_0 with negative scalar curvature in terms of conformal Yamabe metrics, also he has worked on some other geometric operators in case of compact Riemannian manifolds. Some other works have done on evolution of eigenvalues of geometric operators along geometric flows [1, 8, 10].

In this article we investigate the evolution problem for the weighted (p,q)-Laplacian system under the rescaled Yamabe flow (4) and our main results will be classify as below

Theorem 1.2. *Consider $\big(M, g(t), f(t), d\mu = e^{-f}dV\big)$, $t \in [0,T)$ as a solution of the flow (4) on the smooth compact Riemannian manifold (M^n, g_0, f_0) without boundary, and $s(t) > 0$ is scalar function and also $\Delta f - \gamma R \leq 0$, $R \geq 0$ in $M \times [0,T)$. Suppose that $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted (p,q)-Laplacian system then for $p \leq q$ the quantity*

$$\lambda(t)\left(\int_0^t -\rho(\nu)d\nu + \frac{1}{R_{min}(0)}\right)^{\frac{p-2\gamma}{2}} e^{\frac{q}{2}\int_0^t s(\nu)d\nu},$$

is increasing along the flow (4) where $\rho(t) = e^{\int_0^t -s(\nu)d\nu}$, $\gamma < \frac{p-2}{2}$ is constant and also τ is constant which is equal to $\frac{1}{R_{min}(0)}$.

Theorem 1.3. *Consider (M, g_0) as a compact Riemannian manifold of dimension $n \geq 3$ without boundary in a case that $\max\limits_M R_{g_0} < 0$, and g_t, $t \in [0,T)$ is a Yamabe metric which has same volume as g_0. If we denote the first eigenvalue of weighted (p,q)-Laplacian system under the flow (4) with $s(t) = r$, respect to g_0 and g_t by $\lambda_1(g_0)$ and $\lambda_1(g_t)$, respectively, then we have*

$$e^{-c}\lambda_1(g_t) \geq \lambda_1(g_0) \geq e^{c}\lambda_1(g_t),$$

where

$$c = (2n + 2p - 2r)\left[1 - \frac{\min\limits_M R_{g_0}}{\max\limits_M R_{g_0}}\right] - r.$$

Theorem 1.4. *Let $\big(M^2, g(t), f(t), d\mu\big)$, $t \in (0,T)$, be a solution of (4) on the smooth compact surface $\big(M^2, g_0, f_0\big)$ without boundary also we assume that $p \geq q$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of the weighted (p,q)-Laplacian system under the flow (4) with $s(t) = r$, then*

- *If* $R<0$ *and* $\Delta f-\gamma R\leq 0$ *where* $\gamma<\frac{q-n}{2}$ *then*
$$ln\left(\lambda(t)\right)-\left(\frac{p-q}{2}-\gamma\right)rt+\frac{c}{r}\left(\frac{p}{2}-\gamma\right)e^{rt},$$
is inceasing.
- *If* $R<0$ *and* $\Delta f-\gamma R\geq 0$ *where* $\gamma<\frac{q-n}{2}$ *then*
$$ln\left(\lambda(t)\right)+\gamma rt-\left(\frac{q}{2}-\gamma\right)\frac{c}{r}e^{rt},$$
is decreasing.
- *If* $R>0$ *and* $\Delta f-\gamma R\leq 0$ *where* $\gamma<\frac{q-n}{2}$ *then*
$$ln\left(\lambda(t)\right)+\frac{r}{2}pt+\frac{c}{r}\left(\frac{q}{2}-\gamma\right)e^{rt},$$
is increasing.
- *If* $R>0$ *and* $\Delta f-\gamma R\geq 0$ *where* $\gamma<\frac{q-n}{2}$ *then*
$$ln\left(\lambda(t)\right)-\left(\frac{p-q}{2}-\gamma\right)rt-\left(\frac{p}{2}-\gamma\right)\frac{c}{r}e^{rt},$$
is decreasing,

where c is constant and $r=\frac{\int_M RdV}{\int_M dV}$ *is the average of the scalar curvature.*

In other sections we may add some more assumptions for more details.

2 Variation of $\lambda(t)$

In this section we are going to give some useful formulas of variation of $\lambda(t)$ along rescaled Yamabe flow. We start with the below proposition.

Proposition 2.1. *Let $(M^n, g(t), f(t))$ be a solution of the (4) on the smooth closed manifold (M^n, g_0, f_0). If $\lambda(t)$ denotes the evolution of the first non-zero eigenvalue under the flow (4), then we have*

$$\begin{aligned}
\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \frac{n}{2}\lambda(t_0)\int_M R|u|^{\alpha}|v|^{\beta}uv\,d\mu \\
&+ \frac{(\alpha+1)}{2}\int_M (R-s(t))|\nabla u|^p d\mu \\
&+ \frac{(\beta+1)}{2}\int_M (R-s(t))|\nabla v|^q d\mu \\
&- \frac{\alpha+1}{p}\int_M |\nabla u|^p\big[\Delta f + \frac{1}{2}nR\big]d\mu \\
&- \frac{\beta+1}{q}\int_M |\nabla v|^q\big[\Delta f + \frac{1}{2}nR\big]d\mu.
\end{aligned}$$

Proof. From what we explained before, $\lambda(u, v, t)$ is differentiable along the flow (4) then by derivation from the formula (7) respect to time variable t, it satisfies

$$\begin{aligned}
\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \frac{\alpha+1}{2}\Big[\int_M \Big\{-g^{ij}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i u\nabla_j u \\
&+ 2 < \nabla u_t, \nabla u > \Big\}|\nabla u|^{p-2}d\mu\Big] \\
&+ \frac{\beta+1}{2}\Big[\int_M \Big\{-g^{ij}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i v\nabla_j v \\
&+ 2 < \nabla v_t, \nabla v > \Big\}|\nabla v|^{q-2}d\mu\Big] \\
&+ \frac{\alpha+1}{p}\int_M |\nabla u|^p[-f_t d\mu + \frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu] \\
&+ \frac{\alpha+1}{q}\int_M |\nabla v|^q[-f_t d\mu + \frac{1}{2}tr_g(\frac{\partial g}{\partial t})d\mu],
\end{aligned} \tag{8}$$

where $u_t = \frac{\partial u}{\partial t}$ and $f_t = \Delta f$. We can also calculate the term $\int_M < \nabla u_t, \nabla u > |\nabla u|^{p-2} d\mu$ as below

$$\begin{aligned}\int_M < \nabla u_t, \nabla u > |\nabla u|^{p-2} d\mu &= -\int_M u_t div\left(e^{-f}|\nabla u|^{p-2}\nabla u\right) dV,\\ &= -\int_M u_t e^f div\left(e^{-f}|\nabla u|^{p-2}\nabla u\right) d\mu\\ &= -\int_M u_t \Delta_{p,f} u d\mu\\ &= -\int_M u_t\left(-\lambda |u|^{\alpha}|v|^{\beta} v d\mu\right)\\ &= \lambda\int_M |u|^{\alpha}|v|^{\beta} u_t v d\mu.\end{aligned}$$

Similarly we can also calculate

$$\int_M < \nabla v_t, \nabla v > |\nabla v|^{q-2} d\mu = \lambda\int_M |u|^{\alpha}|v|^{\beta} u v_t d\mu.$$

It has been known with $\int_M |u|^{\alpha}|v|^{\beta} uv d\mu = 1$, by derivation from both sides of this equation respect to time variable t, we can see that

$$\begin{aligned}&\int_M [(\alpha+1)|u|^{\alpha}|v|^{\beta} u_t v + (\beta+1)|u|^{\alpha}|v|^{\beta} u v_t] d\mu\\ &+\int_M |u|^{\alpha}|v|^{\beta} uv \frac{1}{2} tr_g(-(R-s)g_{ij}) d\mu = 0,\end{aligned}$$

which finally implies that

$$\begin{aligned}&(\alpha+1)\int_M < \nabla u_t, \nabla u > |\nabla u|^{p-2} d\mu\\ &+(\beta+1)\int_M < \nabla v_t, \nabla v > |\nabla v|^{q-2} d\mu\\ &= \frac{n}{2}\lambda\int_M |u|^{\alpha}|v|^{\beta} uv(R-s) d\mu.\end{aligned} \tag{9}$$

Now by plugging the flow (4), into the formula (8), we have

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t) &= (\alpha+1)\int_M <\nabla u_t, \nabla u> |\nabla u|^{p-2}d\mu \\ &+ (\beta+1)\int_M <\nabla v_t, \nabla v> |\nabla v|^{q-2}d\mu \\ &+ \frac{(\alpha+1)}{2}\int_M (R-s(t))|\nabla u|^p d\mu \\ &+ \frac{(\beta+1)}{2}\int_M (R-s(t))|\nabla v|^q d\mu \\ &+ \frac{\alpha+1}{p}\int_M |\nabla u|^p\big[-\Delta f - \frac{1}{2}nR\big]d\mu \\ &+ \frac{\beta+1}{q}\int_M |\nabla v|^q\big[-\Delta f - \frac{1}{2}nR\big]d\mu + \frac{1}{2}ns(t)\lambda(t).\end{aligned}$$

which by replacing the equality (9), into above equation implies what we looking for. $\square$

Remark 2.2. In special case if we consider $s(t)=r$ where $r=\frac{\int_M RdV}{\int_M dV}$, it gives us the evolution formula under the normalized Yamabe flow (3), as below

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \frac{n}{2}\lambda(t_0)\int_M R|u|^\alpha|v|^\beta uv d\mu \\ &+ \frac{(\alpha+1)}{2}\int_M (R-r)|\nabla u|^p d\mu \\ &+ \frac{(\beta+1)}{2}\int_M (R-r)|\nabla v|^q d\mu \\ &+ \frac{\alpha+1}{p}\int_M |\nabla u|^p\big[-\Delta f - \frac{1}{2}nR\big]d\mu \\ &+ \frac{\beta+1}{q}\int_M |\nabla v|^q\big[-\Delta f - \frac{1}{2}nR\big]d\mu.\end{aligned}$$

Now we are going to give the proof of theorem 1.2 as
Proof.(**Theorem 1.2**) Under consideration $\Delta f - \gamma R \leq 0$ where

$\gamma < \frac{p-n}{2}$, we have

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \frac{n}{2}\lambda(t_0)\int_M R|u|^{\alpha}|v|^{\beta}uvd\mu \\ + \frac{(\alpha+1)}{p}\left(\frac{p}{2}-\left(\gamma+\frac{n}{2}\right)\right)\int_M R|\nabla u|^p d\mu \\ + \frac{(\beta+1)}{q}\left(\frac{q}{2}-\left(\gamma+\frac{n}{2}\right)\right)\int_M R|\nabla v|^q d\mu \\ - \frac{q}{2}s(t_0)\lambda(t_0).\end{aligned} \tag{10}$$

Also the evolution of R under the flow (4) is written as

$$\frac{\partial}{\partial t}R = (n-1)\Delta R + R^2 - Rs(t). \tag{11}$$

Since the solution to the ODE, $\frac{dy}{dt} = y^2 - s(t)y$ is

$$y(t) = \frac{\rho(t)}{\int_0^t -\rho(\nu)d\nu + \tau},$$

where $\rho(t) = e^{\int_0^t -s(\nu)d\nu}$, $\quad y(0) = R_{min}(0)$ and τ is a constant equal to $\frac{1}{R_{min}(0)}$, then by maximum principle to (11), we get $R(x,t) \geq y(t)$, then by (10) and $p \leq q$ we have

$$\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} \geq \lambda(u,v,t_0)\left(\frac{p-2\gamma}{2}y(t_0) - \frac{qs(t_0)}{2}\right),$$

which implies that in any sufficiently small neighborhood of t_0 as I, we get

$$\frac{d}{dt}\lambda(u,v,t) \geq \lambda(u,v,t)\left(\frac{p-2\gamma}{2}y(t) - \frac{qs(t)}{2}\right),$$

and also we have

$$\lambda(u,v,t_0) = \lambda(t_0), \quad \lambda(u,v,t_1) \geq \lambda(t_1). \tag{12}$$

On the other hand, by integration from both sides on $[t_1, t_0]$, it can be easily seen

$$ln\frac{\lambda(t_0)}{\lambda(t_1)} \geq \frac{\eta(t_1)}{\eta(t_0)},$$

where

$$\eta(t)=\left(\int_0^t -\rho(\nu)d\nu+\frac{1}{R_{min}(0)}\right)^{\frac{p-2\gamma}{2}} e^{\frac{q}{2}\int_0^t s(\nu)d\nu}.$$

Now since $t_1<t_0$ and t_0 is arbitrary thus the quantity

$$\lambda(t)\left(\int_0^t -\rho(\nu)d\nu+\frac{1}{R_{min}(0)}\right)^{\frac{p-2\gamma}{2}} e^{\frac{q}{2}\int_0^t s(\nu)d\nu},$$

is increasing. $\square$

And now, we prove the Theorem 1.3.

Proof.(**Theorem 1.3**) It was known that if $g\longrightarrow g_\infty$ as $t\longrightarrow\infty$ under the Yamabe flow (3), in a case that g_∞ is conformal to g_0 and has constant negative scalar curvature, then we have

$$\frac{d}{dt}\left(\int_M dV_g\right)=\int_M\frac{\partial}{\partial t}(dV_g)=-\frac{n}{2}\int_M(R_g-r_g)\,dV_g=0,$$

in particular

$$\int_M dV_{g_\infty}=\int_M dV_{g_0}.$$

On the other hand, $R_{g_t}=\zeta^{\frac{4}{n-2}}R_{g_\infty}$ where we can take ζ to be $\left(\frac{R_{g_t}}{R_{g_\infty}}\right)^{\frac{n-2}{4}}$. This implies that the metric $\zeta^{\frac{4}{n-2}}g_t$ has scalar curvature being equal to

$$R_{\zeta^{\frac{4}{n-2}}g_t}=\zeta^{-\frac{4}{n-2}}R_{g_t}=R_{g_\infty}.$$

By what proved in [9], which says that if g_1 and g_2 are two metrics conformal to g_0 such that $R_{g_1}=R_{g_2}<0$, then $g_1=g_2$. Therefore by what mention above we have

$$\int_M dV_{g_0}=\int_M dV_{g_\infty}=\int_M dV_{\zeta^{\frac{2n}{n-2}}g_t}=\zeta^{\frac{2n}{n-2}}\int_M dV_{g_t}=\zeta^{\frac{2n}{n-2}}\int_M dV_{g_0},$$

where by assumption above this implies that $\zeta=1$ or equivalently $g_t=g_\infty$. Note that by [7], we obtain

$$\min_M R_{g_0}\le r_{g(t)}\le\max_M R_{g_0}\qquad t\ge 0.$$

Also by process of the proof of theorem 1.1, in [7], we conclude that

$$\begin{aligned}&\left(\max_M R_{g_0}\right)\int_0^s\left(\max_M R_{g(\nu)}-r_{g(s)}\right)d\nu\\&\geq\left(r_{g(t)}-r_{g_0}\right)-\left(\max_M R_{g_0}-\min_M R_{g_0}\right)\\&\geq-2\left(\max_M R_{g_0}-\min_M R_{g_0}\right).\end{aligned}$$

As $t\longrightarrow\infty$, by what we mention above $g(t)\longrightarrow g_\infty$, we get

$$-2\left(1-\frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right)\geq\int_0^\infty\left(\max_M R_{g(\nu)}-r_{g(\nu)}\right)d\nu.$$

Similarly

$$2\left(1-\frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right)\leq\int_0^\infty\left(\min_M R_{g(\nu)}-r_{g(\nu)}\right)d\nu.$$

Now by above results and proposition 2.1, we finally obtain

$$ln\frac{\lambda_1(g_t)}{\lambda_1(g_0)}=ln\frac{\lambda_1(g_\infty)}{\lambda_1(g_0)}\geq(2n+2p-2s)\left[1-\frac{\min_M R_{g_0}}{\max_M R_{g_0}}\right]-s.$$

The inverse inequality holds in a similar way, so we prove what we were looking for. $\square$

2.1 Variation of $\lambda(t)$ under the normalized Yamabe flow on the surface

In this section we are going to give the proof of Theorem 1.4.
Proof.(**Theorem 1.4**) We only give a proof for first section of theorem 1.4, the other sections follow similar process. First of all, we have to mention that if $\left(M^2,g(t),f(t),d\mu\right)$ denotes the solution of the flow (4) with $s(t)=r$, on the smooth Riemannian compact surface, then we can find some bounds for the scalar curvature tensor R as below

- $r < 0$; $\quad r - ce^{rt} \leq R \leq r + ce^{rt}$,
- $r = 0$; $\quad -\frac{c}{1+ct} \leq R \leq c$,
- $r > 0$; $\quad -ce^{rt} \leq R \leq r + ce^{rt}$,

where c is constant and r is as similar as we found in the normalized Yamabe flow (3).
Now under $\Delta f \leq \gamma R$, $R < 0$, and $p \geq q$, we have

$$\frac{d}{dt}\lambda(u, v, t)|_{t=t_0} \geq \lambda(t_0)\left(r\left(\frac{p-q}{2} - \gamma\right) - ce^{rt_0}\left(\frac{p}{2} - \gamma\right)\right),$$

which implies that in any sufficient small neighborhood of t_0 as $I = [t_1, t_0]$ we have

$$\lambda(u, v, t_0) = \lambda(t_0), \quad \lambda(u, v, t_1) \geq \lambda(t_1),$$

where implies that

$$\begin{aligned} & ln\left(\lambda(t_0)\right) - \left(\frac{p-q}{2} - \gamma\right) rt_0 + \frac{c}{r}\left(\frac{p}{2} - \gamma\right) e^{rt_0} \\ & \geq ln\left(\lambda(t_1)\right) - \left(\frac{p-q}{2} - \gamma\right) rt_1 \\ & + \frac{c}{r}\left(\frac{p}{2} - \gamma\right) e^{rt_1}, \end{aligned}$$

which means

$$ln\left(\lambda(t)\right) - \left(\frac{p-q}{2} - \gamma\right) rt + \frac{c}{r}\left(\frac{p}{2} - \gamma\right) e^{rt},$$

is increasing. $\square$

3 Homogeneous 3-manifolds

Locally homogeneous 3-manifolds have been contained into 9 classes which are divided in two groups. The first is contained $H(3)$, $H(2) \times \mathbb{R}^1$ and $SO(3) \times \mathbb{R}^1$, and the other includes $\mathbb{R}^3$, SU(2), SL(2, $\mathbb{R}$), $Heisenberg$, E(1, 1) and E(2), in which the second group are called Bianchi classes. In this section we are going to give evolution of the first eigenvalue of the weighted (p, q)-Laplacian system (5), in a case of Bianchi classes.

Remark 3.1. Consider the evolution formula of $\lambda(t)$ under the flow (4), then in homogeneous condition where in this case R is independent from the volume element, we have

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &= \frac{\alpha+1}{2}\left(R-s(t_0)\right)\int_M |\nabla u|^p d\mu \\ &+ \frac{\beta+1}{2}\left(R-s(t_0)\right)\int_M |\nabla v|^q d\mu \\ &- \frac{\alpha+1}{p}\int_M \Delta f |\nabla u|^p d\mu \\ &- \frac{\beta+1}{q}\int_M \Delta f |\nabla v|^q d\mu.\end{aligned}$$

Now under consideration $\Delta f \leq R$ and $p \leq q$ under the Yamabe flow (2), where $s(t) = 0$, we finally get

$$\begin{aligned}\frac{d}{dt}\lambda(u,v,t)|_{t=t_0} &\geq R\left(\frac{p}{2}-1\right)\frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu \\ &+ R\left(\frac{q}{2}-1\right)\frac{\beta+1}{q}\int_M |\nabla v|^q d\mu \\ &\geq R\left(\frac{p}{2}-1\right)\lambda\left(u,v,t\right).\end{aligned} \tag{13}$$

First let us consider g_0 as a given metric in the Bianchi classes, [12] has provided before a frame $\{X_i\}_{i=1}^3$ in which both the Ricci tensors and metric are diagonalized and this property is preserved by Ricci flow. In this case, we consider the metric g as

$$g(t) = A(t)\left(\theta_1\right)^2 + B(t)\left(\theta_2\right)^2 + C(t)\left(\theta_3\right)^2,$$

where $\{\theta_i\}_{i=1}^3$ is the frame of 1-forms dual to $\{X_i\}_{i=1}^3$. Now we study the behavior of the first eigenvalue of weighted (p,q)-Laplacian in each classes separately.

Case 1: $\mathbb{R}^3$

In this case all metrics are flat, so for all $t \geq 0$ we have $g(t) = g_0$ where g_0 is constant, therefore $\lambda(t)$ is constant.

Case 2: Heisenberg
This class is isomorphic to the set of upper-triangular 3×3 matrices endowed with the usual matrix multiplication. Under the metric g_0 we choose a frame $\{X_i\}_{i=1}^3$ in which

$$[X_2,X_3]=X_1,\quad [X_3,X_1]=0,\quad [X_1,X_2]=0,$$

also under the normalization $A_0B_0C_0=1$ we have

$$R=-\frac{1}{2}A^2,\quad R_{11}=\frac{1}{2}A^3,\quad R_{22}=-\frac{1}{2}A^2B,\quad R_{33}=-\frac{1}{2}A^2C,$$
$$||Ric||^2=\frac{3}{4}A^4,$$

where under the Yamabe flow (2) we find

$$\frac{\partial g_{ij}}{\partial t}=\frac{1}{2}A^2g_{ij},$$

now set $g_{11}(t)=A(t)$, it concludes that

$$A'(t)=\frac{1}{2}A^3(t),\quad A(0)=A_0,$$

this is the equation depended to A and has the solution

$$A^2(t)=\frac{1}{A_0^{-2}-t},$$

by replacing R into the inequality (13) we get

$$\begin{aligned}\frac{\frac{d}{dt}\lambda(u,v,t)\,|_{t=t_0}}{\lambda(u,v,t)}&\geq\left(\frac{p}{2}-1\right)R\\&=\left(\frac{p}{2}-1\right)\left(-\frac{1}{2}A^2\right)\\&=\left(\frac{p}{4}-\frac{1}{2}\right)\left(\frac{-1}{A_0^{-2}-t_0}\right).\end{aligned}$$

Thus for the sufficient neighborhood of t_0 like I we have

$$\frac{\frac{d}{dt}\lambda(u,v,t)}{\lambda(u,v,t)}\geq\left(\frac{p}{4}-\frac{1}{2}\right)\left(\frac{-1}{A_0^{-2}-t_0}\right).$$

By integrating on $[t_1, t_0] \subset I$ from both sides of above inequality we see

$$ln\frac{\lambda(t_0)}{\lambda(t_1)} \geq ln\left(\frac{A_0^{-2} - t_0}{A_0^{-2} - t_1}\right)^{\left(\frac{p}{4} - \frac{1}{2}\right)},$$

thus

$$\lambda(t_0)\left(A_0^{-2} - t_0\right)^{-\left(\frac{p}{4} - \frac{1}{2}\right)} \geq \lambda(t_1)\left(A_0^{-2} - t_1\right)^{-\left(\frac{p}{4} - \frac{1}{2}\right)}.$$

Since t_0 is arbitrary, then

$$\lambda(t)\left(A_0^{-2} - t\right)^{-\left(\frac{p}{4} - \frac{1}{2}\right)},$$

is increasing.

Case 3: E(2)

Manifold E(2) is the group of isometries of Euclidian plane. In this case we have an Einstein metric and Ricci flow converges exponentially to flat metrics. Dependent to the metric g_0 we choose the frame $\{X_i\}_{i=0}^{3}$ such that

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = 0,$$

In this case under the normalization $A_0B_0C_0 = 1$ we have

$$R = -\frac{1}{2}\left(1 - \frac{B_0}{A_0}\right)^2 A^2, \quad R_{11} = \frac{1}{2}A\left(A^2 - B^2\right),$$
$$R_{22} = \frac{1}{2}B\left(B^2 - A^2\right), \quad R_{33} = -\frac{1}{2}C\left(A - B\right)^2,$$
$$||Ric||^2 = \frac{1}{2}\left(A^2 - B^2\right)^2 + \frac{1}{4}\left(A - B\right)^4,$$

and also under the Yamabe flow (2), we obtain

$$\frac{\partial g_{ij}}{\partial t} = \frac{1}{2}\left(1 - \frac{B_0}{A_0}\right)^2 A^2 g_{ij},$$

and similare to the case of *Heisenberg* by solving the equation $A'(t) = \frac{1}{2}\left(1 - \frac{B_0}{A_0}\right)^2 A^3$ we find that

$$A^2 = \frac{A_0^2}{1 - \left(A_0 - B_0\right)^2 t}.$$

Then by replacing R into the inequality (13) we conclude

$$\frac{\lambda'(u,v,t)\,|_{t=t_0}}{\lambda(u,v,t)} \geq -\frac{1}{2}\left(1-\frac{B_0}{A_0}\right)^2\left(\frac{p}{2}-1\right)\left(\frac{A_0^2}{1-(A_0-B_0)^2\,t_0}\right).$$

Now by replacing A and integrating on $[t_1,t_0]\subset I$ and since t_0 is arbitrary then the quantity

$$\lambda(t)\left(1-(A_0-B_0)^2\,t\right)^{-\left(\frac{p}{4}-\frac{1}{2}\right)},$$

is increasing.

Case 4: E(1,1)

Manifold E(1,1) is the group of isometries of the plane with flat Lorentz metric, there is no Einstein metric here and Ricci flow fails to converge, they all are asymptotically cigar degeneracies. For a given metric g_0 similarly by a frame $\{X_i\}_{i=0}^{3}$ we have

$$[X_1,X_2]=0,\quad [X_2,X_3]=-X_1,\quad [X_3,X_1]=X_2.$$

Also under the normalization $A_0B_0C_0=1$ we conclude

$$\begin{aligned}
&R=-\frac{1}{2}\left(1+\frac{B_0}{A_0}\right)^2A^2,\quad R_{11}=\frac{1}{2}A\left(A^2-B^2\right),\\
&R_{22}=\frac{1}{2}B\left(B^2-A^2\right),\quad R_{33}=-\frac{1}{2}C\left(A+B\right)^2,\\
&||Ric||^2=\frac{3}{4}A^4,
\end{aligned}$$

where under the Yamabe flow (2), we get

$$A^2=\frac{A_0^2}{1-(A_0+B_0)^2\,t}.$$

Now by replacing R into the inequality (13) and integrating, we conclude that

$$\lambda(t)\left(1-(A_0+B_0)^2\,t\right)^{-\left(\frac{p}{4}-\frac{1}{2}\right)},$$

is increasing.

Case 5: SU(2)
Similarly in this class we have Einstein metrics and Ricci flow converges exponentially in to these metrics, also by the frame $\{X_i\}_{i=0}^3$ we have

$$[X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3,$$

In this case under the normalization $A_0B_0C_0 = 1$, we get

$$\begin{aligned}
&R = \eta A^2, \quad R_{11} = \frac{1}{2}A[A^2 - (B-C)^2],\\
&R_{22} = \frac{1}{2}B[B^2 - (A-C)^2], \quad R_{33} = \frac{1}{2}C[C^2 - (A-B)^2],\\
&||Ric||^2 = \frac{1}{4}\Big[\Big(A^2 - (B-C)^2\Big)^2 + \Big(B^2 - (A-C)^2\Big)^2\\
&+ \Big(C^2 - (A-B)^2\Big)^2\Big],
\end{aligned}$$

where:

$$\begin{aligned}
\eta = \frac{1}{2}\Big\{1 - \Big(\frac{B_0}{A_0} - \frac{C_0}{A_0}\Big)^2 + \Big(\frac{B_0}{A_0}\Big)^2 - \Big(1 - \frac{C_0}{A_0}\Big)^2\\
+ \Big(\frac{C_0}{A_0}\Big)^2 - \Big(1 - \frac{B_0}{A_0}\Big)^2\Big\},
\end{aligned}$$

and under the Yamabe flow (2), we have $A^2 = \frac{1}{A_0^{-2}+\eta t}$, by replacing R into the inequality (13) and integrating, we find that if $A_0 \geq 4B_0 = 4C_0$ then

$$\lambda(t)\left(A_0^{-2} + \eta t\right)^{-\left(\frac{p}{2}-1\right)},$$

is increasing.

Case 6: SL(2,$\mathbb{R}$)
There is no Einstein metric here and the Ricci flow doesn't converge and develops a pancake degeneracy, also by the frame $\{X_i\}_{i=0}^3$, we get

$$[X_2, X_3] = -X_1, \quad [X_3, X_1] = X_2, \quad [X_1, X_2] = X_3,$$

in this case we also have

$$
\begin{aligned}
&R = \eta A^2, \quad R_{11} = \frac{1}{2}A[A^2 - (B-C)^2],\\
&R_{22} = \frac{1}{2}B[B^2 - (A+C)^2]\\
&R_{33} = \frac{1}{2}C[C^2 - (A+B)^2],\\
&||Ric||^2 = \frac{1}{4}\big\{[A^2 - (B-C)^2]^2 + [B^2 - (A+C)^2]^2\\
&+[C^2 - (A+B)^2]^2\big\},
\end{aligned}
$$

in which

$$
\eta = -\frac{1}{2}\Big\{1 + \left(\frac{B_0}{A_0}\right)^2 + \left(\frac{C_0}{A_0}\right)^2 + 2\frac{B_0}{A_0} + 2\frac{C_0}{A_0} - 2\frac{B_0 C_0}{A_0}\Big\},
$$

and also under the Yamabe flow (2), we find

$$
A^2 = \frac{1}{A_0^{-2} - \eta t}.
$$

Now if $B_0 = C_0$ then by replacing R into the inequality (13) and integrating, we conclude

$$
\lambda(t)\left(A_0^{-2} - \eta t\right)^{\left(\frac{p}{2}-1\right)},
$$

is increasing.

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